

EXAM 3

Math 103, Fall 2006-2007, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number _____

1a. _____ (/10 points)

1b. _____ (/15 points)

1c. _____ (/15 points)

1d. _____ (/15 points)

"I have adhered to the Duke Community
Standard in completing this
examination."

2a. _____ (/15 points)

Signature: _____

2b. _____ (/15 points)

2c. _____ (/15 points)

Total _____ (/100 points)

1. For each of the following fields \vec{F} and paths C , compute the line integral

$$\int_C \vec{F} \cdot \vec{T} ds$$

(a) $\vec{F} = \langle 2xy + y^2, 2xy + x^2 \rangle$, C is parametrized by $\vec{r}(t) = (e^{t^2-t}, t^3 + t^2)$, $t \in [0, 1]$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (2y+2x) - (2x+2y) = 0$$

So $\vec{F} = \nabla f$ for some f

$$\frac{\partial f}{\partial x} = 2xy + y^2 \Rightarrow f = x^2y + xy^2 + c_1(y)$$

$$\frac{\partial f}{\partial y} = 2xy + x^2 \Rightarrow f = x^2y + xy^2 + c_2(x)$$

$$\text{Choose } f = x^2y + xy^2$$

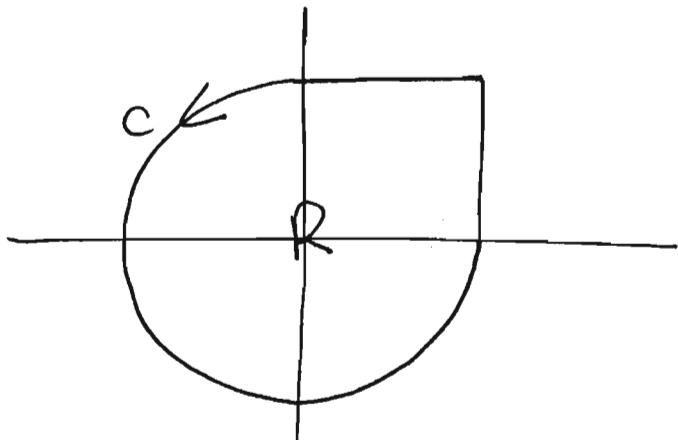
$$\text{Then } \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1, 2) - f(1, 0)$$

$$= 6 - 0$$

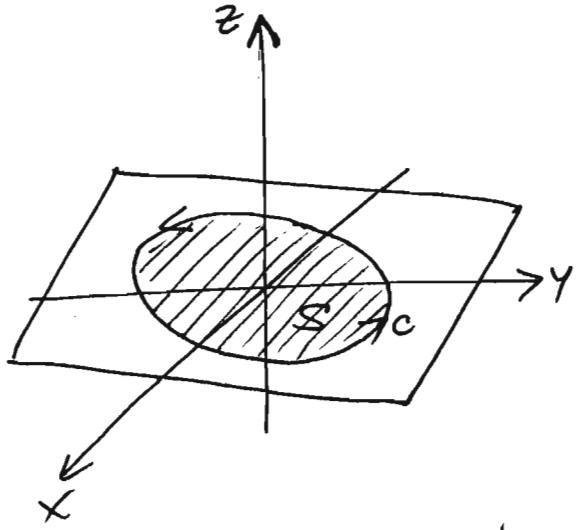
$$= \boxed{6}$$

(b) $\vec{F} = \langle e^x + y, -x + e^{y^2} \rangle$, C is the boundary of the set R , which is the union of the solid unit square $[0, 1] \times [0, 1]$ and the solid unit disk.



$$\begin{aligned}
 \int_C \vec{F} \cdot d\vec{r} &= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\
 &= \iint_R (-1 - 1) dA \\
 &= (-2)(\text{area of } R) \\
 &= (-2)\left(1 + \frac{3\pi}{4}\right) \\
 &= \boxed{-2 - \frac{3\pi}{2}}
 \end{aligned}$$

- (c) $\vec{F} = \langle x^2 e^x, x + y^2, y + z^3 \rangle$, C is parametrized by $\vec{r}(t) = (\cos t, \sin t, \sin t + \cos t)$, $t \in [0, 2\pi]$. (Hint: Notice that the curve C is entirely in the plane $x + y - z = 0$.)



Note that C is a closed curve, and is entirely in the given plane. So it is the boundary of a region S in that plane.

Then, we can write the unit normal vector for S as $\vec{n} = (-1, -1, 1)/\sqrt{3}$

Stokes curl theorem then gives us

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS \\ &= \iint_S (1, 0, 1) \cdot (-1, -1, 1)/\sqrt{3} \, dS \\ &= \iint_S 0 \, dS \\ &= \boxed{0}\end{aligned}$$

- (d) $\vec{F} = \left\langle (x^3y^4z^3 + 3x^2y^3z^3)e^{xy}, (x^4y^3z^3 + 3x^3y^2z^3)e^{xy}, 3x^3y^3z^2e^{xy} + 1 \right\rangle$, C is parametrized by $\vec{r}(t) = (-\sin t, \cos t, t)$, $t \in [0, 2\pi]$. (Hint: You may use the fact that $\nabla \times \vec{F} = \vec{0}$.)

$\nabla \times \vec{F} = \vec{0} \Rightarrow \vec{F}$ is path independent.

So we consider instead the straight line path

from $\vec{r}(0) = (0, 1, 0)$ to $\vec{r}(2\pi) = (0, 1, 2\pi)$,
parametrized by

$$\vec{r}_1(t) = (0, 1, t), t \in [0, 2\pi]$$

Since $x = 0$ for all points on this path, we
can rewrite \vec{F} there as $\vec{F} = (0, 0, 1)$.

Then

$$\begin{aligned} \int \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F} \cdot \vec{r}' dt = \int_0^{2\pi} (0, 0, 1) \cdot (0, 0, 1) dt \\ &= \int_0^{2\pi} 1 dt = \boxed{2\pi} \end{aligned}$$

2. For each of the following fields \vec{F} and surfaces S , compute the surface integral

$$\iint_S \vec{F} \cdot \vec{n} dS$$

(a) $\vec{F} = \langle x, z, y \rangle$, S is parametrized by $\vec{r}(u, v) = (uv, u+v, u-v)$, $u \in [0, 1]$, $v \in [0, 1]$.

The surface is not a boundary, and the field is not a curl (since $\nabla \cdot \vec{F} = 1 \neq 0$), so we compute from the definition.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N} du dv$$

$$\vec{r}_u = (v, 1, 1) \Rightarrow \vec{N} = (-2, u+v, v-u)$$

$$\vec{r}_v = (u, 1, -1)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_0^1 (uv, u-v, u+v) \cdot (-2, u+v, v-u) du dv$$

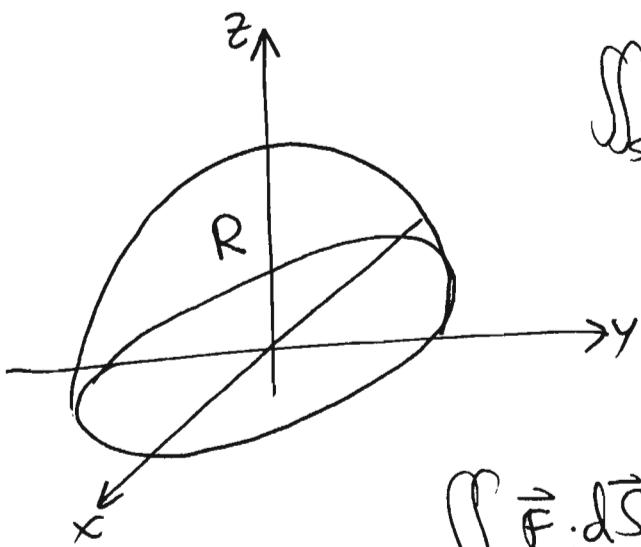
$$= \int_0^1 \int_0^1 -2uv du dv$$

$$= \int_0^1 \left(-u^2 v \Big|_{u=0}^{u=1} \right) dv$$

$$= \int_0^1 -v dv$$

$$= \boxed{-\frac{1}{2}}$$

(b) $\vec{F} = (x^2 + y^2 z, z^2 e^x, x^3)$, S is the boundary of the solid $R = \{x^2 + y^2 + z^2 \leq 1, z \geq 0\}$.



$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \nabla \cdot \vec{F} \, dV$$

$$\begin{aligned}\nabla \cdot \vec{F} &= (2x) + (0) + (0) \\ &= 2x\end{aligned}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_R 2x \, dV$$

The integrand has odd symmetry through the yz -plane, and the domain is symmetric through that same plane, so the integral is zero by symmetry.

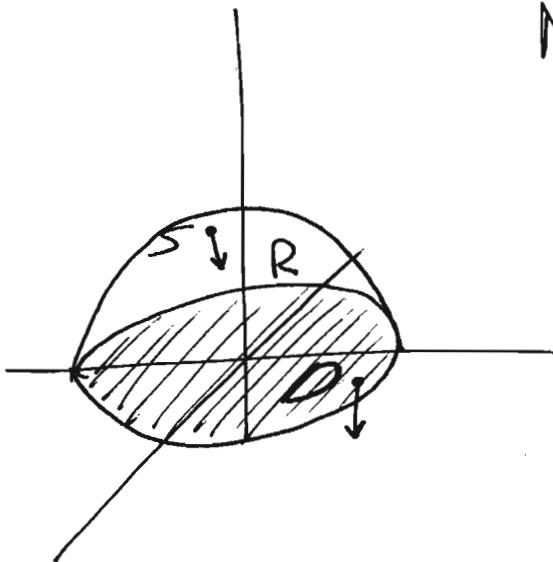
So

$$\iint_S \vec{F} \cdot d\vec{S} = \boxed{0}$$

(c) $\vec{F} = (ye^{z^2}, x^3z^2, 2)$, $S = \{x^2 + y^2 + (z+4)^2 = 25, z \geq 0\}$ oriented downward.

Note $\nabla \cdot \vec{F} = 0$, and with D and R defined as in the picture, we have

$$\partial R = D - S$$



Then

$$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R D \cdot \vec{F} dV$$

$$\iint_D \vec{F} \cdot d\vec{S} - \iint_S \vec{F} \cdot d\vec{S} = 0$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S}$$

For D , the unit normal vector is $\vec{n} = (0, 0, -1)$,

so we have

$$\begin{aligned} \iint_D (\vec{F} \cdot \vec{n}) dS &= \iint_D (-2) dS \\ &= (-2) (\text{area of } D) \end{aligned}$$

D is a disk of radius 3, so its area is 9π .

So

$$\iint_S \vec{F} \cdot d\vec{S} = \boxed{-18\pi}$$