EXAM 1

Math 103, Summer 2006, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

Total __________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

1. Consider the following three points: $\vec{P} = (2,3,4)$, $\vec{Q} = (4,-7,1)$, $\vec{R} = (-1,-1,4)$.

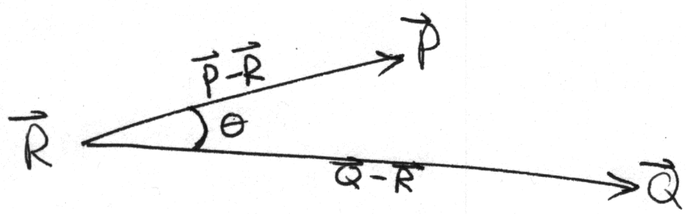
(a) Compute the distance from $\vec{P}$ to $\vec{Q}$.

$$\text{dist.} = \|\vec{PQ}\| = \|\vec{Q} - \vec{P}\| = \left\| \begin{pmatrix} 2 \\ -10 \\ -3 \end{pmatrix} \right\|$$

$$= \sqrt{2^2 + (-10)^2 + (-3)^2}$$

$$= \boxed{\sqrt{113}}$$

(b) Compute the angle between the two line segments attaching the point $\vec{R}$ to the points $\vec{P}$ and $\vec{Q}$.



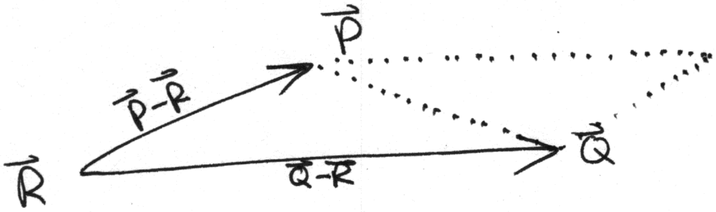
$$(\vec{P}-\vec{R}) \cdot (\vec{Q}-\vec{R}) = \|(\vec{P}-\vec{R})\| \, \|(\vec{Q}-\vec{R})\| \cos\theta$$

$$\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -6 \\ -3 \end{pmatrix} = \left(\sqrt{3^2+4^2+0^2}\right)\left(\sqrt{5^2+(-6)^2+(-3)^2}\right) \cos\theta$$

$$-9 = \left(\sqrt{25}\right)\left(\sqrt{70}\right) \cos\theta$$

$$\theta = \arccos\left(\frac{-9}{5\sqrt{70}}\right)$$

(c) Compute the area of the triangle whose vertices are at the points $\vec{P}$, $\vec{Q}$ and $\vec{R}$.



$$\text{area of } \| \text{gram} = \left\| (\vec{P}-\vec{R}) \times (\vec{Q}-\vec{R}) \right\|$$

$$\text{area of tri.} = \frac{1}{2} \left\| (\vec{P}-\vec{R}) \times (\vec{Q}-\vec{R}) \right\|$$

$$= \frac{1}{2} \left\| \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 4 & 0 \\ 5 & -6 & -3 \end{pmatrix} \right\|$$

$$= \frac{1}{2} \left\| \begin{pmatrix} -12 \\ 9 \\ -38 \end{pmatrix} \right\|$$

$$= \frac{1}{2} \sqrt{12^2 + 9^2 + 38^2}$$

$$= \frac{1}{2} \sqrt{144 + 81 + 1444}$$

$$= \frac{1}{2} \sqrt{1669}$$

2. Find the equation for the plane that is parallel to and equidistant from each of the following two parametric lines:

$$\begin{bmatrix} 4t \\ t \\ -t \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3-t \\ 2+2t \\ 3+3t \end{bmatrix}$$

$$\begin{pmatrix} 4t \\ t \\ -t \end{pmatrix} = \vec{0} + t\begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 3-t \\ 2+2t \\ 3+3t \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Let $\vec{n} = \begin{pmatrix} 4 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

$$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & -1 \\ -1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 9 \end{pmatrix}$$

Let $\vec{x}_0 = \dfrac{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}}{2} = \begin{pmatrix} 3/2 \\ 1 \\ 3/2 \end{pmatrix}$

Eqn of plane is $\quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$

$$5x + (-6)y + 9z = 15$$

$$\boxed{5x - 6y + 9z = 15}$$

3. Suppose that a particle begins at rest at $t = 0$, and that the acceleration as a function of time is given by

$$\vec{a}(t) = \begin{bmatrix} 6t+1 \\ 3t^2 \\ 4t \end{bmatrix}$$

Find the curvature of the path that this particle follows, ~~as a function of time~~ at $t=1$

$$\vec{a}(t) = \begin{pmatrix} 6t+1 \\ 3t^2 \\ 4t \end{pmatrix}$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt$$

$$= \begin{pmatrix} 3t^2+t \\ t^3 \\ 2t^2 \end{pmatrix} + \vec{c}$$

$$@t=0,\ \vec{v}=\vec{0}: \quad \vec{v}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \vec{c} \Rightarrow \vec{c} = \vec{0}$$

$$\vec{v}(t) = \begin{pmatrix} 3t^2+t \\ t^3 \\ 2t^2 \end{pmatrix}$$

$$\text{So } \vec{v}(1) = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{a}(1) = \begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix}$$

$$\kappa(1) = \frac{\|\vec{v}\times\vec{a}\|}{v^3} \qquad \vec{v}\times\vec{a} = \det\begin{pmatrix} \vec{i} & \vec{j} & k \\ 4 & 1 & 2 \\ 7 & 3 & 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix}$$

$$= \frac{\sqrt{(-2)^2+(-2)^2+5^2}}{\left(\sqrt{4^2+1^2+2^2}\right)^3} = \frac{\sqrt{33}}{(\sqrt{21})^3}$$

4. Compute each of the following limits, if it exists.

(a) $\lim_{\vec{x}\to(1,2,3)} \frac{(x+y-z)^3}{(x^2+y^2+z^2)}$

This is a rat'l function, and at $\begin{pmatrix}1\\2\\3\end{pmatrix}$ the fn is defined, with value $= \frac{0^3}{1^2+2^2+3^2} = 0$.
So the fn is continuous, and thus the limit is the value of the fn.

So, $\boxed{\lim = 0}$

(b) $\lim_{\vec{x}\to\vec{0}} \frac{x^3y-y^3x}{x^2+y^2}$ $\qquad X = r\cos\theta \qquad y = r\sin\theta$

$$= \lim_{r\to 0} \frac{(r\cos\theta)^3 (r\sin\theta) - (r\sin\theta)^3 (r\cos\theta)}{r^2}$$

$$= \lim_{r\to 0} (r^2)(\cos^3\theta \sin\theta - \sin^3\theta \cos\theta)$$

For any path approaching $\vec{0}$, $(\cos^3\theta \sin\theta - \sin^3\theta\cos\theta)$ is bounded and $(r^2) \to 0$.

So, $\boxed{\lim = 0}$

(c) $\lim_{\vec{x}\to\vec{0}} \frac{x^2-xy}{x^2+y^2}$

Along $y = mx$:

$$\lim_{x\to 0} \frac{x^2 - x(mx)}{x^2 + (mx)^2}$$

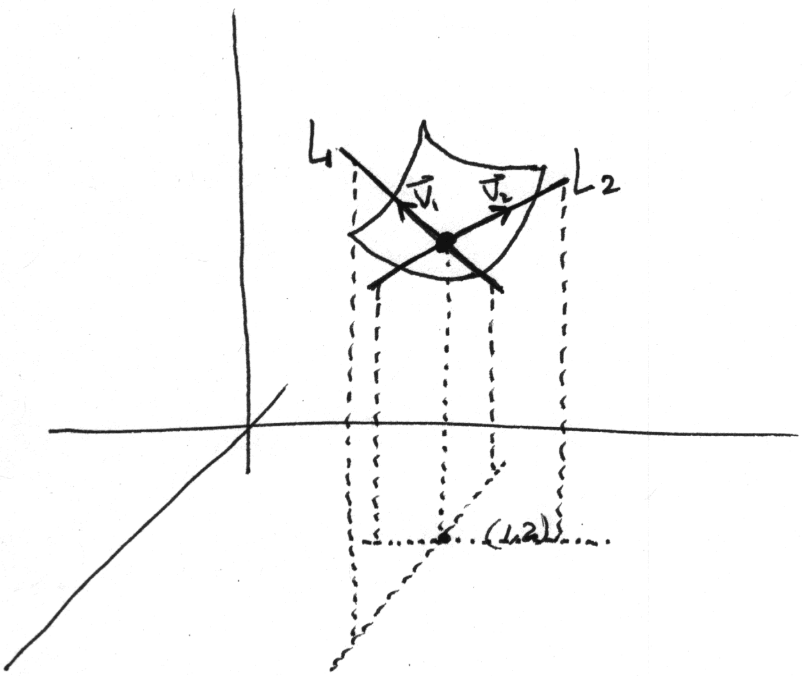
$$= \lim_{x\to 0} \frac{1-m}{1+m^2}$$

So along lines with different slopes, we get different values, so $\boxed{\text{lim DNE.}}$

5. Consider the surface S that is the graph of the function $f(x, y) = x^2 + y^2$. In this problem we will find an alternative derivation for the equation of the tangent plane to the surface, at the specific point $(1, 2, 5)$.

   (a) Let L_1 be the line that is tangent to this surface at the point $(1, 2, 5)$ and whose projection to the xy-plane is parallel to the x-axis. Similarly, let L_2 be the line that is tangent to this surface at the point $(1, 2, 5)$ and whose projection to the xy-plane is parallel to the y-axis.

   Let the direction vectors $\vec{v}_1$ and $\vec{v}_2$ for these lines be given by $(1, 0, a)$ and $(0, 1, b)$, respectively. Use your knowledge of partial derivatives to compute the values for a and b. (Make sure to explain your reasoning clearly.)



$a = \frac{a}{1}$ is the $\frac{\text{rise}}{\text{run}}$ of the line L_1. Since L_1 is tangent to the surface and has projection in the x-direction, this is the partial derivative, $\frac{\partial f}{\partial x}$. So $a = \frac{\partial f}{\partial x}$.

Similarly, $b = \frac{\partial f}{\partial y}$.

$$a = \frac{\partial f}{\partial x}(1,2) = 2x\Big|_{(1,2)} = \boxed{2}$$

$$b = \frac{\partial f}{\partial y}(1,2) = 2y\Big|_{(1,2)} = \boxed{4}$$

(b) Use the fact that $\vec{v}_1$ and $\vec{v}_2$ are tangent to the surface to find a normal vector $\vec{n}$ for the tangent plane, and then write down the equation of the tangent plane. (Make sure to explain your reasoning clearly.)

From part (a), we know that $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

These vectors are tangent to the surface, and thus are parallel to the tangent plane. So their cross product is normal to the tangent plane.

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ -4 \\ 1 \end{pmatrix}$$

The equation of the tangent plane is then $\vec{n} \cdot \vec{x} = \vec{n} \cdot \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$

$$-2x - 4y + z = (-2)(1) + (-4)(2) + (1)(5)$$

$$\boxed{z = (2)(x-1) + (4)(y-2) + 5}$$

$$\left(\text{or } \; -2x - 4y + z = -5\right)$$