EXAM 1

Math 103, Spring 2008-2009, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions ______________________

ID number______________________

1. ____________
2. ____________
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9. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

Total Score ____________ (/100 points)

Consider the vectors

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

1. *(12 pts)* What is the volume of the parallelepiped that is defined by these three vectors?

$$\det \begin{pmatrix} 3 & 2 & 4 \\ 0 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix} = 3 \det \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} - 2 \det \begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} + 4 \det \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$$

$$= 3 \cdot 2 - 2 \cdot (-3) + 4(-6)$$

$$= -12$$

$$V = |\det| = \boxed{12}$$

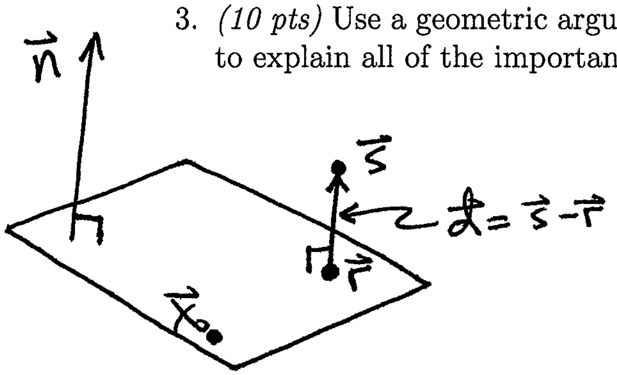
2. *(10 pts)* Is the ordered list of vectors $\vec{u}, \vec{v}, \vec{w}$ in right hand order, or not in right hand order? Make sure you explain your reasoning.

The above determinant is negative, so the list $\vec{u}, \vec{v}, \vec{w}$ is not in right hand order.

Suppose we are given a point $\vec{s}$, and a plane P defined by the equation $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$, given a normal vector $\vec{n}$ (pointing to the same side of P that the point $\vec{s}$ is on) and a point $\vec{x}_0$ in P.

On this page you will derive the formula for the distance from $\vec{s}$ to P (in terms of the givens $\vec{n}$, $\vec{s}$, and $\vec{x}_0$) by considering the point $\vec{r}$ on P that is closest to $\vec{s}$, and the difference vector $\vec{d} = \vec{s} - \vec{r}$. (You should draw a figure representing these objects.)

3. *(10 pts)* Use a geometric argument to compute $\vec{n} \cdot \vec{d}$ in terms of $\|\vec{n}\|$ and $\|\vec{d}\|$. (Make sure to explain all of the important points in your reasoning!)



$$\vec{n} \cdot \vec{d} = \|\vec{n}\| \, \|\vec{d}\| \cos \theta$$

$\vec{n}$ and $\vec{d}$ are both $\perp$ to plane, so they are parallel, so $\theta = 0$.

$$\Rightarrow \vec{n} \cdot \vec{d} = \|\vec{n}\| \, \|\vec{d}\| \cos(0)$$
$$= \|\vec{n}\| \, \|\vec{d}\|$$

4. *(10 pts)* Use the above definition of $\vec{d}$ to compute $\vec{n} \cdot \vec{d}$ in terms of dot products of $\vec{n}$, $\vec{s}$, and $\vec{x}_0$, and then combine with the above result to solve for the desired distance as $\|\vec{d}\|$ in terms of the givens.

$$\vec{n} \cdot \vec{d} = \vec{n} \cdot (\vec{s} - \vec{r})$$
$$= \vec{n} \cdot \vec{s} - \vec{n} \cdot \vec{r}$$

$\vec{r}$ is in the plane with equation $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$, so we have $\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{x}_0$. So

$$\vec{n} \cdot \vec{d} = \vec{n} \cdot \vec{s} - \vec{n} \cdot \vec{x}_0$$

Combining with the result of (3), we have

$$\|\vec{n}\| \, \|\vec{d}\| = \vec{n} \cdot \vec{s} - \vec{n} \cdot \vec{x}_0$$

$$\|\vec{d}\| = \frac{\vec{n} \cdot \vec{s} - \vec{n} \cdot \vec{x}_0}{\|\vec{n}\|}$$

Consider the parametric curve $\vec{x}(t)$ for which we know the velocity is given by $\vec{v}(t) = (2t, 6t^2, \pi \sin(\pi t))$.

5. *(12 pts)* Suppose that we know $\vec{x}(0) = \vec{0}$. Compute $\vec{x}(1)$.

$$\vec{x}(1) - \vec{x}(0) = \int_0^1 \vec{v}(t)\,dt = \int_0^1 \left(2t, 6t^2, \pi \sin(\pi t)\right) dt$$

$$= \left(t^2, 2t^3, -\cos(\pi t)\right)\Big]_0^1$$

$$= \left(1, 2, 1-(-1)\right) = (1,2,2)$$

$$\vec{x}(1) = (1,2,2) + \vec{0} = \boxed{(1,2,2)}$$

6. *(12 pts)* Find the curvature of this path at the point where $t = 1$.

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3}$$

$$\vec{v}(1) = (2,6,0)$$

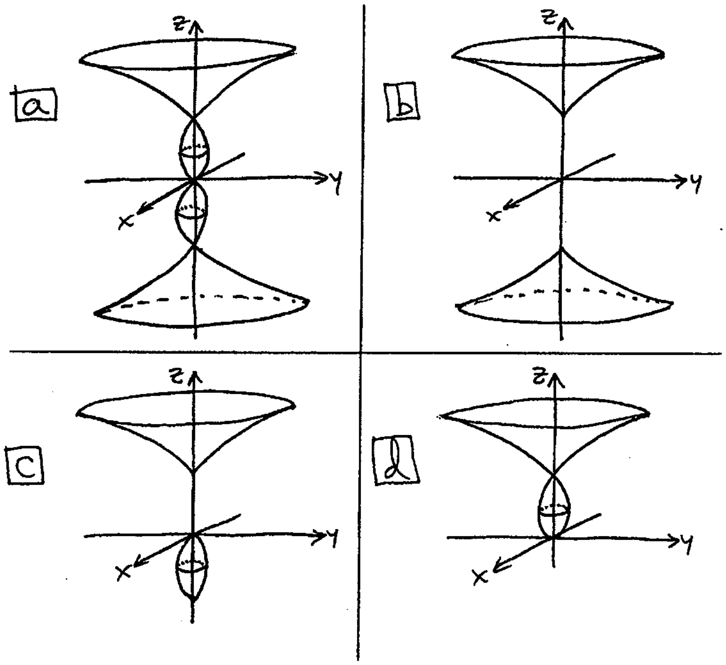
$$\vec{a}(t) = \vec{v}'(t) = \left(2, 12t, \pi^2 \cos(\pi t)\right)$$

$$\vec{a}(1) = \left(2, 12, -\pi^2\right)$$

$$\vec{v}(1) \times \vec{a}(1) = \det \begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 2 & 6 & 0 \\ 2 & 12 & -\pi^2 \end{pmatrix}$$

$$= \left(-6\pi^2, 2\pi^2, 12\right)$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{\|\vec{v}\|^3} = \frac{\sqrt{144 + 40\pi^4}}{\left(\sqrt{40}\right)^3} = \boxed{\frac{\sqrt{36 + 10\pi^4}}{4\left(10^{3/2}\right)}}$$

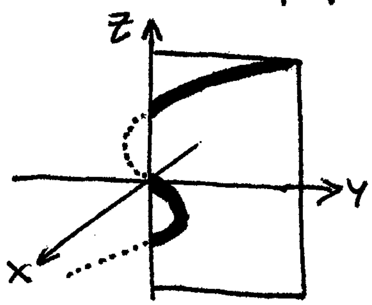


7. *(12 pts)* Which one of the surfaces above (a, b, c, or d) is the most accurate representation of the solution set to the equation

$$\sqrt{x^2+y^2} = z^3 - z$$

(Make sure to explain your reasoning!)

The surface is a rotation around the z-axis. Looking in the y-positive part of the yz-plane, we have:



$$\sqrt{x^2+y^2} = z^3 - z$$

$$x = 0,\ y \geq 0$$

$$\Rightarrow y = z^3 - z$$

Rotating this half-plane cross section around the z-axis gives surface $\boxed{c}$ above.

8. *(12 pts)* Compute the following limit:

$$\lim_{\vec{x}\to\vec{0}} \frac{x^2y^2 - xy^3}{x^4 + y^4}$$

Check along lines $y = mx$:

$$\lim_{x\to 0} \frac{x^2(mx)^2 - x(mx)^3}{x^4 + (mx)^4} = \lim \frac{x^4(m^2-m^3)}{x^4(1+m^4)}$$

$$= \frac{m^2-m^3}{1+m^4}$$

This gives different values for different lines, so the original limit does not exist.

9. *(10 pts)* Find a function g such that one of the level sets of g is the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}^1$, given by

$$f(x, y) = \sin x - e^{xy}$$

Graph of f is $z = f(x,y)$

$$z = \sin x - e^{xy}$$

This is equivalent to

$$z - \sin x + e^{xy} = 0$$

which is the level set $g = 0$ for the function

$$\boxed{g(x,y,z) = z - \sin x + e^{xy}}$$