EXAM 3

Math 103, Spring 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score __________ (/100 points)

1. *(16 pts)* Let R be the region in xyz-space bounded by the six surfaces $x+y+z=1$, $x+y+z=3$, $2x+y=0$, $2x+y=1$, $y+4z=2$, and $y+4z=5$. Compute the integral

$$\iiint_R (2x+y)^2\,dx\,dy\,dz$$

$u = x+y+z$
$v = 2x+y$
$w = y+4z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 4 \end{pmatrix} = -2$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2}$$

$$\iiint_R (2x+y)^2\,dx\,dy\,dz = \int_2^5\int_0^1\int_1^3 (v^2)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

$$= \int_2^5\int_0^1\int_1^3 \frac{1}{2}v^2\,du\,dv\,dw$$

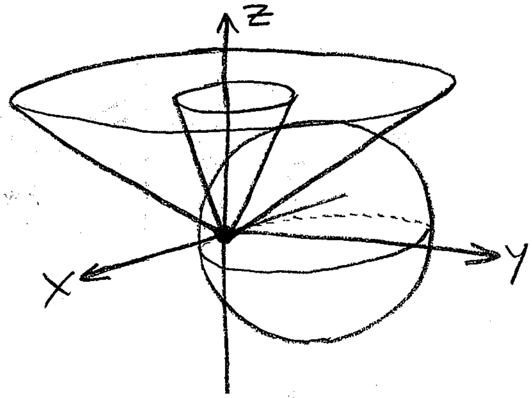
$$= \int_2^5\int_0^1 v^2\,dv\,dw$$

$$= \int_2^5 \frac{1}{3}v^3\Big]_{v=0}^{v=1}\,dw$$

$$= \int_2^5 \frac{1}{3}\,dw$$

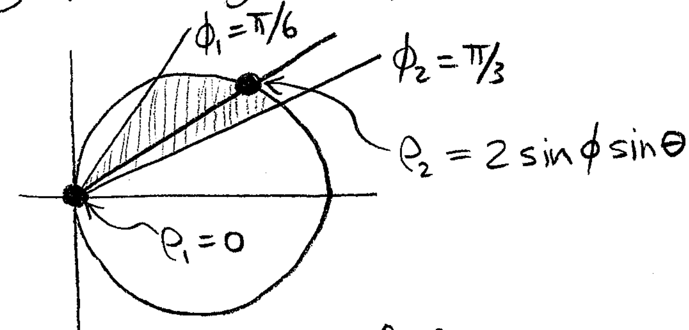
$$= 1$$

2. *(14 pts)* Write down and simplify, but do not evaluate, a *single* triple nested integral (in whatever coordinate system you can) that represents the volume of the part of the ball $x^2 + (y-1)^2 + z^2 \leq 1$ that is outside of the cone $\phi = \pi/6$ and inside the cone $\phi = \pi/3$.



① $\theta \in [0, \pi]$

② For a given θ, cross-section is:



$x^2 + (y-1)^2 + z^2 = 1$

$\rho^2 - 2y + 1 = 1$

$\rho^2 - 2\rho\sin\phi\sin\theta = 0$

$\rho = 2\sin\phi\sin\theta$

$$V = \iiint dV = \iiint \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\boxed{= \int_0^{\pi} \int_{\pi/6}^{\pi/3} \int_0^{2\sin\phi\sin\theta} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

3. *(14 pts)* Write down and simplify, but do not evaluate, a nested integral representing the area of the part of the surface $4 + x^2 - y - z^2 = 0$ that is inside the cylinder $x^2 + z^2 = 2$.

$y = 4 + x^2 - z^2$, param. by $\vec{r}(u,v) = (u, 4+u^2-v^2, v)$

over $u^2 + v^2 \leq 2$.

$\vec{r}_u = (1, 2u, 0)$

$\vec{r}_v = (0, -2v, 1)$

$\vec{N} = (2u, 1, -2v)$

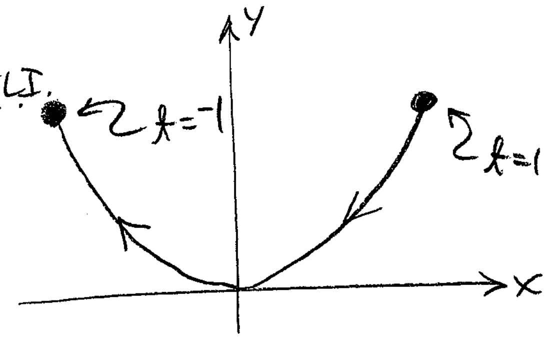
$\|\vec{N}\| = \sqrt{4u^2 + 1 + 4v^2}$

$$S = \iint dS = \iint_S \|\vec{N}\| \, du \, dv = \boxed{\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-v^2}}^{\sqrt{2-v^2}} \sqrt{1 + 4u^2 + 4v^2} \, du \, dv}$$

4. *(14 pts)* Let C be the part of the curve $y = x^2$ with $0 \le y \le 1$, oriented to the left. Compute the line integral over this curve of the field $\vec{F} = (3y, -2x)$.

$\text{grn}\vec{F} = Q_x - P_y = (-2) - (3) = -5 \neq 0$

So $\vec{F}$ is <u>not</u> a gradient, can't use FTLI.



Parametrize C by $\vec{r}(t) = (t, t^2)$

t starts at 1, ends at -1.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \begin{pmatrix} 3y \\ -2x \end{pmatrix} \cdot \vec{r}\,' \, dt = \int_1^{-1} \begin{pmatrix} 3t^2 \\ -2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$

$$= \int_1^{-1} (-t^2)\, dt = \left(-\frac{1}{3}t^3\right]_1^{-1} = \frac{2}{3}$$

5. *(14 pts)* Let C be the curve parametrized by $\vec{r}(t) = (t^2 - 4t + 5, \cos(2\pi t)e^{t^4 - t}, \sin(2\pi t))$, with $t \in [0, 1]$. Compute the line integral over this curve of the field $\vec{F} = (2xe^{yz}, x^2ze^{yz}, x^2ye^{yz})$.

$$\nabla \times \vec{F} = \left(x^2e^{yz} - x^2e^{yz},\ 2xye^{yz} - 2xye^{yz},\ 2xze^{yz} - 2xze^{yz}\right)$$

$= (0, 0, 0) \implies \vec{F}$ <u>is</u> a gradient of some f.

$$\left.\begin{aligned} f &= x^2e^{yz} + c_1(y,z) \\ f &= x^2e^{yz} + c_2(x,z) \\ f &= x^2e^{yz} + c_3(x,y) \end{aligned}\right\} \implies \text{choose } f = x^2e^{yz}$$

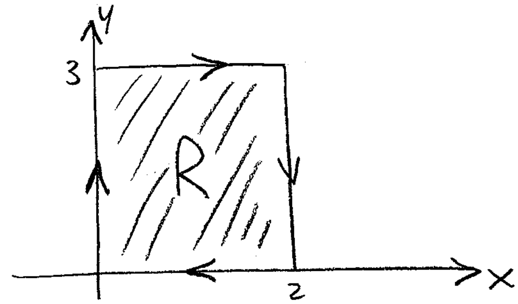
$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - f\begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$$

$$= 4e^0 - 25e^0 = -21$$

6. *(14 pts)* Let C be the rectangle with vertices at $(0,0)$, $(2,0)$, $(0,3)$, $(2,3)$, oriented clockwise. Compute the line integral over this curve of the field $\vec{F} = (x^2y + xe^{x^3}, xy - \sin^2(e^y))$.

$$\operatorname{grn}\vec{F} = Q_x - P_y = y - x^2$$



$$\int_C \vec{F}\cdot d\vec{r} = -\iint_R (y - x^2)\,dx\,dy$$

b/c of clockwise or. of C.

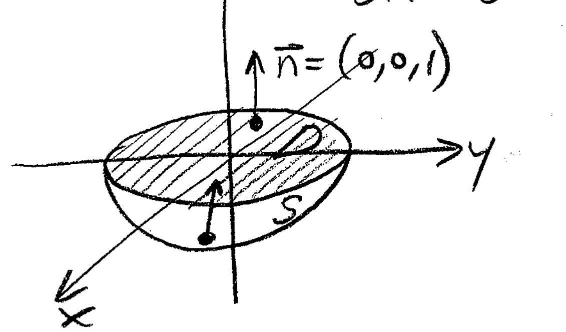
$$= \int_0^3\int_0^2 x^2 - y\,dx\,dy = \int_0^3 \left(\tfrac{1}{3}x^3 - xy\right]_{x=0}^{x=2} dy$$

$$= \int_0^3 \frac{8}{3} - 2y\,dy = \left(\frac{8}{3}y - y^2\right]_0^3 = -1$$

7. *(14 pts)* Let S be the part of the surface $z = (x^2 + y^2 - 1)(x^4 + y^4 + 1)$ that is below the plane $z = 0$, oriented upward. Compute the flux through this surface of the field $\vec{F} = (xy^3 + y^2z, x^2z^2 - x^2y, zx^2 - zy^3)$.

$$\partial R = D - S$$

$$\nabla\cdot\vec{F} = y^3 - x^2 + x^2 - y^3 = 0$$



$$0 = \iiint_R \nabla\cdot\vec{F}\,dV = \iint_{\partial R}\vec{F}\cdot d\vec{S}$$

$$= \iint_D \vec{F}\cdot d\vec{S} - \iint_S \vec{F}\cdot d\vec{S}$$

$$\Rightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S}$$

On D, we have $z=0$, and so also $\vec{F} = (xy^3, -x^2y, 0)$

Then $\iint_D \vec{F}\cdot\vec{n}\,dS = \iint 0\,dS = 0$