

EXAM 3

Math 103, Spring 2006, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number _____

1. _____ (/30 points)

2. _____ (/20 points)

"I have adhered to the Duke Community
Standard in completing this
examination."

3. _____ (/40 points)

Signature: _____

4. _____ (/10 points)

Total _____ (/100 points)

1. For each of the following fields $\vec{F} = (P, Q)$ and paths C , compute the line integral

$$\int_C \vec{F} \cdot \vec{T} ds = \int_C P dx + Q dy$$

(a) $\vec{F} = (y+1, -x)$, $C = \{x^2 + y^2 = 4, y \geq 0\}$ oriented to the left.

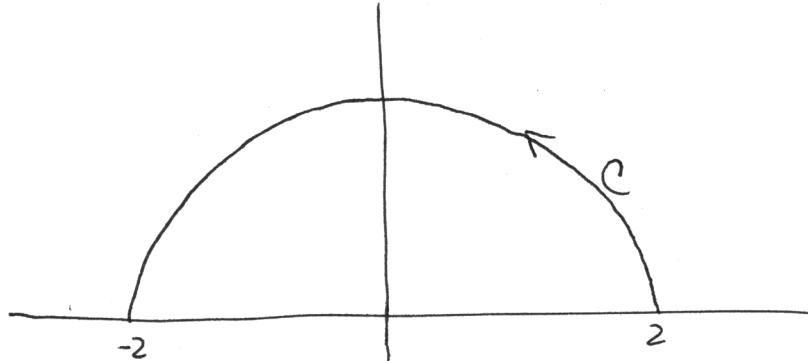
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (-1) - (1) = -2 \neq 0$$

$\Rightarrow \vec{F} \neq \nabla f$ for any f .

So can't use F.T.L.I.

And $C \neq \partial D$ for any D , so can't use Green's Thm.

We parametrize C by $\vec{r}(t) = (2\cos t, 2\sin t)$, $t \in [0, \pi]$



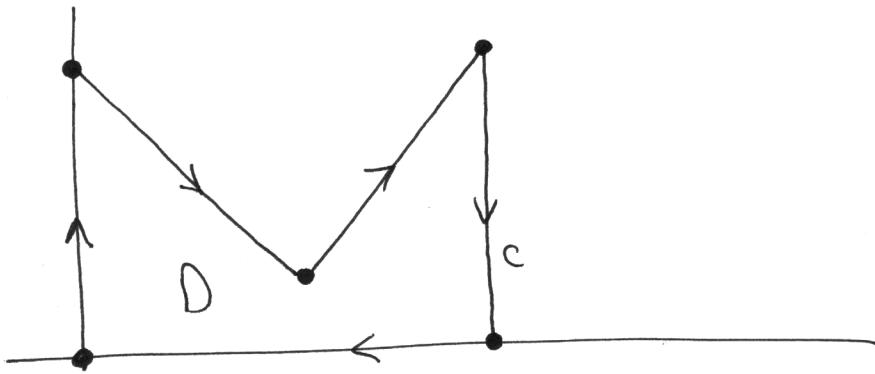
$$\int_C \vec{F} \cdot \vec{T} ds = \int_0^\pi \vec{F} \cdot \vec{r}' dt = \int_0^\pi (y+1, -x) \cdot (-2\sin t, 2\cos t) dt$$

$$= \int_0^\pi (2\sin t + 1, -2\cos t) \cdot (-2\sin t, 2\cos t) dt = \int_0^\pi -4 - 2\sin t dt$$

$$= -4t + 2\cos t \Big|_0^\pi = (-4\pi + 2(-1)) - (0 + 2(1))$$

$$= \boxed{-4\pi - 4}$$

- (b) $\vec{F} = (y+1, -x)$, C is the polygonal path with vertices (in order) $(0, 0), (0, 3), (2, 1), (4, 3), (4, 0), (0, 0)$.



The boundary of D is the path C with the opposite orientation, so by Green's theorem,

$$\begin{aligned}\int_C \vec{F} \cdot \vec{T} ds &= - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_D ((1) - (-1)) dx dy \\ &= 2 \cdot (\text{area of } D)\end{aligned}$$

Simple geometry shows D has area 8, so

$$\int_C \vec{F} \cdot \vec{T} ds = 2 \cdot 8 = \boxed{16}$$

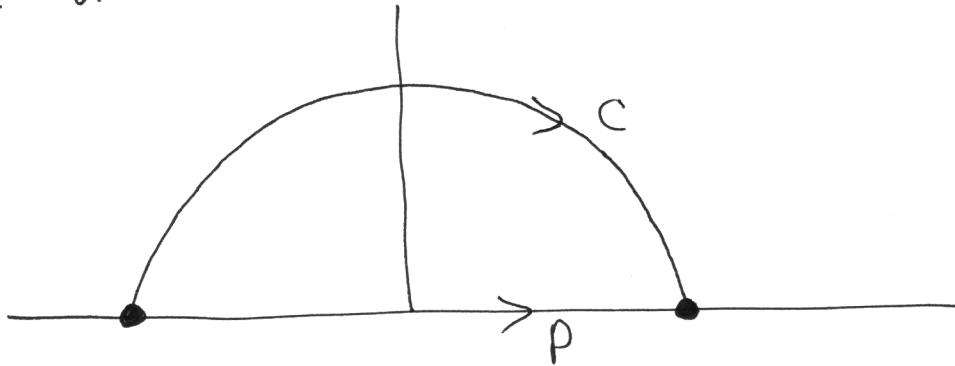
- (c) $\vec{F} = (3 + (2x^2y^3 + y)e^{x^2y^2}, (2x^3y^2 + x)e^{x^2y^2})$, C is the upper half of the unit circle, oriented to the right. (Hint: Is this field path-independent? How might that help?)

$$\frac{\partial Q}{\partial x} = (6x^2y^2 + 1)(e^{x^2y^2}) + (2x^3y^2 + x)(2xy^2 e^{x^2y^2})$$

$$\frac{\partial P}{\partial y} = (6x^2y^2 + 1)(e^{x^2y^2}) + (2x^2y^3 + y)(2x^3y e^{x^2y^2})$$

These are equal, so $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$, so \vec{F} is conservative and thus path independent.

So, consider instead the straight line path P with $\partial P = \partial C$:



$$\int_C \vec{F} \cdot \vec{T} ds = \int_P \vec{F} \cdot \vec{T} ds = \int_P \vec{F} \cdot \vec{r}' dt$$

Parametrize P by $\vec{r}(t) = (t, 0)$, $t \in [-1, 1]$
 $\vec{r}' = (1, 0)$

$$\text{Then } \int_P \vec{F} \cdot \vec{r}' dt = \int_{-1}^1 \left(\begin{pmatrix} 3 \\ t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) dt = \int_{-1}^1 3 dt$$

$$= \boxed{6}$$

2. For each of the following fields $\vec{F} = (P, Q, R)$ and paths C , compute the line integral

$$\int_C \vec{F} \cdot \vec{T} ds = \int_C P dx + Q dy + R dz$$

(a) $\vec{F} = (yz, xz, xy)$, C is parametrized by $\vec{r}(t) = (t(t+2), t^2, e^{t^3})$, $t \in [0, 1]$.

$$\nabla \times \vec{F} = \begin{pmatrix} x-x \\ y-y \\ z-z \end{pmatrix} = \vec{0}, \text{ so } \vec{F} \text{ is a gradient.}$$

By inspection, $f = xyz$ satisfies $\nabla f = \vec{F}$.

The F.T.L.I. then tells us

$$\begin{aligned} \int_C \vec{F} \cdot \vec{T} ds &= f(\vec{r}(1)) - f(\vec{r}(0)) \\ &= f\left(\begin{pmatrix} 3 \\ 1 \\ e \end{pmatrix}\right) - f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \end{aligned}$$

$$= 3e - 0$$

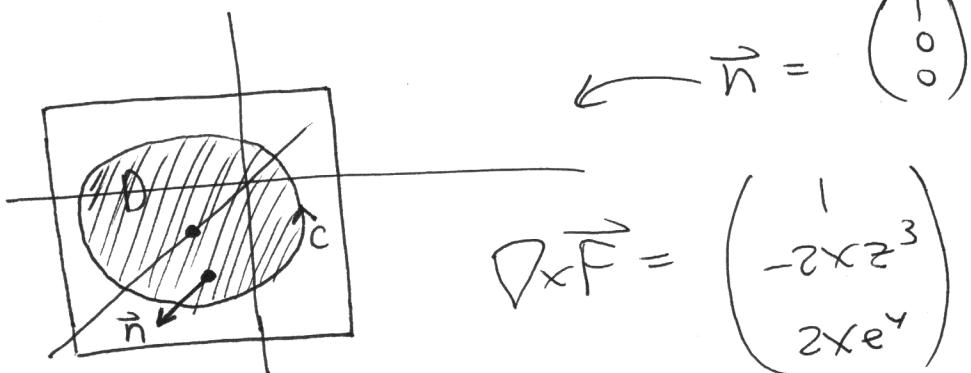
$$= \boxed{3e}$$

(b) $\vec{F} = (e^x, x^2 e^y, y + x^2 z^3)$, C is parametrized by $\vec{r}(t) = (2, \cos(t), \sin(t))$, $t \in [0, 2\pi]$.

We note C is a closed loop, namely the unit circle in the plane $x=2$, ~~which we will call~~. Let D be the disk in that same plane with $\partial D = C$.

Then by Stokes theorem,

$$\int_C \vec{F} \cdot \vec{T} ds = \iint_D (\nabla \times \vec{F}) \cdot \vec{n} dS$$



$$\iint_D (\nabla \times \vec{F}) \cdot \vec{n} dS = \iint_D (1) dS = \text{area of } D$$

$$= \boxed{\pi}$$

3. For each of the following fields \vec{F} and surfaces S , compute the surface integral

$$\iint_S \vec{F} \cdot \vec{n} dS$$

(a) $\vec{F} = (x, y, z)$, S is parametrized by $\vec{r}(u, v) = (u^2, v^2, u + v)$, $u \in [0, 1]$, $v \in [0, 1]$.

$$D.F = 1+1+1 = 3 \neq 0, \text{ so } \vec{F} \neq D \times \vec{G}.$$

And S is not a boundary.

We compute directly then :

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint \vec{F} \cdot \vec{N} du dv$$

$$\vec{r}_u = (2u, 0, 1)$$

$$\vec{r}_v = (0, 2v, 1)$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = (-2v, -2u, 4uv)$$

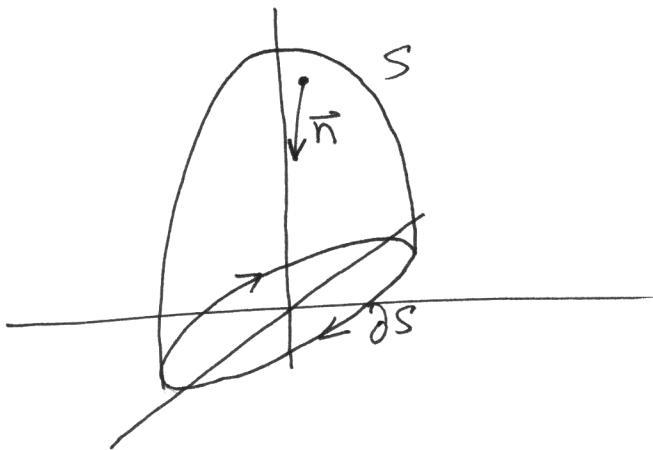
$$\iint \vec{F} \cdot \vec{N} du dv = \iint \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2v \\ -2u \\ 4uv \end{pmatrix} du dv$$

$$= \iint_0^1 \iint_0^1 \begin{pmatrix} u^2 \\ v^2 \\ uv \end{pmatrix} \cdot \begin{pmatrix} -2v \\ -2u \\ 4uv \end{pmatrix} du dv$$

$$= \iint_0^1 \iint_0^1 2u^2v + 2uv^2 du dv = \int_0^1 \left(\frac{2}{3}u^3v + u^2v^2 \right)_0^1 dv$$

$$= \int_0^1 \left(\frac{2}{3}v + v^2 \right) dv = \left(\frac{1}{3}v^2 + \frac{1}{3}v^3 \right)_0^1 = \boxed{\frac{2}{3}}$$

(b) $\vec{F} = \nabla \times (0, 0, e^{\sin(x^3y)})$, $S = \{z = 36 - x^2 - 9y^2, z \geq 0\}$ oriented downward.



By Stokes theorem,

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \oint_S \left[\nabla \times \begin{pmatrix} 0 \\ 0 \\ e^{\sin(x^3y)} \end{pmatrix} \right] \cdot \vec{n} \, dS$$

$$= \oint_{\partial S} \begin{pmatrix} 0 \\ 0 \\ e^{\sin(x^3y)} \end{pmatrix} \cdot \vec{T} \, ds$$

Since ∂S is entirely in the plane $z=0$, \vec{T} must have 0 as its third component. So the above integral is zero, and thus

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \boxed{0}$$

- (c) $\vec{F} = (y^2z, y + xe^z, x^2y^3)$, S is the unit cube, oriented inward.

S is clearly a boundary (though with negative orientation), so we can use Gauss' theorem. Let R be the solid unit cube. Then

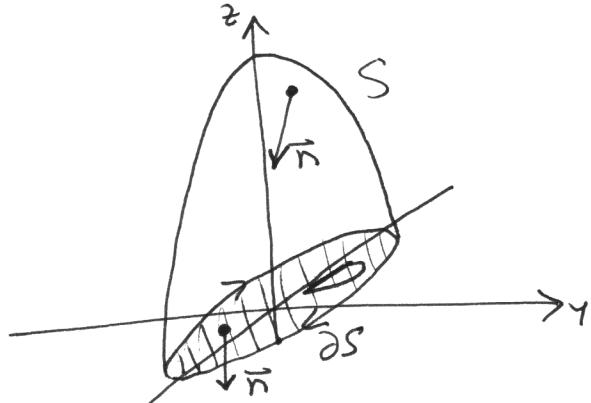
$$\oint_{S=\partial R} \vec{F} \cdot \vec{n} dS = - \iiint_R (\nabla \cdot \vec{F}) dV$$

$$= \iiint_R - (0+1+0) dV = \iiint_R -1 dV$$

$$= -(\text{volume of } R)$$

$$= \boxed{-1}$$

(d) $\vec{F} = (y^2 z, x e^z, x^2 y^3)$, $S = \{z = 36 - x^2 - 9y^2, z \geq 0\}$ oriented downward.



Note $\nabla \cdot \vec{F} = 0$, so $\vec{F} = \nabla \times \vec{G}$ for some \vec{G} . Then let D be the region in the xy -plane with $\partial D = \partial S$. We have

$$\iint_S \vec{F} \cdot \vec{n} dS = \iint_S (\nabla \times \vec{G}) \cdot \vec{n} dS = \iint_{\partial S} \vec{G} \cdot \vec{T} ds$$

$$\iint_D \vec{F} \cdot \vec{n} dS = \iint_D (\nabla \times \vec{G}) \cdot \vec{n} dS = \iint_D \vec{G} \cdot \vec{T} ds$$

On D we note $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, so we have

$$= \iint_D \begin{pmatrix} y^2 z \\ x e^z \\ x^2 y^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} dS = \iint_D (-x^2 y^3) dS$$

D is symmetric over the x -axis, and $(-x^2 y^3)$ has odd symmetry over the x -axis, so this is O.

4. Show that the volume of a solid region R in xyz -space, whose boundary is the surface ∂R , can be computed with the formula

$$\text{volume}(R) = \iint_{\partial R} x \, dy \, dz$$

$$\iint_{\partial R} x \, dy \, dz = \iint_{\partial R} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} dy \, dz \\ \frac{dz}{dx} \, dx \\ \frac{dy}{dx} \, dy \end{pmatrix}$$

$$= \iint_{\partial R} \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \cdot \vec{n} \, dS$$

Applying Gauss' theorem, we get

$$= \iiint_R (1 + 0 + 0) \, dv$$

$$= \iiint_R 1 \, dv$$

$$= (\text{volume of } R)$$