EXAM 1

Math 103, Fall 2006-2007, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

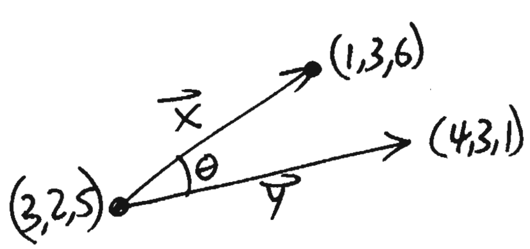
"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total Score: __________ (/100 points)

1. *(20 points)* Points: ____________

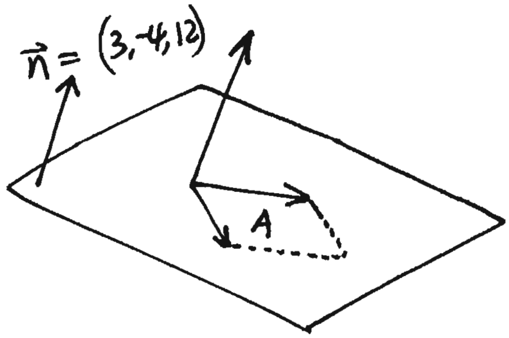
(a) Compute the cosine of the angle between the vectors $\vec{x}$ and $\vec{y}$, which both have tails at the point $(3, 2, 5)$ and have heads at the points $(1, 3, 6)$ and $(4, 3, 1)$, respectively.



$$\vec{x} = (-2, 1, 1)$$
$$\vec{y} = (1, 1, -4)$$

$$\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$$
$$-5 = (\sqrt{6})(\sqrt{18}) \cos\theta$$
$$\boxed{\frac{-5}{6\sqrt{3}} = \cos\theta}$$

(b) The two vectors $\vec{v}$ and $\vec{w}$ are parallel to the plane with equation $3x - 4y + 12z = 7$, they define a parallelogram with area 26, and $\vec{v} \times \vec{w}$ has negative z-coordinate. Compute $\vec{v} \times \vec{w}$.



$\vec{v}, \vec{w} \parallel$ plane $\Rightarrow \vec{v}, \vec{w} \perp \vec{n}$

$\Rightarrow \vec{v} \times \vec{w} \parallel \vec{n}$

$\Rightarrow \vec{v} \times \vec{w} = k\vec{n}$ for some k

$$\|\vec{v} \times \vec{w}\| = A$$
$$\|k\vec{n}\| = A$$
$$|k| \|\vec{n}\| = A$$
$$|k| (13) = 26$$
$$|k| = 2$$

$\Rightarrow \vec{v} \times \vec{w} = 2(3, -4, 12)$ or $(-2)(3, -4, 12)$

Since z-coord. is negative, we have

$$\boxed{\vec{v} \times \vec{w} = (-6, 8, -24)}$$

2. *(20 points)* Points: ____________

(a) The curve C is defined by the equation $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$. Compute the curvature of this curve at the point $(3, 0)$. *(Hint: Parametrize this curve by starting with the parametrization* $(\cos t, \sin t)$ *of the unit circle, and then note that an ellipse is obtained from that by stretching* x *and* y *by factors.)*

C parametrized by $\vec{r}(t) = (x, y) = (3\cos t, 2\sin t)$

$$x' = -3\sin t \Rightarrow x'(0) = 0$$
$$y' = 2\cos t \Rightarrow y'(0) = 2$$
$$x'' = -3\cos t \Rightarrow x''(0) = -3$$
$$y'' = -2\sin t \Rightarrow y''(0) = 0$$

$$\kappa = \frac{|x'y'' - x''y'|}{\left((x')^2 + (y')^2\right)^{3/2}}$$

$$\kappa(0) = \frac{|0 \cdot 0 - (-3)(2)|}{(0^2 + 2^2)^{3/2}} = \boxed{3/4}$$

(b) What kind of curves are the horizontal cross-sections (traces) of the surface described by the equation $x^2(z^2+1)^2 + xz^2 - y^2e^z = e^z$? *(Make sure to explain your reasoning!)*

Horizontal cross-sections have constant $z = c$; so the equations of those cross-sections can be written

$$k_1 x^2 + k_2 x - k_3 y^2 = k_4$$ for some constants $k_1, \ldots, k_4$

Completing the square gives us

$$k_1 (x-a)^2 - k_3 (y)^2 = k_5$$

Since k_1 and k_3 are positive, this represents a $\boxed{\text{hyperbola}}$.

3. *(20 points)* Points: __________

Compute the following limits, or show that they do not exist:

(a)

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + y^2e^x}{x^2+e^y}$$

The argument is a composition of continuous functions, and is defined at $(0,0)$, so the argument is continuous.

So, $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = f(\vec{0}) = \frac{0}{1} = \boxed{0}$

(b)

$$\lim_{(x,y)\to(0,0)} \frac{(x^2-y^2)(x+y)}{x^2+y^2}$$

$$= \lim_{r\to 0} \frac{(r^2\cos^2\theta - r^2\sin^2\theta)(r\cos\theta - r\sin\theta)}{r^2}$$

$$= \lim_{r\to 0} (r)(\cos^2\theta - \sin^2\theta)(\cos\theta - \sin\theta)$$

$= \boxed{0}$ since $r\to 0$, and other factors are bounded.

(c)

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

Along x-axis: $y=0$, so $\rightarrow = \lim_{x\to 0} \frac{(x^3)(0)}{x^6+0} = \lim 0 = 0$

Along $y=x^3$: $\rightarrow = \lim_{x\to 0} \frac{(x^3)(x^3)}{x^6+(x^3)^2} = \lim_{x\to 0} \frac{x^6}{2x^6} = \lim \frac{1}{2} = \frac{1}{2}$

These values are different, so this $\lim_{\vec{x}\to\vec{0}}$ DNE.

4. *(20 points)* Points: ________

(a) Consider the graph of $f(x, y)$, and the tangent plane to this graph above the fixed point (a, b) in the domain. Use the fact that the tangent plane must pass through the point $(a, b, f(a, b))$ to show, as was claimed in class, that the equation of the tangent plane can be put into the form

$$z = f(a, b) + P(x - a) + Q(y - b)$$

for some constants P and Q.

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x_0}$ We let $\vec{n} = (n_1, n_2, n_3)$,

$\vec{x_0} = (a, b, f(a,b))$

$$\Rightarrow (n_1, n_2, n_3) \cdot (x, y, z) = (n_1, n_2, n_3) \cdot (a, b, f(a,b))$$

$$(n_1)(x-a) + (n_2)(y-b) + (n_3)(z - f(a,b)) = 0$$

$$z - f(a,b) = \left(-\frac{n_1}{n_3}\right)(x-a) + \left(-\frac{n_2}{n_3}\right)(y-b)$$

$$\boxed{z = f(a,b) + P(x-a) + Q(y-b) \quad \text{where } P = -\frac{n_1}{n_3},\ Q = -\frac{n_2}{n_3}}$$

(b) Use the geometric properties of the tangent plane to derive the values for these constants P and Q. Make sure to explain your reasoning.

Since the plane is tangent to the graph, their cross-sections must have the same slopes in all directions. So the partial derivatives must be equal. So

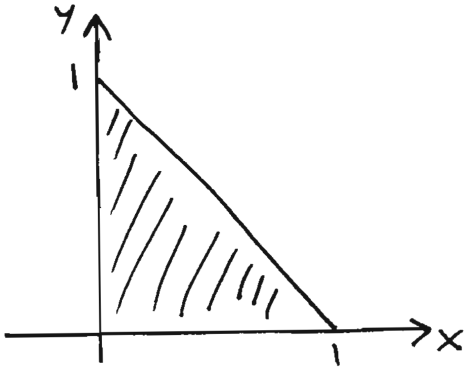
$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$ and thus $\boxed{P = \frac{\partial f}{\partial x}}$

$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$ and thus $\boxed{Q = \frac{\partial f}{\partial y}}$

So we can conclude $z = f(a,b) + \left(\frac{\partial f}{\partial x}\right)(x-a) + \left(\frac{\partial f}{\partial y}\right)(y-b)$ is the eqn. of the tangents plane.

5. *(20 points)* Points: ___________

The domain D is the solid triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$, $(0,1)$. Find the absolute maximum value of the function $f(x,y) = x^2y$ over the domain D.



First, note D is closed and bounded and f is continuous, so the absolute max must be achieved somewhere on D.

Interior: $\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2$

These are defined everywhere, and are never both zero in the interior of D. So, no critical points in the interior.

Boundary: On the horizontal and vertical edges, f is identically zero, while $f > 0$ on the interior; so the absolute max is not on those edges.

All that remains is the angled edge, which we can parametrize by $\vec{r}(t) = (t, 1-t) \quad t \in [0,1]$

Then $f(\vec{r}(t)) = x^2y = (t)^2(1-t) = t^2 - t^3, \ t \in [0,1]$

$f' = 2t - 3t^2 = 0$

$t = 2/3$

$\Rightarrow (x,y) = (2/3, 1/3)$

This is thus the only point on D we cannot rule out as the absolute max., so this must be it.

$$\boxed{f(2/3, 1/3) = \frac{4}{27}}$$