EXAM 2

Math 103, Fall 2005, Clark Bray.

You have 50 minutes.

No notes, no books.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name Solutions

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

Total __________ (/100 points)

1. (a) Compute the value of $D_{\vec{v}}f(\vec{a})$ in terms of p, q, and r, where

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xyz \\ x^2y - z^2 \end{bmatrix} \quad \text{and} \quad \vec{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{v} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

f is continuously differentiable, so

$$D_{\vec{v}}f(\vec{a}) = J_{f,\vec{a}}\,\vec{v}$$

$$J_f = \begin{pmatrix} yz & xz & xy \\ 2xy & x^2 & -2z \end{pmatrix}$$

$$J_{f,\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

$$D_{\vec{v}}f(\vec{a}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p+q+r \\ 2p+q-2r \end{pmatrix}$$

(b) Suppose that we require $p^2 + 4q^2 + r^2 \leq 4$; what velocity vector $\vec{v}$ in the domain causes f_2 to increase most quickly?

We want to maximize $\frac{df_2}{dt} = 2p + q - 2r$,

subject to the constraint $p^2 + 4q^2 + r^2 \leq 4$.

Call $h = 2p + q - 2r$.

Interior: $Dh = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \neq \vec{0}$, so

there are no interior critical points.

Boundary: $p^2 + 4q^2 + r^2 = 4$

(note, $Dg = \vec{0}$ only at $\vec{0}$, not on bdry.)

Lagrange: $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \lambda \begin{pmatrix} 2p \\ 8q \\ 2r \end{pmatrix}$

$\Rightarrow \lambda = \frac{2}{2p} = \frac{1}{8q} = \frac{-2}{2r}$

$\Rightarrow p = 8q = -r$

Plugging this into the bdry condition,

$$(8q)^2 + 4q^2 + (-8q)^2 = 4$$

$$132\, q^2 = 4$$

$$q = \pm\sqrt{\frac{4}{132}} = \frac{\pm 1}{\sqrt{33}}$$

So

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 8/\sqrt{33} \\ 1/\sqrt{33} \\ -8/\sqrt{33} \end{pmatrix} \text{ or } \begin{pmatrix} -8/\sqrt{33} \\ -1/\sqrt{33} \\ 8/\sqrt{33} \end{pmatrix}$$

(over)

For the first point:

$$\frac{df_2}{dt} = h = \frac{17}{\sqrt{33}}$$

For the second point:

$$\frac{df_2}{dt} = h = \frac{-17}{\sqrt{33}}$$

So $\frac{df_2}{dt}$ is maximized at

$$\boxed{\vec{v} = \begin{pmatrix} 8/\sqrt{33} \\ 1/\sqrt{33} \\ -8/\sqrt{33} \end{pmatrix}}$$

2. Consider the functions $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$, where

$$f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} s^3t^2 - 2t \\ 2t^2 - s \\ t^4 - 2st^2 \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

Suppose that when composed as $g \circ f$, we know that $(\partial g_2/\partial t)(1,1) = 12$; what is $(\partial g_2/\partial y)(-1,1,-1)$?

$$\underset{(s,t)}{\mathbb{R}^2} \xrightarrow{f} \underset{(x,y,z)}{\mathbb{R}^3} \xrightarrow{g} \underset{(g_1,g_2)}{\mathbb{R}^2}$$

$$J_{g\circ f,(1,1)} = J_{g,f(1,1)} \, J_{f,(1,1)} \qquad f(1,1) = (-1,1,-1)$$

$$J_{g\circ f,(1,1)} = J_{g,(-1,1,-1)} \, J_{f,(1,1)}$$

$$\begin{pmatrix} \cdot & \cdot \\ \cdot & \frac{\partial g_2}{\partial t} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \end{pmatrix} \begin{pmatrix} \cdot & \frac{\partial x}{\partial t} \\ \cdot & \frac{\partial y}{\partial t} \\ \cdot & \frac{\partial z}{\partial t} \end{pmatrix}$$

We know

$$\begin{aligned} x &= s^3t^2 - 2t & & \frac{\partial x}{\partial t} = 2s^3t - 2 & &= 0 \ @(1,1) \\ y &= 2t^2 - s & \Rightarrow & \frac{\partial y}{\partial t} = 4t & &= 4 \ @(1,1) \\ z &= t^4 - 2st^2 & & \frac{\partial z}{\partial t} = 4t^3 - 4st & &= 0 \ @(1,1) \end{aligned}$$

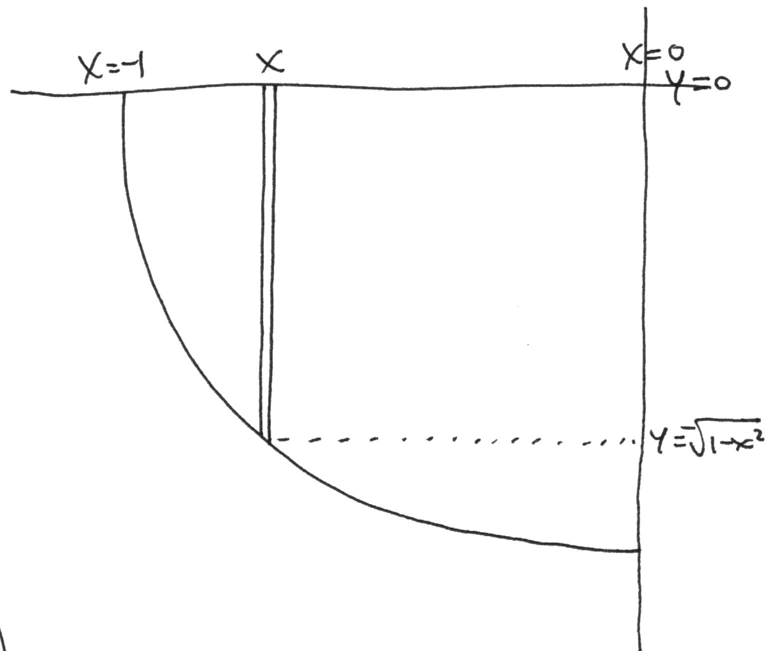
So the relevant dot product in the above matrix becomes

$$\underbrace{\left(\frac{\partial g_2}{\partial t}\right)(1,1)}_{=12} = \left(\frac{\partial g_2}{\partial y}\right)(-1,1,-1) \underbrace{\left(\frac{\partial y}{\partial t}\right)(1,1)}_{=4}$$

So, $\left(\frac{\partial g_2}{\partial y}\right)(-1,1,-1) = 3$

3. Evaluate the following, using any techniques from this course.

(a) $\iint_R 2x^2y\,dA$, where R is the part of the unit disk in the third quadrant.



$x \in [-1, 0]$

$y \in [-\sqrt{1-x^2}, 0]$

$$\iint_R 2x^2y\,dA$$

$$= \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} 2x^2y\,dy\,dx$$

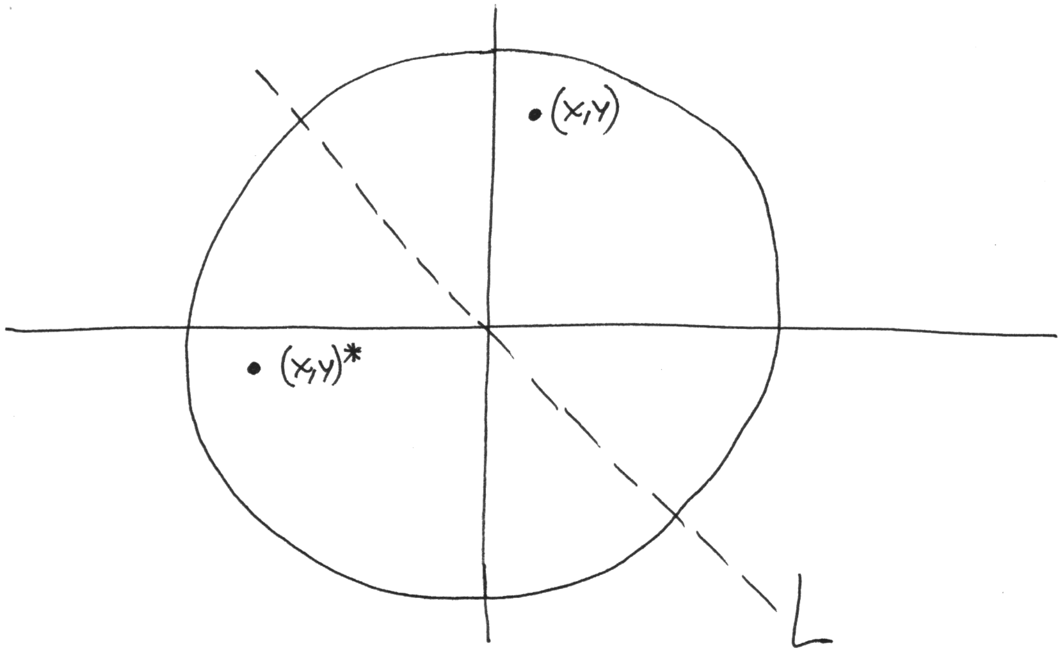
$$= \int_{-1}^{0}\left(x^2y^2 \Big]_{y=-\sqrt{1-x^2}}^{0} \right) dx$$

$$= \int_{-1}^{0} -x^2(1-x^2)\,dx$$

$$= \frac{1}{5}x^5 - \frac{1}{3}x^3 \Big]_{-1}^{0}$$

$$= (0-0) - \left(\frac{-1}{5} - \frac{-1}{3}\right) = \frac{1}{5} - \frac{1}{3} = \boxed{\frac{-2}{15}}$$

(b) $\iint_D \sin(x+y)\,dA$, where D is the entire unit disk.



Let L be the line $x+y=0$. Define $\vec{x}^*$ to be the reflection of $\vec{x}$ over L, so $(x,y)^* = (-y,-x)$.

We have:

① D is symmetric over L

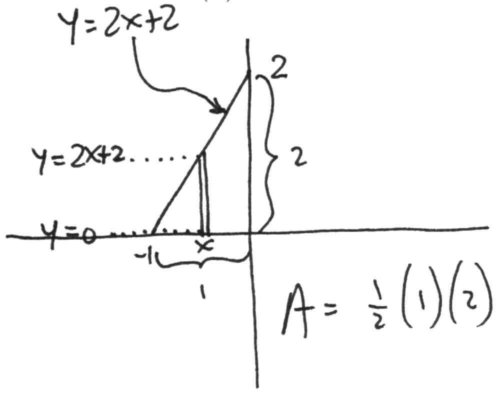
② $f(x,y) = \sin(x+y)$ has odd symmetry over L because

$$f(-y,-x) = \sin\big((-y)+(-x)\big) = \sin\big(-(x+y)\big) = -\sin(x+y)$$
$$= -f(x,y)$$

So, this integral is zero by symmetry.

4. Write down, but do not evaluate, explicit iterated integrals representing the following quantities. Make sure that you clearly demonstrate how you arrived at your result.

(a) The centroid of the region in the second quadrant below the line $y = 2x + 2$.

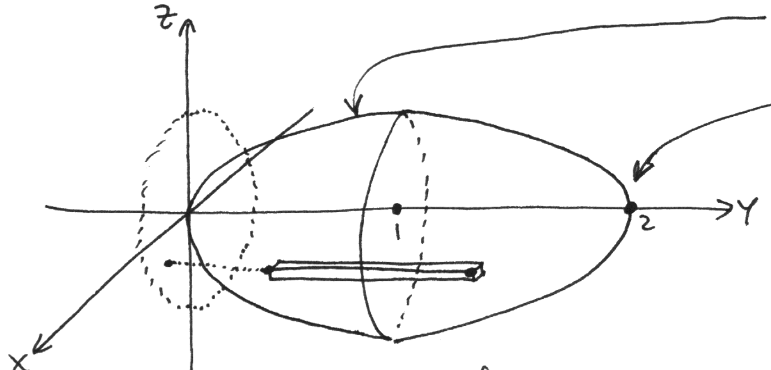


$A = \frac{1}{2}(1)(2) = 1$

$$\bar{x} = \frac{1}{A}\iint x \, dA = \int_{-1}^{0}\int_{0}^{2x+2} x \, dy \, dx$$

$$\bar{y} = \frac{1}{A}\iint dA = \int_{-1}^{0}\int_{0}^{2x+2} y \, dy \, dx$$

(b) The mass of the region in $\mathbb{R}^3$ bounded by the surfaces $\underbrace{x^2 - y + z^2 = 0}_{y = x^2+z^2}$ and $\underbrace{x^2 + y + z^2 = 2}_{y = 2-(x^2+z^2)}$, where the density is given by $\delta = x^2 + y^2 + z^2$.



For every pt. in the dotted circle in the xz-plane, y ranges over a clear set of values. The dotted circle is the intersection of the surfaces :

$$x^2+z^2 = 2-(x^2+z^2)$$

$$\Rightarrow x^2+z^2 = 1 \quad \Rightarrow z \in \left[-\sqrt{1-x^2}, \sqrt{1-x^2}\right]$$

So,

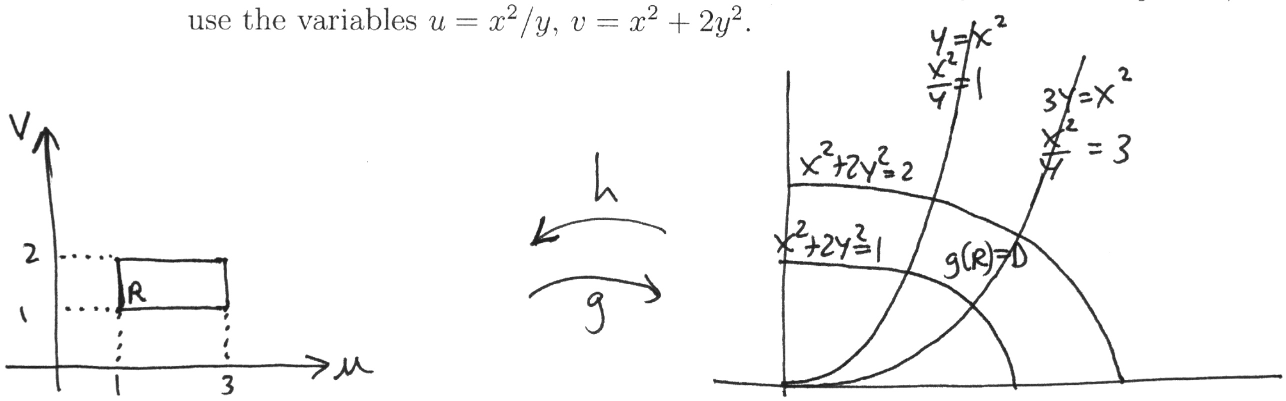
$$M = \iiint dm = \iiint \delta \, dV$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+z^2}^{2-(x^2+z^2)} (x^2+y^2+z^2) \, dy \, dz \, dx$$

5. Compute the following integral using the given change of variables.

$$\iint_D \left(2 - \frac{x^2}{y}\right)\left(8x + \frac{2x^3}{y^2}\right)\, dx\, dy$$

D is the region in the first quadrant bounded by $y = x^2$, $3y = x^2$, $x^2+2y^2 = 1$, $x^2+2y^2 = 2$; use the variables $u = x^2/y$, $v = x^2 + 2y^2$.



$$\iint_R f(g(\vec{u}))\,|\det J_g|\, dA \quad = \quad \iint_D f(\vec{x})\, dS$$

$$\iint_R f(g(\vec{u}))\,\frac{1}{|\det J_h|}\, du\, dv \quad = \quad \iint_D \left(2-\frac{x^2}{y}\right)\left(8x+\frac{2x^3}{y^2}\right) dx\,dy$$

$$J_h = \begin{pmatrix} 2x/y & -x^2/y^2 \\ 2x & 4y \end{pmatrix} \Rightarrow |\det J_h| = 8x + \frac{2x^3}{y^2}$$

So,

$$\iint = \iint_R \left(2-\frac{x^2}{y}\right)\left(8x+\frac{2x^3}{y^2}\right)\frac{1}{\left(8x+\frac{2x^3}{y^2}\right)}\, du\, dv$$

$$= \iint_R (2-u)\, du\, dv$$

(over)

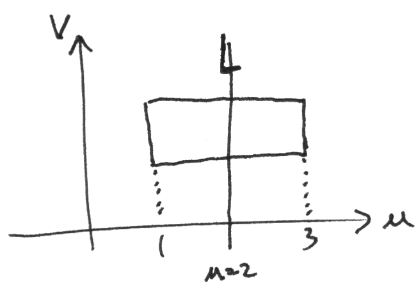
Let L be the line $u = 2$. Then we can compute reflections over L with

$$(u, v)^* = (4-u, v)$$

The integrand, $I(u,v) = 2-u$, thus has odd symmetry over L, since

$$I\left((u,v)^*\right) = I(4-u, v) = 2-(4-u) = u-2$$
$$= -(2-u) = -I(u,v)$$

Since we also have that R is symmetric over L,



we conclude that

$$\iint_R (2-u)\, du\, dv = 0 \quad \text{by symmetry}$$

So the original integral is zero.