EXAM 1

Math 212, 2013-2014 Spring, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines
on the class webpages are in effect on this exam.

Good luck!

Name Solutions

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________

Total Score ____________ (/100 points)

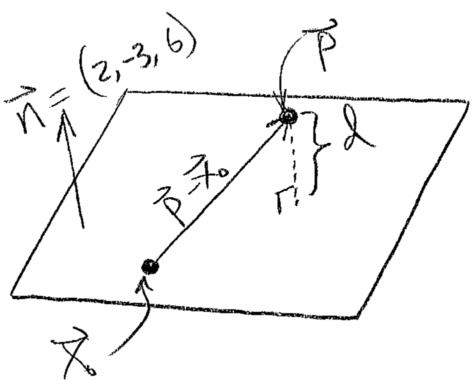
1. *(10 pts)* For the vectors below, find their magnitudes and the cosine of the angle between them.

$$\vec{v} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

$$\|\vec{v}\| = \sqrt{(1)^2 + (-4)^2 + (2)^2} = \boxed{\sqrt{21}} \qquad \|\vec{w}\| = \sqrt{(5)^2 + (1)^2 + (3)^2} = \boxed{\sqrt{35}}$$

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{1 \cdot 5 + (-4) \cdot 1 + 2 \cdot 3}{\sqrt{21}\ \sqrt{35}} = \frac{7}{7\sqrt{15}} = \boxed{\frac{1}{\sqrt{15}}}$$

2. *(15 pts)* Find the distance between the point $\vec{p} = (3, -2, 5)$ and the plane with equation $2x - 3y + 6z = 4$. *(Hint: How does this relate to the $\vec{n}$ component of the vector $\vec{p} - \vec{x}_0$, where $\vec{x}_0$ is in the plane and $\vec{n}$ is normal to the plane?)*



$$d = \text{comp}_{\vec{n}}(\vec{p} - \vec{x}_0) = \frac{\vec{n} \cdot (\vec{p} - \vec{x}_0)}{\|\vec{n}\|}$$

Arbitrarily we choose $\vec{x}_0 = (2,0,0)$. Then

$$d = \frac{\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}}{\sqrt{2^2 + (-3)^2 + 6^2}} = \boxed{\frac{38}{7}}$$

3. *(15 pts)* Consider the vectors

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}$$

(a) Find the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.

$$\vec{u} \times \vec{v} = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 1 & 2 & 3 \\ 2 & 3 & 0 \end{pmatrix} = (-9, 6, -1)$$

$$\text{area} = \|\vec{u} \times \vec{v}\| = \boxed{\sqrt{118}}$$

(b) Find the volume of the parallelepiped defined by $\vec{u}$, $\vec{v}$, and $\vec{w}$, and determine if the listing $\vec{u}$, $\vec{w}$, $\vec{v}$ *(observe carefully!)* is in right hand order or left hand order.

$$\det \begin{pmatrix} \vec{w} \\ \vec{u} \\ \vec{v} \end{pmatrix} = \vec{w} \cdot (\vec{u} \times \vec{v}) = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ 6 \\ -1 \end{pmatrix} = -8$$

$\Rightarrow$ Volume $= |\det| = 8$, and $\vec{w}, \vec{u}, \vec{v}$ is in LHO.

So $\vec{u}, \vec{w}, \vec{v}$ is in RHO.

(c) Use a clever application of the equation below to show that for all vectors $\vec{x}, \vec{y} \in \mathbb{R}^3$ the listing $\vec{x} \times \vec{y}$, $\vec{x}$, $\vec{y}$ is never in left hand order. *Be sure to explain your reasoning.*

$$\det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right)$$

$$\det \begin{pmatrix} \vec{x} \times \vec{y} \\ \vec{x} \\ \vec{y} \end{pmatrix} = (\vec{x} \times \vec{y}) \times (\vec{x} \times \vec{y}) = \|\vec{x} \times \vec{y}\|^2 \geq 0$$

This det is never negative, so $\vec{x} \times \vec{y}, \vec{x}, \vec{y}$ is never in left hand order.

4. *(15 pts)* Compute the velocity, acceleration, and curvature of the parametric curve $\vec{x}(t) = (t^2+1, 2t-1, t^3-5)$ at the point $(2, 1, -4)$.

$$\vec{x} = \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix} \Rightarrow y = 2t - 1 = 1 \Rightarrow t = 1$$

$$\vec{v} = \vec{x}\,' = \begin{pmatrix} 2t \\ 2 \\ 3t^2 \end{pmatrix} \qquad \vec{v}(1) = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{a} = \vec{v}\,' = \begin{pmatrix} 2 \\ 0 \\ 6t \end{pmatrix} \qquad \vec{a}(1) = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix}$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} \qquad \kappa(1) = \frac{\|(12, -6, -4)\|}{\left(\sqrt{2^2+2^2+3^2}\right)^3}$$

$$= \boxed{\frac{14}{17\sqrt{17}}}$$

5. *(15 pts)*

(a) The curve C in the xy-plane has equation $y = x^2 - 5$. Find the equation of the surface S_1 obtained by rotating C around the y-axis in xyz-space.

$\boxed{y = x^2 + z^2 - 5}$ has (1) correct rotational symmetry

(2) correct cross section in xy-plane.

So this is the equation of the desired surface

(b) The hyperboloid H has equation $x^2 - 3y^2 + 5z^2 = 1$. Find the equation of the surface S_2 obtained by stretching H by a factor of 4 in the x-direction.

Stretch is achieved by replacing "x" with "$\frac{x}{4}$", so S_2 has equation

$$\boxed{\left(\frac{x}{4}\right)^2 - 3y^2 + 5z^2 = 1}$$

(c) Find the equation of the surface S_3 obtained by translating H (from the previous part) by the vector $\vec{v} = (1, 3, 6)$ and then reflecting the result through the xz-plane.

translating gives $(x-1)^2 - 3(y-3)^2 + 5(z-6)^2 = 1$

reflecting gives $(x-1)^2 - 3((-y)-3)^2 + 5(z-6)^2 = 1$

$$\boxed{(x-1)^2 - 3(y+3)^2 + 5(z-6)^2 = 1}$$

6. *(15 pts)* In this question we consider the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x, y) = (y - x^2)^2$.

(a) Find the equation of the level set of this function that includes the point $(1, 2)$.

$f(1,2) = 1$, so the level set has equation $f(x,y) = 1$,
which is $(y-x^2)^2 = 1$

(b) Explain how you know the level set from the previous part is not the graph of any other function.

For $x=0$, we have that both $y=1$ and $y=-1$ work in the equation. So this curve fails the vertical line test.

(c) Find the only function g whose graph is also a level set of f.

$f=c$ will be solved when $y = x^2 \pm \sqrt{c}$

For $c>0$, this causes failure of vert. line test.

For $c<0$, there are no solutions.

For $c=0$, we have $y=x^2+0$ — graph of $\boxed{g(x)=x^2}$

7. *(15 pts)* Compute (or show it does not exist) the limit below.

$$\lim_{\vec{x}\to\vec{0}} \frac{x^2}{x^2+y^2}$$

Consider paths $y=mx$ to $\vec{0}$. Along these paths,

$$\lim_{x\to 0} \frac{x^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{1}{1+m^2} = \frac{1}{1+m^2}$$

This is different for different m. So the limit DNE.