

# EXAM 2

Math 103, Spring 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING  
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number \_\_\_\_\_

1. \_\_\_\_\_ "I have adhered to the Duke Community Standard in completing this examination."
2. \_\_\_\_\_
3. \_\_\_\_\_ Signature: \_\_\_\_\_
4. \_\_\_\_\_
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6. \_\_\_\_\_
7. \_\_\_\_\_
8. \_\_\_\_\_ Total Score \_\_\_\_\_ (/100 points)

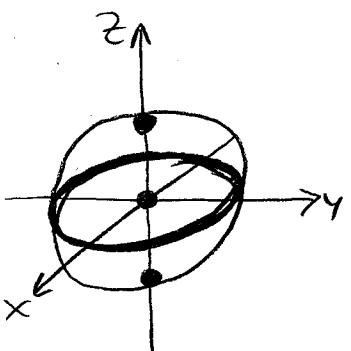
1. (16 pts) The solid  $D$  is bounded by the surface with equation  $z^2 - \ln(2 - x^2 - y^2) = 0$ . Find the absolute maximum value of the function  $f(x, y, z) = x^2 + y^2 + z^2$  over the solid  $D$ .

Interior: Need  $\nabla f = \vec{0}$

$$\Rightarrow \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{x} = \vec{0} \text{ crit. pt.}$$

Boundary: Let  $g = z^2 - \ln(2 - x^2 - y^2)$

Need either  $\nabla g = \vec{0}$  or  $\nabla f = \lambda \nabla g$ .



$$\nabla g = \vec{0} \Rightarrow \begin{pmatrix} 2x/z^2 \\ 2y/z^2 \\ 2z \end{pmatrix} = \vec{0} \Rightarrow x = 0$$

Not c.p. b/c not on  $g = 0$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x/z^2 \\ 2y/z^2 \\ 2z \end{pmatrix}$$

$$3\text{rd coord} \Rightarrow \lambda = 1 \text{ or } z = 0$$

$$\text{If } z = 0, \text{ then } \ln(2 - x^2 - y^2) = 0$$

$$\Rightarrow x^2 + y^2 = 1 \text{ crit. pts}$$

$$\text{If } \lambda = 1 \text{ and } z \neq 0, \text{ then } 2 - x^2 - y^2 \neq 1$$

$$\Rightarrow x = y = 0, z = \pm \sqrt{\ln 2} \text{ crit. pt.}$$

Checking values, we get:

$$f(0, 0, 0) = 0$$

$$f(0, 0, \pm \sqrt{\ln 2}) = \ln 2 < 1$$

$$\text{on } x^2 + y^2 = 1, z = 0, \Rightarrow f = 1 \leftarrow \text{absolute max}$$

2. (12 pts) Classify the critical point  $(1, 1)$  of the function  $f(x, y) = (y - x^2)(2x - y - 1)$ .

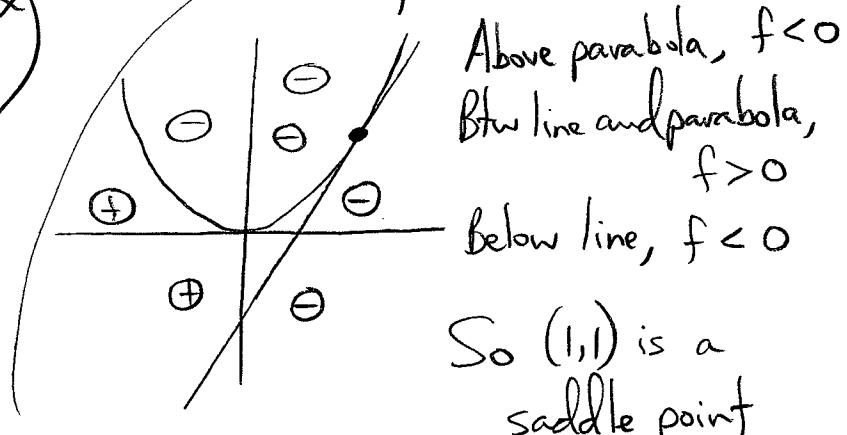
$$f = 2xy - y^2 - y - 2x^3 + x^2y + x^2$$

2nd der test fails, consider nearby values :

$$H = \begin{pmatrix} -12x + 2y + 2 & 2 + 2x \\ 2 + 2x & -2 \end{pmatrix}$$

$$H|_{(1,1)} = \begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix}$$

$$\Delta = (-8)(-2) - (4)(4) = 0$$



3. (12 pts) Find the unit vector representing the direction of fastest increase of the function  $h(x, y, z) = e^{xyz}$  at the point  $(1, 2, 3)$ .

$$\nabla h = \begin{pmatrix} yz e^{xyz} \\ xz e^{xyz} \\ xy e^{xyz} \end{pmatrix} \Rightarrow \nabla h \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right) = \begin{pmatrix} 6e^6 \\ 3e^6 \\ 2e^6 \end{pmatrix}$$

Dir. of fastest increase is dir. of  $\nabla h$ ,

$$\vec{\mu} = \frac{\nabla h}{\|\nabla h\|} = \frac{\nabla h}{e^6 \sqrt{6^2 + 3^2 + 2^2}} = \frac{\nabla h}{7e^6}$$

$$= \begin{pmatrix} 6/7 \\ 3/7 \\ 2/7 \end{pmatrix}$$

4. (12 pts) Use the total derivative to approximate the value of the function  $P(x, y) = e^{xy} - x(y+1)^2$  at the point  $(3.98, 0.01)$ .

$$\vec{a} = (4, 0) \quad d\vec{x} = (-.02, .01)$$

$$dP = \frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy$$

$$= (ye^{xy} - (y+1)^2) dx + (xe^{xy} - 2x(y+1)) dy$$

At  $\vec{a}$ , this becomes

$$dP = (-1)(-.02) + (-4)(.01) = -.02$$

$$\begin{aligned} \text{So } P(\vec{x}) &\approx P(\vec{a}) + dP \\ &\approx -3 + (-.02) \\ &\approx \boxed{-3.02} \end{aligned}$$

5. (12 pts) Suppose that  $z = f(x, y)$ ,  $x = u - 3v$ , and  $y = 3u + v$ . Find the mixed second partial  $z_{uv}$  in terms of partials of  $f$ . (Assume  $f$  has continuous 2nd partials)

$$z_u = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\begin{matrix} z & z_x & z_y \\ x & & y \\ u & & v \end{matrix}$$

$$z_u = z_x + 3z_y$$

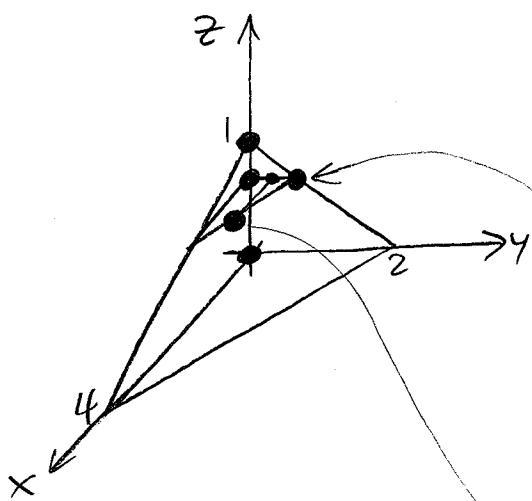
$$z_{uv} = \frac{\partial}{\partial v} (z_u) = \frac{\partial}{\partial v} (z_x) + 3 \frac{\partial}{\partial v} (z_y)$$

$$= \left( \frac{\partial z_x}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z_x}{\partial y} \frac{\partial y}{\partial v} \right) + 3 \left( \frac{\partial z_y}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z_y}{\partial y} \frac{\partial y}{\partial v} \right)$$

$$= (-3z_{xx} + z_{xy}) + 3(-3z_{xy} + z_{yy})$$

$$= -3z_{xx} - 8z_{xy} + 3z_{yy}$$

6. (16 pts) Compute the  $z$ -coordinate of the centroid of the solid region  $T$ , bounded by the surfaces  $x = 0$ ,  $y = 0$ ,  $z = 0$ , and  $x + 2y + 4z = 4$ .



$$z \in [0, 1]$$

For given  $z$ ,  $y \in [0, y_2]$ ,  
where  $y_2$  def. by:  $x = 0$   
 $x + 2y + 4z = 4$   
 $z = z$   
 $\Rightarrow y_2 = 2 - 2z$

For given  $y, z$ ,  $x \in [0, x_2]$ ,  
where  $x_2$  def. by:  $x + 2y + 4z = 4$   
 $y = y, z = z$

$$\Rightarrow x_2 = 4 - 2y - 4z$$

$$\bar{z} = \frac{1}{V} \iiint z \, dv = \frac{1}{\frac{1}{6}(1)(2)(4)} \int_0^1 \int_0^{2-2z} \int_0^{4-2y-4z} z \, dx \, dy \, dz$$

$$= \frac{3}{4} \int_0^1 \int_0^{2-2z} 4z - 2yz - 4z^2 \, dy \, dz$$

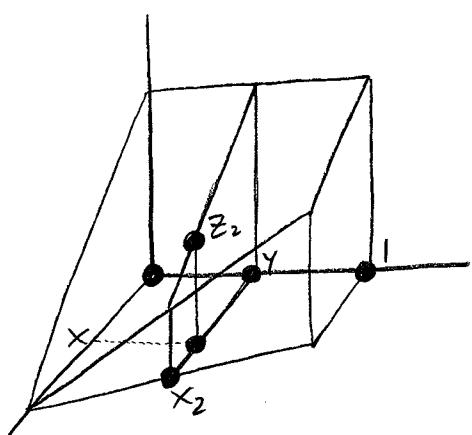
$$= \frac{3}{4} \int_0^1 \left( (4z - 4z^2)y - y^2 z \Big|_{y=0}^{y=2-2z} \right) dz$$

$$= \frac{3}{4} \int_0^1 8(z^3 - 2z^2 + z) - 4(z^3 - 2z^2 + z) \, dz$$

$$= \left(\frac{3}{4}\right)(4) \left( \frac{1}{4}z^4 - \frac{2}{3}z^3 + \frac{1}{2}z^2 \Big|_0^1 \right)$$

$$= \boxed{\frac{1}{4}}$$

7. (10 pts) Set up, but do not evaluate, an iterated integral representing the mass of the solid part of the first octant bounded by the coordinate planes, and the surfaces  $y = 1$ ,  $2x+y = 2$ , and  $x+z = 1$ , where the density function is given by  $\delta(x, y, z) = \ln(1+x^2+3y^2+2z^2)$ .

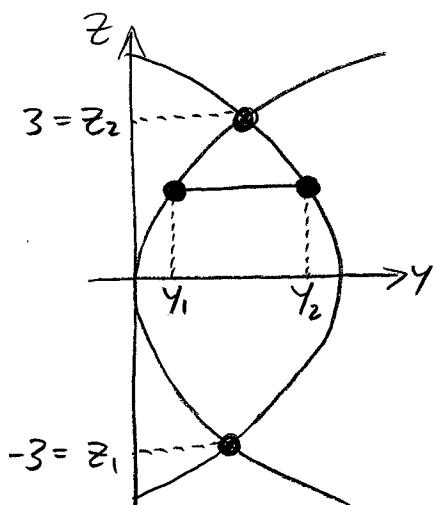


$$m = \iiint \delta \, dV$$

$$y \in [0, 1] ; x \in [0, \frac{2-y}{2}] ; z \in [0, 1-x]$$

$$m = \int_0^1 \int_0^{2-y} \int_0^{1-x} \ln(1+x^2+3y^2+2z^2) \, dz \, dx \, dy$$

8. (10 pts) Set up, but do not evaluate, an iterated integral representing the moment of inertia around the  $z$ -axis of the region in the  $yz$ -plane bounded by the curves  $y = z^2$  and  $y = 18 - z^2$ , where density is given by  $\delta(x, y, z) = y^2$ .



$$I = \iint r^2 \, dm = \iint y^2 \delta \, dA$$

$$= \iint y^4 \, dA$$

$$= \int_{-3}^3 \int_{-z^2}^{18-z^2} y^4 \, dy \, dz$$