EXAM 3

Math 103, Fall 2005, Clark Bray.

You have 50 minutes.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/30 points)

4. __________ (/15 points)

5. __________ (/15 points)

Total __________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. (a) Let S_1 and S_2 be the spheres of radius a with centers at $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix}$, respectively.

Derive the spherical equations of S_1 and S_2.

$\underline{S_1}$: $(x-a)^2 + y^2 + z^2 = a^2$

$x^2 - 2ax + a^2 + y^2 + z^2 = a^2$

$(x^2+y^2+z^2) - 2ax = 0$

$\rho^2 - 2a\rho\cos\theta\sin\phi = 0$

$\boxed{\rho(\rho - 2a\cos\theta\sin\phi) = 0}$

$\underline{S_2}$: $x^2 + (y-a)^2 + z^2 = a^2$

$x^2 + y^2 - 2ay + a^2 + z^2 = a^2$

$(x^2+y^2+z^2) - 2ay = 0$

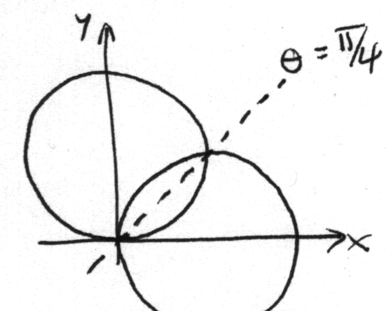
$\rho^2 - 2a\rho\sin\theta\sin\phi = 0$

$\boxed{\rho(\rho - 2a\cos\theta\sin\phi) = 0}$

(b) Let D be the set of points interior to both S_1 and S_2. Suppose that the density inside D is given by the function $\delta(\vec{x})$. Write down, but do not evaluate, an iterated integral in spherical coordinates representing the mass contained in D.

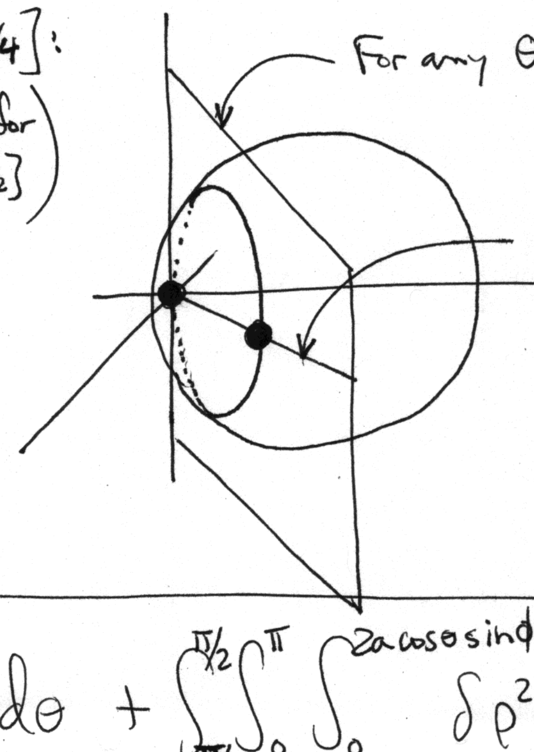
From above:

Need separate integrals for $\theta \in [0, \pi/4]$, $\theta \in [\pi/4, \pi/2]$



$\theta \in [0, \pi/4]$:

(Similarly for $\theta \in [\pi/4, \pi/2]$)



For any θ, we have $\phi \in [0, \pi]$

For a given θ, ϕ, ρ goes between the two solutions in above equation for S_2

$M = \iiint dm = \iiint \delta\, dV$

$= \iiint \delta \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$\boxed{= \int_0^{\pi/4}\int_0^{\pi}\int_0^{2a\sin\theta\sin\phi} \delta \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\pi}\int_0^{2a\cos\theta\sin\phi} \delta \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$

2. (a) Show by direct computation that if C is the line beginning at $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and ending at $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, then

$$\int_C x\, dy = \bar{x} \Delta y$$

$$\vec{r}(t) = \left(x_1 + t(x_2 - x_1),\ y_1 + t(y_2 - y_1)\right), \quad t \in [0,1]$$

$$\int_C x\,dy = \int_0^1 \left(x_1 + t(x_2 - x_1)\right)(y_2 - y_1)\, dt$$

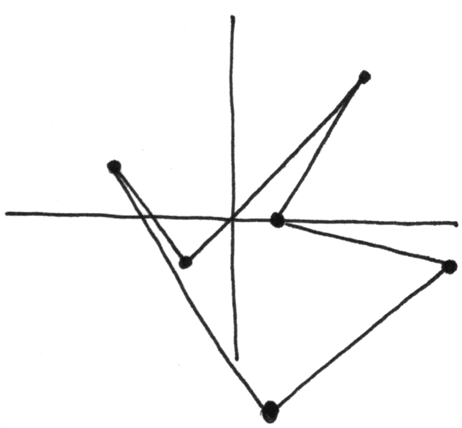
$$= x_1(y_2 - y_1) + (x_2 - x_1)(y_2 - y_1)\int_0^1 t\, dt$$

$$= x_1(y_2 - y_1) + \frac{1}{2}(x_2 - x_1)(y_2 - y_1)$$

$$= \left(\frac{x_1 + x_2}{2}\right)(y_2 - y_1)$$

$$= \bar{x}\, \Delta y$$

(b) Let P be the hexagon with vertices (moving counterclockwise around P) at the points $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Use the result above to compute the area of P, and make sure to explain how Green's Theorem relates that line integral to area.



$$\int_{\partial P} x\,dy = \int_{\partial P} 0\,dx + x\,dy = \int_P \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0)\, dx\,dy$$

Green's Theorem

$$= \int_P 1\, dA = \underline{\text{area}}$$

segment	start	end	$\bar{x}$	Δy	$\int x\,dy$
1	(1,0)	(3,4)	2	4	8
2	(3,4)	(-1,-1)	1	-5	-5
3	(-1,-1)	(-3,1)	-2	2	-4
4	(-3,1)	(1,-4)	-1	-5	5
5	(1,-4)	(5,-1)	3	3	9
6	(5,-1)	(1,0)	3	1	3

$$\int_{\partial P} x\,dy = \boxed{16} = \text{area}$$

3. (a) Let $\vec{F}$ be the field $\begin{bmatrix} e^{yz} \\ xze^{yz} \\ xye^{yz}+1 \end{bmatrix}$, and C the oriented curve defined parametrically by $\vec{r} = \begin{bmatrix} t^3 \\ t^2 \\ t \end{bmatrix}$, $t \in [0,1]$. Compute

$$\int_C \vec{F} \cdot d\vec{r}$$

Can check directly that $\vec{F} = \nabla\big(\underbrace{xe^{yz}+z}_{\text{call this } f}\big)$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot dr = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f\begin{pmatrix}1\\1\\1\end{pmatrix} - f\begin{pmatrix}0\\0\\0\end{pmatrix} = (e+1) - (0) = \boxed{e+1}$$

(b) Let S be the closed surface defined by $\rho = 2 + \sin^3(\theta)\sin(6\phi)$, and let $\vec{F}$ be the field $\begin{bmatrix} y^2z^3 \\ 2yz \\ xy - z^2 \end{bmatrix}$. Compute

$$\iint_S \vec{F} \cdot d\vec{S}$$

Since S is a closed surface, it is the boundary of a region R. Then the divergence thm. gives us

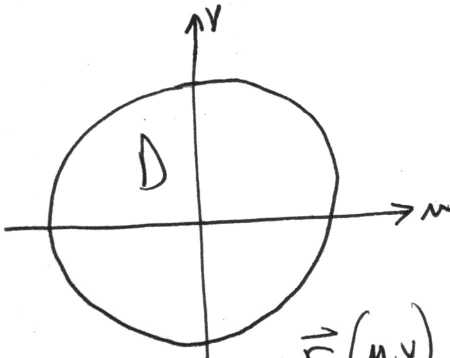
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_R (\nabla \cdot \vec{F})\, dV$$

But $\nabla \cdot \vec{F} = 0$, so this becomes

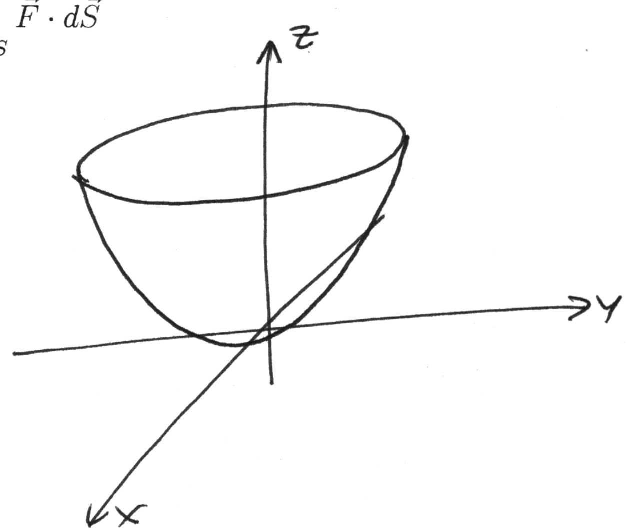
$$= \iiint_R 0 \, dV = \boxed{0}$$

4. Let S be the part of the graph $z = x^2 + y^2$ directly above the unit circle in the xy-plane, oriented upwards, and let $\vec{F}$ be the field $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Parametrize S, and then with that parametrization directly compute (without using any boundary theorems)

$$\iint_S \vec{F} \cdot d\vec{S}$$



$\vec{r}$



$$\vec{r}(u,v) = (u, v, u^2+v^2)$$

$$\vec{r}_u = (1, 0, 2u)$$

$$\vec{r}_v = (0, 1, 2v)$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = (-2u, -2v, 1)$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{N} \, du\, dv$$

$$\iint_D \vec{F} \cdot \vec{N} \, du\, dv = \iint_D \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix} du\, dv = \iint_D \begin{pmatrix} u \\ v \\ u^2+v^2 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix} du\, dv$$

$$= \iint_D -(u^2+v^2)\, du\, dv$$

Switch to polar coordinates in uv-plane:

$$= \int_0^{2\pi} \int_0^1 (-r^2)\, r\, dr\, d\theta = -\int_0^{2\pi} \left(\frac{1}{4} r^4 \Big]_{r=0}^{r=1} \right) d\theta$$

$$= \left(-\frac{1}{4}\right)(2\pi) = \boxed{\frac{-\pi}{2}}$$

5. Let $\vec{F}$ be the field $\begin{bmatrix} x^2z^3 \\ e^y(z^2+1) \\ xyz^2 \end{bmatrix}$, and C the oriented curve defined parametrically by $\vec{r} = \begin{bmatrix} 1+e^{\sin(t)} \\ \cos^3(t) \\ 1 \end{bmatrix}$, $t \in [0, 2\pi]$. Compute

$$\int_C \vec{F} \cdot d\vec{r}$$

Notice that **C** is a closed curve, and that it is in the horizontal plane $z=1$. Then we conclude that $C = \partial S$ for some surface S in that plane.

So, we can apply Stokes' theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

Since S is in the plane $z=1$, we have $\vec{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

And $\nabla \times \vec{F} = \begin{pmatrix} xz^2 - 2ze^y \\ 3x^2z^2 - yz^2 \\ 0 - 0 \end{pmatrix}$

So $(\nabla \times \vec{F}) \cdot \vec{n} = 0$, and thus

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = 0$$