

EXAM 2

Math 103, Spring 2007-2008, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number _____

1. _____ "I have adhered to the Duke Community Standard in completing this examination."

2. _____
3. _____ Signature: _____

4. _____

5. _____

6. _____

7. _____

8. _____ Total Score _____ (/100 points)

1. (12 pts) If the following limit exists, compute its value. If it does not exist, show how you know that it does not.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^3y - xy^3}{x^4 + y^4}$$

On $y = mx$:

$$\lim_{x \rightarrow 0} \frac{(x^3)(mx) - (x)(mx)^3}{x^4 + (mx)^4} = \lim_{x \rightarrow 0} \frac{m - m^3}{1 + m^4}$$

$$= \frac{m - m^3}{1 + m^4}$$

This value depends on m , thus giving different limits along different paths. So the original limit does not exist.

2. (12 pts) Use the total derivative to estimate the value of the function

$$f(x, y, z) = x + e^{3y+z} \ln(x-z)$$

at the point $(x, y, z) = (1.01, .03, -.02)$. Choose $\vec{a} = (1, 0, 0)$

$$\frac{\partial f}{\partial x} = 1 + e^{3y+z} \cdot \frac{1}{x-z} \quad \frac{\partial f}{\partial x}(\vec{a}) = 2$$

$$\frac{\partial f}{\partial y} = 3e^{3y+z} \ln(x-z) \quad \frac{\partial f}{\partial y}(\vec{a}) = 0$$

$$\frac{\partial f}{\partial z} = e^{3y+z} \ln(x-z) + e^{3y+z} \frac{-1}{x-z} \quad \frac{\partial f}{\partial z}(\vec{a}) = -1$$

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ &= (2)(.01) + (0)(.03) + (-1)(-.02) = .04 \end{aligned}$$

$$f(\vec{x}) \approx f(\vec{a}) + df = 1 + .04 = \boxed{1.04}$$

assume w has continuous 2nd partials

3. (14 pts) Suppose that $w = w(x, y, z)$. Use the chain rule to compute the partial derivatives $\frac{\partial w}{\partial s}$ and $\frac{\partial^2 w}{\partial s \partial t}$, both when $s = 1$ and $t = 2$, given the relationships $x = 2st - t^2$, $y = t^2 - 3s$, $z = t^2$. Your answer will be written in terms of the partial derivatives of w with respect to x, y , and z . (DO NOT compute the composition explicitly to answer this question.)

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

$$= (w_x)(2t) + (w_y)(-3) + (w_z)(0)$$

$$\Rightarrow \frac{\partial w}{\partial s}(1, 2) = \boxed{4w_x - 3w_y}$$

$$\frac{\partial^2 w}{\partial s \partial t} = \frac{\partial}{\partial t} \left(\frac{\partial w}{\partial s} \right) = \frac{\partial}{\partial t} (2t w_x - 3w_y)$$

$$= 2w_x + 2t \left(\frac{\partial w_x}{\partial t} \right) - 3 \left(\frac{\partial w_y}{\partial t} \right)$$

$$= 2w_x + 2t \left(\frac{\partial w_x}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w_x}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w_x}{\partial z} \frac{\partial z}{\partial t} \right) - 3 \left(\frac{\partial w_y}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w_y}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w_y}{\partial z} \frac{\partial z}{\partial t} \right)$$

$$= 2w_x + 2t (w_{xx}(2s-2t) + w_{xy}(2t) + w_{xz}(2t))$$

$$- 3 (w_{xy}(2s-2t) + w_{yy}(2t) + w_{yz}(2t))$$

$$\frac{\partial^2 w}{\partial s \partial t}(1, 2) = \boxed{2w_x - 8w_{xx} + 22w_{xy} + 16w_{xz} - 12w_{yy} - 12w_{yz}}$$

4. (12 pts) What is the rate of change of the value of the function $f(x, y, z) = e^{(x+2y)} - (yz^2)$ with respect to distance traveled, when at the point $(3, 0, 2)$ and moving in the direction indicated by the vector $(1, 3, 2)$?

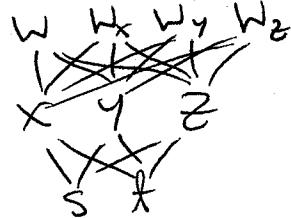
$$\vec{v} = (1, 3, 2) \quad \vec{\mu} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\frac{df}{ds} = D_{\vec{\mu}} f(3, 0, 2)$$

$$= \nabla f(3, 0, 2) \cdot \vec{\mu}$$

$$= \begin{pmatrix} e^3 \\ 2e^3 - 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} / \sqrt{14}$$

$$= \boxed{\frac{7e^3 - 12}{\sqrt{14}}}$$



$$\nabla f = \begin{pmatrix} e^{x+2y} \\ 2e^{x+2y} - z^2 \\ -2yz \end{pmatrix}$$

5. (14 pts) Find the absolute maximum value of the function $f(x, y, z) = x^2 - y^2 + z^2$ on the solid unit ball centered at the origin.

$$\{g(x, y, z) = x^2 + y^2 + z^2 \leq 1\}$$

Interior

$$\nabla f = \begin{pmatrix} 2x \\ -2y \\ 2z \end{pmatrix} = \vec{0} \Rightarrow \vec{x} = \vec{0} \Leftrightarrow \text{critical point.}$$

Boundary

$$\textcircled{1} \quad \nabla g = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = \vec{0} \Rightarrow x = 0, \text{ not on bdry; ignore.}$$

$$\textcircled{2} \quad \nabla f = \lambda \nabla g \Rightarrow \vec{x} \neq \vec{0} \Rightarrow \lambda = 1 \text{ or } -1$$

$$\begin{pmatrix} 2x \\ -2y \\ 2z \end{pmatrix} = \lambda \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$\left. \begin{array}{l} x(1-\lambda) = 0 \\ y(1+\lambda) = 0 \\ z(1-\lambda) = 0 \end{array} \right\}$$

Check values:

$$f(\vec{0}) = 0$$

$$f(0, 1, 0) = -1$$

$$f(0, -1, 0) = -1$$

$$f(x, 0, z) \text{ (with } x^2 + z^2 = 1)$$

$$\boxed{= 15}$$

Abs. max = 1, attained
on entire circle on bdry.

critical
points

6. (12 pts) Consider the function $f(x, y) = x^3y - xy^3$. Confirm that f has a critical point at the origin, and classify that critical point as being a local maximum, local minimum, or saddle point.

$$\nabla f = \begin{pmatrix} 3x^2y - y^3 \\ x^3 - 3xy^2 \end{pmatrix} \Rightarrow \nabla f(\vec{0}) = \vec{0} \Rightarrow \boxed{\vec{0} \text{ is a critical point}}$$

$$H = \begin{pmatrix} 6xy & 3x^2 - 3y^2 \\ 3x^2 - 3y^2 & -6x^2 \end{pmatrix} \Rightarrow \Delta = \det H(\vec{0}) = 0 \Rightarrow \text{2nd der. test fails.}$$

Check behavior along lines $y = mx$:

$$f(x, mx) = x^3(mx) - x(mx)^3 = x^4(m - m^3)$$

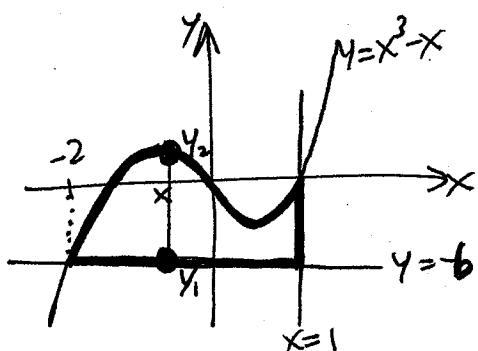
for $m = \frac{1}{2}$, this is curving upward

for $m = 2$, this is curving downward

$\Rightarrow \boxed{\text{saddle point}}$

$$y = -6$$

7. (12 pts) Compute the volume under the surface $z = x + 3$, above the region in the xy -plane bounded by the curves $y = x^3 - x$, ~~$y = x$~~ , and $x = 1$.



$$V = \iint_D x+3 \, dA$$

We slice Γ to x -axis, so
 x ranges between -2 and 1 .

$$V = \int_{-3}^1 \int_{y_1}^{y_2} (x+3) dy dx$$

$$\begin{aligned}
 & \int_{-2}^1 \int_{y=1}^{y=x^3-x} x+3 \, dy \, dx = \int_{-2}^1 \left((x+3)y \Big|_{y=1}^{y=x^3-x} \right) dx \\
 &= \int_{-2}^1 ((x+3)(x^3-x+6)) \, dx = \int_{-2}^1 x^4 + 3x^3 - x^2 + 3x + 18 \, dx \\
 & \boxed{[8x]_{-2}^1 = \left[\frac{33}{5} - \frac{45}{4} - 3 - \frac{9}{2} + 54 \right]}
 \end{aligned}$$

$$= \frac{1}{5}x^5 + \frac{3}{4}x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 + 18x \Big|_{-2}^1 = \boxed{\frac{33}{5} - \frac{45}{4} - 3 - \frac{9}{2} + 54}$$

8. (12 pts) Find the volume that is bounded between the graphs of the functions $f(x, y) = x^2 + e^{xy}$ and $g(x, y) = e^{xy} - y^2 + 1$. (Leave answer as a single variable integral w.r.t. x)

$$f \geq g \iff x^2 + e^{xy} \geq e^{xy} - y^2 + 1 \iff x^2 + y^2 \geq 1$$

$$g \geq f \iff x^2 + y^2 \leq 1$$

So, volume bounded above by $z=g$, below by $z=f$,
over the unit disk.

$$V = \iint_D (g - f) dA = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 - x^2 + y^2 dy dx$$

$$= \int_{-1}^1 \left[y - x^2y - \frac{1}{3}y^3 \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-1}^1 2\sqrt{1-x^2} - 2x^2\sqrt{1-x^2} - \frac{2}{3}(1-x^2)^{3/2} dx$$