EXAM 3

Math 103, Spring 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number ______________________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

9. ____________

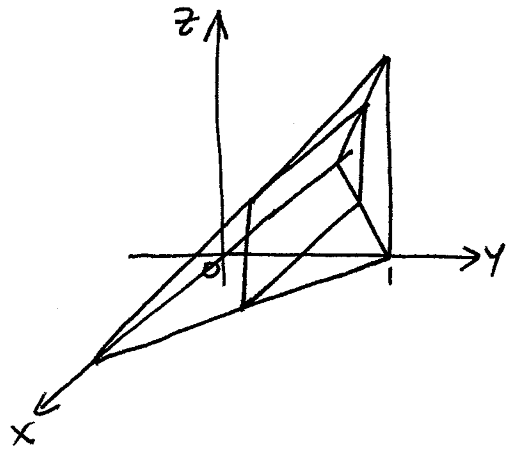
10. ____________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

Total Score ____________ (/100 points)

1. *(10 pts)* Find the moment of inertia around the x-axis of the solid bounded by the planes $z = 0$, $x + y = 1$, $y = 1 + x$, and $y = z$, with constant density δ.



$$\int_0^1 \int_{y-1}^{1-y} \int_0^y (y^2+z^2)\delta \; dz\, dx\, dy$$

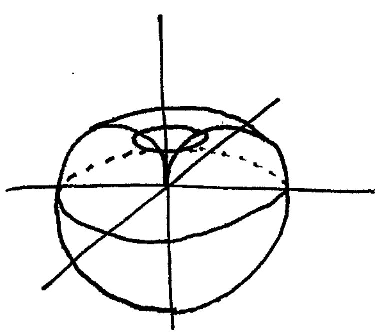
$$= \delta \int_0^1 \int_{y-1}^{1-y} y^2 z + \frac{1}{3} z^3 \Big]_{z=0}^{z=y} dx\, dy$$

$$= \delta \int_0^1 \int_{y-1}^{1-y} \frac{4}{3} y^3 \, dx\, dy$$

$$= \frac{8\delta}{3} \int_0^1 1-y \; dy$$

$$= \frac{4\delta}{3}$$

2. *(10 pts)* Find the volume of the solid bounded by the surface with spherical equation $\rho = 1 - \cos\phi$.



$$V = \int_0^{2\pi} \int_0^{\pi} \int_0^{1-\cos\phi} 1 \; \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

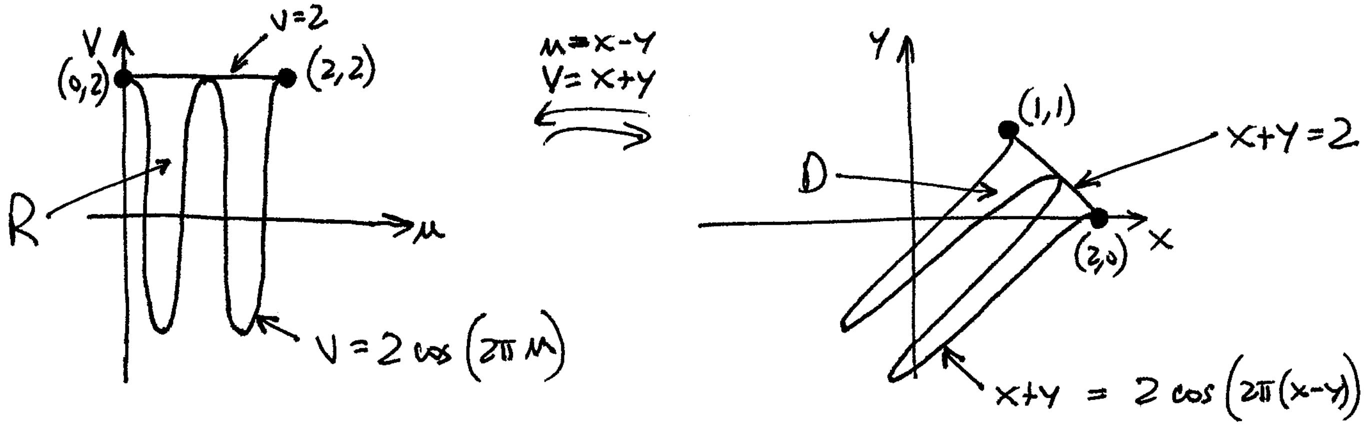
$$= \int_0^{2\pi} \int_0^{\pi} \left(\frac{1}{3} \rho^3 \sin\phi \Big]_{\rho=0}^{\rho=1-\cos\phi} \right) d\phi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \frac{1}{3} (1-\cos\phi)^3 \sin\phi \; d\phi\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{12} (1-\cos\phi)^4 \Big]_{\phi=0}^{\phi=\pi} \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (16-0) \, d\theta = \frac{8\pi}{3}$$

3. *(12 pts)* Find the area of the region in the xy-plane bounded between the curve $x + y = 2\cos(2\pi(x - y))$ and the line segment from the point $(1, 1)$ to the point $(2, 0)$. *(Hint: Use change of variables with $u = x - y$ and $v = x + y$.)*



$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$$

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Big/ \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}$$

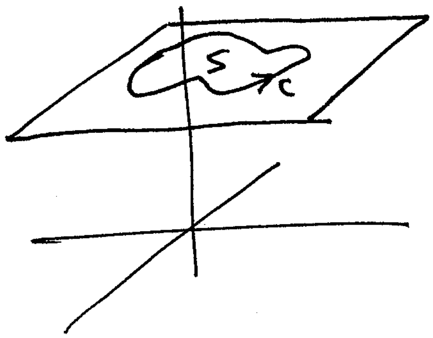
$$A = \iint_D 1 \, dx\,dy = \iint_R 1 \cdot \frac{\partial(x,y)}{\partial(u,v)} \cdot du\,dv$$

$$= \iint_R \frac{1}{2}\, du\,dv = \int_0^2 \int_{2\cos(2\pi u)}^{2} \frac{1}{2}\, dv\,du$$

$$= \int_0^2 1 - \cos(2\pi u)\, du = 2$$

4. (10 pts) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (x^3 - z, y^3, y + z^3)$ and C is parametrized by $\vec{r}(t) = (\cos^4 t, \sin^3 t, 7)$ with $t \in [0, 2\pi]$.

C is closed, and is in plane $z=7$. $\Rightarrow \vec{n} = \begin{pmatrix}0\\0\\1\end{pmatrix}$



$$\nabla\times\vec{F} = (1, -1, 0)$$

By Stokes: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, dS$

$$= \iint_S \begin{pmatrix}1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} dS$$

$$= \iint_S 0\, dS = 0$$

5. (10 pts) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (ze^{xz} + e^x, 2yz, xe^{xz} + y^2)$ and C is parametrized by $\vec{r}(t) = (e^{t^2}, e^{t^3}, t^4)$ with $t \in [0, 1]$.

$$\nabla\times\vec{F} = \left(2y - 2y, (xze^{xz} + e^{xz}) - (xze^{xz} + e^{xz}), 0\right) = \vec{0}$$

$\Rightarrow \vec{F} = \nabla f$ for some f.

$$f = \int ze^{xz} + e^x\, dx = e^{xz} + e^x + k_1(y,z)$$

$$f = \int 2yz\, dy = y^2 z + k_2(x,z)$$

$$f = \int xe^{xz} + y^2\, dz = e^{xz} + y^2 z + k_3(x,y)$$

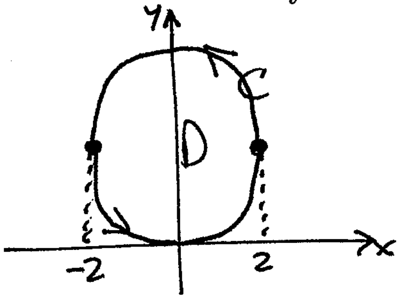
$$\Longrightarrow f = e^{xz} + e^x + y^2 z$$

F.T.L.I.: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$$= f(e, e, 1) - f(1, 1, 0)$$

$$= (e^e + e^e + e^2) - (1 + e + 0) = 2e^e + e^2 - e - 1$$

6. *(10 pts)* Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = (y+\sin x, x^2+e^y)$ and C is the boundary of the region between the curves $y = x^4$ and $32 - x^4 = y$.



By Green's thm:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint_D (2x-1)\,dx\,dy$$

$$= \int_{-2}^{2}\int_{x^4}^{32-x^4} (2x-1)\,dy\,dx = \int_{-2}^{2} (2x-1)(32-2x^4)\,dx$$

$$= \int_{-2}^{2} -4x^5 + 2x^4 + 64x - 32\,dx$$

$$= -\frac{512}{5}$$

7. *(10 pts)* Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is parametrized by $\vec{r}(t) = (t + t(t-1)e^t, (t-1)\sin t)$ with $t \in [0,1]$, and the field $\vec{F} = (P, Q)$ is known only to satisfy the equations

$\vec{r}(0) = \vec{0}$

$\vec{r}(1) = (1,0)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$
$$P(x,0) = 3x^2$$
$$Q(x,0) = x^3$$

$\vec{F}$ is path-independent by , so we can change the path to: $\vec{r}_1(t) = (t, 0)$ $\quad t \in [0,1]$

Then $\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}\cdot\vec{r}_1{}'\,dt = \int_0^1 \begin{pmatrix} 3x^2 \\ x^3 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} dt$

$$= \int_0^1 3t^2\,dt = 1$$

8.[redacted] *(10 pts)* Compute the flux given by $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F}=(y,-x,z)$ and the surface S is defined by $z=\theta$, $0\le\theta\le\pi$ and $1\le x^2+y^2\le 4$.

$$\vec{r}(r,\theta)=(x,y,z)=(r\cos\theta,\ r\sin\theta,\ z)=(r\cos\theta,\ r\sin\theta,\ \theta)$$

$$\vec{r}_r=(\cos\theta,\sin\theta,0)$$

$$\vec{r}_\theta=(-r\sin\theta,\ r\cos\theta,\ 1)$$

$$\vec{N}=(\sin\theta,\ -\cos\theta,\ r)$$

$$\vec{F}=(r\sin\theta,\ -r\cos\theta,\ \theta)$$

$$\vec{F}\cdot\vec{N}=r+r\theta=r(1+\theta)$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS=\iint \vec{F}\cdot\vec{N}\,dr\,d\theta$$

$$=\int_0^\pi\int_1^2 r(1+\theta)\,dr\,d\theta$$

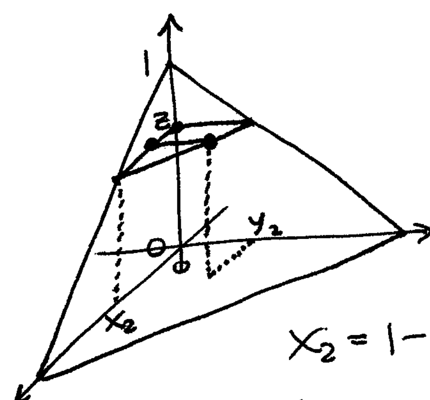
$$=\frac{3}{2}\int_0^\pi(1+\theta)\,d\theta$$

$$=\frac{3}{2}\left(\pi+\frac{1}{2}\pi^2\right)$$

$$=\frac{3}{2}\pi+\frac{3}{4}\pi^2$$

9.[redacted] *(10 pts)* Compute the flux given by $\iint_S \vec{F}\cdot\vec{n}\,dS$, where $\vec{F}=(y^3+z^2, xy-xz^2, xe^y)$ and the surface S is the boundary of the solid defined by $x,y,z\ge 0$ and $x+y+z\le 1$.

$S=\partial R$, so by Gauss,



$$X_2=1-z$$

$$Y_2=1-x-z$$

$$\iint_S \vec{F}\cdot\vec{n}\,dS=\iiint_R \nabla\cdot\vec{F}\,dV$$

$$=\iiint_R(x)\,dV$$

$$=\int_0^1\int_0^{1-z}\int_0^{1-x-z}x\,dy\,dx\,dz$$

$$=\int_0^1\int_0^{1-z}x-x^2-xz\,dx\,dz$$

$$=\int_0^1\left(\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{2}x^2z\right]_{x=0}^{x=1-z}$$

$$=\int_0^1\frac{1}{6}(1-z)^3\,dz$$

$$=\frac{1}{24}$$