EXAM 1

Math 103, Spring 2006, Clark Bray.

You have 50 minutes.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total __________ (/100 points)

1. Consider the three vectors $\vec{p} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \vec{q} = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}, \vec{r} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$ in $\mathbb{R}^3$.

(a) Compute the lengths of each of these vectors.

$$\|\vec{p}\| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$$

$$\|\vec{q}\| = \sqrt{7^2+4^2+4^2} = \sqrt{81} = 9$$

$$\|\vec{r}\| = \sqrt{8^2+4^2+1^2} = \sqrt{81} = 9$$

(b) Compute the cosines of the three angles between the three pairs of these three vectors.

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$$

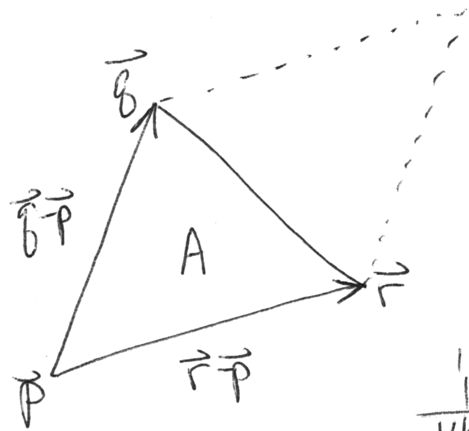
$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$$\cos(\theta_{pq}) = \frac{\vec{p} \cdot \vec{q}}{\|\vec{p}\| \|\vec{q}\|} = \frac{14+8+4}{3 \cdot 9} = \frac{26}{27}$$

$$\cos(\theta_{pr}) = \frac{\vec{p} \cdot \vec{r}}{\|\vec{p}\| \|\vec{r}\|} = \frac{16+8+1}{3 \cdot 9} = \frac{25}{27}$$

$$\cos(\theta_{qr}) = \frac{\vec{q} \cdot \vec{r}}{\|\vec{q}\| \|\vec{r}\|} = \frac{56+16+4}{9 \cdot 9} = \frac{76}{81}$$

(c) Compute the area of the triangle with vertices $\vec{p}, \vec{q}, \vec{r}$.



$$2A = \left\| (\vec{q}-\vec{p}) \times (\vec{r}-\vec{p}) \right\|$$

$$A = \frac{1}{2} \left\| (5,2,3) \times (6,2,0) \right\|$$

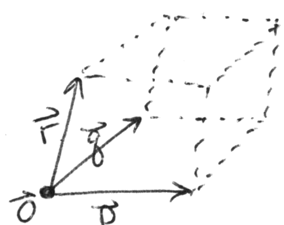
$$= \frac{1}{2} \left\| (-6, 18, -2) \right\|$$

$$\begin{array}{r} 18 \\ 18 \\ \hline 144 \\ 18 \\ \hline 324 \end{array}$$

$$= \frac{1}{2} \sqrt{36 + 324 + 4}$$

$$= \frac{1}{2} \sqrt{364} = \boxed{\sqrt{91}}$$

(d) Compute the volume of the parallelopiped with one vertex at the origin and three edges defined by the vectors $\vec{p}, \vec{q}, \vec{r}$.



$$V = \left| \det \begin{pmatrix} \vec{p} \\ \vec{q} \\ \vec{r} \end{pmatrix} \right|$$

$$= \left| \det \begin{pmatrix} 2 & 2 & 1 \\ 7 & 4 & 4 \\ 8 & 4 & 1 \end{pmatrix} \right|$$

$$= \left| (2)(-12) - (2)(-25) + (1)(-4) \right|$$

$$= \left| -24 + 50 - 4 \right|$$

$$= 22$$

2. Let H be the hyperbola in the yz-plane defined by $z^2 - y^2 = 16$, and let S be the surface obtained by rotating H around the z-axis.

(a) Find an equation whose solution set is the surface S.

$g(y,z) = z^2 - y^2 = 16$ in yz-plane

equation of rotation around z-axis is

$g\left(\sqrt{x^2+y^2}, z\right) = 16$

$\Rightarrow \boxed{z^2 - (x^2+y^2) = 16}$

(b) Find a function $f : \mathbb{R}^2 \to \mathbb{R}^1$ whose graph is the part of S above the xy-plane. Identify the variables x, y, z individually as being independent or dependent variables for that graph.

$z^2 - (x^2+y^2) = 16$

$z^2 = 16 + (x^2+y^2)$

$z = \pm\sqrt{16 + (x^2+y^2)}$

To get points above xy-plane, choose "+".

$z = \sqrt{16 - (x^2+y^2)}$

This is the graph of the function

$f(x,y) = \sqrt{16 - (x^2+y^2)}$

where x, y are independent vars and $z = f(x,y)$ is the dependent variable.

(c) At what point does the parametric curve defined by $\vec{r}(t) = (t, t^2, 3t)$ intersect the part of S above the xy-plane?

Need $g(\vec{r}(t)) = 16$, so

$$(3t)^2 - \left((t)^2 + (t^2)^2\right) = 16$$

$$0 = t^4 - 8t^2 + 16$$

$$0 = (t^2 - 4)^2$$

above xy-plane

$$\Rightarrow t^2 = 4 \Rightarrow t = \pm 2 \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boxed{\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}} \text{ or } \begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}$$

(d) Compute the curvature and the principal unit normal vector for $\vec{r}(t)$ at the point computed in the previous part.

$$\vec{r} = (t, t^2, 3t) \qquad \vec{v} = (1, 2t, 3) \qquad \vec{a} = (0, 2, 0)$$

$$\vec{r}(2) = (2, 4, 6) \qquad \vec{v}(2) = (1, 4, 3) \qquad \vec{a}(2) = (0, 2, 0)$$

$$\kappa = \frac{\|\vec{v} \times \vec{a}\|}{v^3} = \frac{\|(1,4,3) \times (0,2,0)\|}{\left(\sqrt{1^2 + 4^2 + 3^2}\right)^3} = \frac{\|(-6, 0, 2)\|}{(\sqrt{26})^3} = \boxed{\frac{\sqrt{40}}{(\sqrt{26})^3}}$$

$$a_T = \frac{\vec{v} \cdot \vec{a}}{v} = \frac{8}{\sqrt{26}} \qquad a_N = \frac{\|\vec{v} \times \vec{a}\|}{v} = \frac{\sqrt{40}}{\sqrt{26}}$$

$$\vec{T} = \frac{\vec{v}}{v} = \frac{(1, 4, 3)}{\sqrt{26}}$$

$$N = \frac{\vec{a} - a_T \vec{T}}{a_N} = \frac{(0, 2, 0) - \frac{(8, 32, 24)}{26}}{\sqrt{40}/\sqrt{26}} = \boxed{\left(-\frac{8}{\sqrt{26}\sqrt{40}}, \frac{20}{\sqrt{26}\sqrt{40}}, \frac{-24}{\sqrt{26}\sqrt{40}}\right)}$$

3. Compute the following, or show that they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}$

Let $y = mx$

$$\lim_{x\to 0} \frac{m^2x^4}{x^4+m^4x^4} = \lim_{x\to 0} \frac{m^2}{1+m^4} = \frac{m^2}{1+m^4}$$

This is <u>not</u> independent of m, so the limit cannot exist.

(b) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

Let $x = r\cos\theta$ $y = r\sin\theta$

$$\lim_{r\to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2} = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta)$$

$= 0$ since $r\to 0$, and $|\cos^3\theta - \sin^3\theta| \le 1$

So, $\boxed{\lim = 0}$

(c) $\lim_{(x,y)\to(1,3)} \frac{(x^2-y)(y^2-x)}{(x-2y)(y-2x)}$

This rational function is continuous where it is defined, which it is at $(1,3)$. So

$$\lim = \frac{(1^2-3)(3^2-1)}{(1-2\cdot3)(3-2\cdot1)} = \frac{(-2)(8)}{(-5)(1)} = \boxed{\frac{16}{5}}$$

(d) Let $f(x,y) = \frac{x^2y^2}{x^4+y^4}$ for $(x,y) \neq (0,0)$, and $f(0,0) = 0$. Compute $\frac{\partial}{\partial x}\Big|_{(0,0)} f(x,y)$.

Cannot use the quotient rule on $\frac{x^2y^2}{x^4+y^4}$, since that is not the def'n of f at $(0,0)$.

By def'n,

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

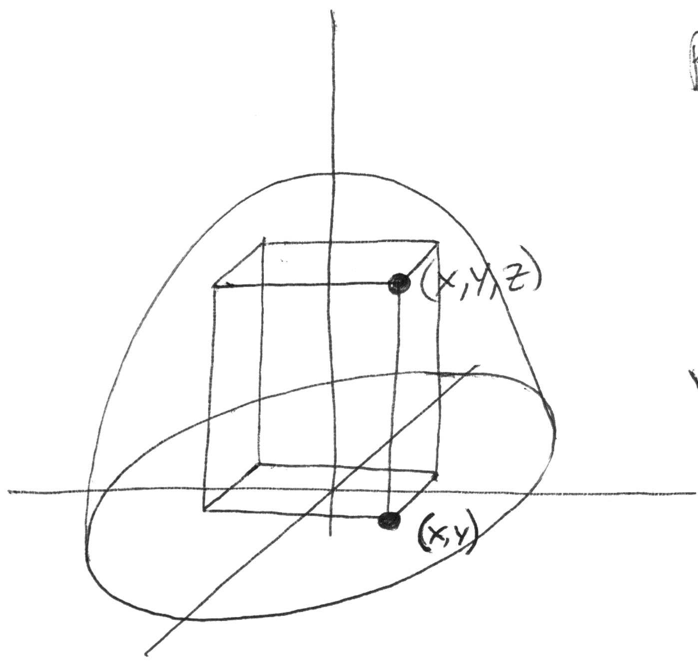
$$= \lim_{h\to 0} \frac{0 - 0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= \boxed{0}$$

4. The house that Bob lives in has an unusual shape – the floor is a large disk of radius 50 feet in the xy-plane centered at the origin, and the ceiling and walls consist of an upside-down paraboloid with a radius of 50 feet at the bottom and a height of 50 feet (given by the equation $x^2 + y^2 + 50z = 2500$).

   Having grown tired of having a curved ceiling and walls, he decides to build a rectangular box inside of the paraboloid to live in instead (and to use the space outside the box for storage). Assuming that the floor of the box has center at the origin and that the box will be built as tall as possible so that the upper corners touch the paraboloid, what are the dimensions of such a box that will give him the maximum possible volume to live in?



Box defined by the point (x,y) defining the floor corner in the first quadrant.

$$V = 4xyz$$
$$= 4xy\left(\frac{2500 - x^2 - y^2}{50}\right)$$
$$= 200xy - \frac{4}{50}x^3y - \frac{4}{50}xy^3$$

$$\frac{\partial V}{\partial x} = 200y - \frac{12}{50}x^2y - \frac{4}{50}y^3 = 0$$

$$\frac{\partial V}{\partial y} = 200x - \frac{4}{50}x^3 - \frac{12}{50}xy^2 = 0$$

$$\left.\begin{array}{l} 200 \cdot 50 = 12x^2 + 4y^2 \\ 200 \cdot 50 = 4x^2 + 12y^2 \end{array}\right\} \text{difference} \Rightarrow x = y$$

$$20{,}000 = 16(x^2 + y^2)$$

$$\sqrt{\frac{20{,}000}{16 \cdot 2}} = x = y = \boxed{25}$$

So, $\boxed{\text{box has } 50 \times 50 \text{ base, height} = 25}$

5. Let $w = xy + yz + xz$, $x = r^2 - s^2$, $y = r^2 + s^2$, $z = 2rs$, $r = \cos(t)$, $s = \sin(t)$.

(a) Compute $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ completely in terms of r and s.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (y+z)(2r) + (x+z)(2r) + (x+y)(2s)$$

$$= (r^2+s^2+2rs)(2r) + (r^2-s^2+2rs)(2r) + (2r^2)(2s)$$

$$= 2r^3 + \cancel{2rs^2} + 4r^2s + 2r^3 - \cancel{2rs^2} + 4r^2s + 4r^2s$$

$$= \boxed{4r^3 + 12r^2s}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (y+z)(-2s) + (x+z)(2s) + (x+y)(2r)$$

$$= (r^2+s^2+2rs)(-2s) + (r^2-s^2+2rs)(2s) + (2r^2)(2r)$$

$$= -\cancel{2r^2s} - 2s^3 - \cancel{4rs^2} + \cancel{2r^2s} - 2s^3 + \cancel{4rs^2} + 4r^3$$

$$= \boxed{4r^3 - 4s^3}$$

(b) Compute $\frac{dw}{dt}$ completely in terms of t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial r}\frac{dr}{dt} + \frac{\partial w}{\partial s}\frac{ds}{dt}$$

$$= \left(4r^3 + 12r^2 s\right)\left(-\sin t\right) + \left(4r^3 - 4s^3\right)\left(\cos t\right)$$

$$= \left(4\cos^3 t + 12\cos^2 t\sin t\right)\left(-\sin t\right) + \left(4\cos^3 t - 4\sin^3 t\right)\left(\cos t\right)$$

$$= 4\cos t\left(\cos^3 t - \cos^2 t\sin t - 3\cos t\sin^2 t - \sin^3 t\right)$$