EXAM 1

Math 103, Fall 2005, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name Solutions

ID number______________________

1. __________ (/30 points)

2. __________ (/20 points)

3. __________ (/30 points)

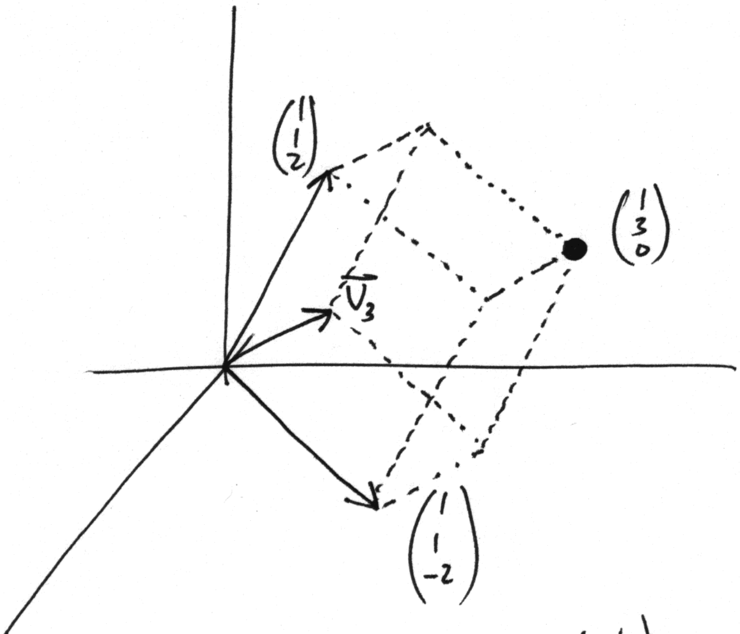
4. __________ (/20 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ______________________

Total __________ (/100 points)

1. (a) The parallelopiped P has one vertex at the origin and the opposite vertex at $(1, 3, 0)$. Two of the edges touching the origin are defined by the vectors $(1, 1, 2)$ and $(1, 1, -2)$, respectively. What is the volume of P?



$$\begin{pmatrix}1\\1\\2\end{pmatrix} + \begin{pmatrix}1\\1\\-2\end{pmatrix} + \vec{v}_3 = \begin{pmatrix}1\\3\\0\end{pmatrix}$$

So $\vec{v}_3 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

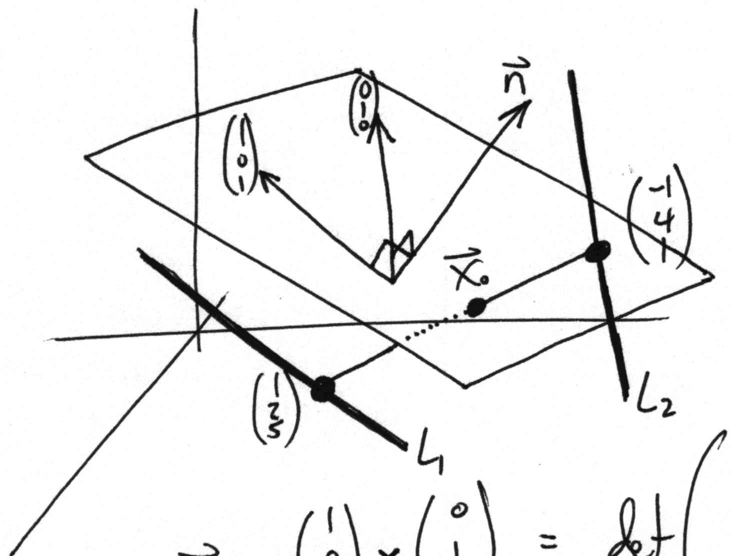
By the volume formula, we then have

$$V = \det\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & -2\\ -1 & 1 & 0\end{bmatrix} = 1\begin{vmatrix}1 & -2\\ 1 & 0\end{vmatrix} - 1\begin{vmatrix}1 & -2\\ -1 & 0\end{vmatrix} + 2\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix}$$

$$= 1(2) - 1(-2) + 2(2) = 8$$

(b) Find the equation for the plane L in $\mathbb{R}^3$ that is parallel to and equidistant from the two lines

$$\begin{bmatrix}1\\2\\5\end{bmatrix} + t\begin{bmatrix}1\\0\\1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-1\\4\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\0\end{bmatrix}$$



$$\vec{n} = \begin{pmatrix}1\\0\\1\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \det\begin{pmatrix}\vec{e}_1 & \vec{e}_2 & \vec{e}_3\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Eqn of plane is then $\vec{n}\cdot\vec{X} = \vec{n}\cdot\vec{X_0}$

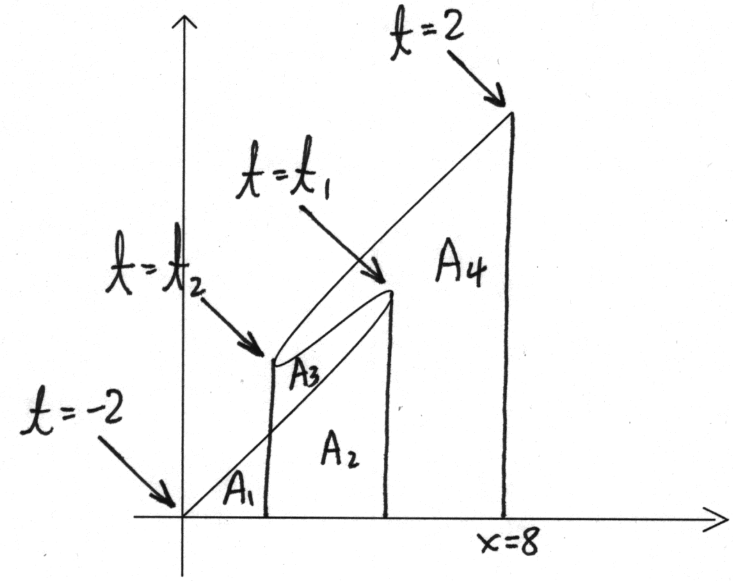
If plane is equidistant from the two lines, it must contain the midpoint of the segment from $\begin{pmatrix}1\\2\\5\end{pmatrix}$ to $\begin{pmatrix}-1\\4\\1\end{pmatrix}$, which is $\begin{pmatrix}0\\3\\3\end{pmatrix}$; call this $\vec{X}_0$. Then the equation becomes

$$\begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\3\\3\end{pmatrix}$$

$$\boxed{-x + z = 3}$$

2. The point $\vec{x}(t)$ moves along the path shown below according to the parametric equations

$$x(t) = t^3 - 2t + 4 \quad y(t) = t^3 - t + 6$$



Find the area between this curve, the positive part of the x axis, and the line $x = 8$.

$$① \int_{t=-2}^{t=t_1} y\,dx = \int_{-2}^{t_1} yx'\,dt = \int_{-2}^{t_1} dA = A_1 + A_2$$

$$② \int_{t_1}^{t_2} yx'\,dt = \int_{t_1}^{t_2} (-dA) = -A_2 - A_3$$

$$③ \int_{t_2}^{2} yx'\,dt = \int_{t_2}^{2} dA = A_2 + A_3 + A_4$$

Adding these we get $\int_{-2}^{2} yx'\,dt = A_1 + A_2 + A_4$ = desired area

$$= \int_{-2}^{2} (t^3 - t + 6)(3t^2 - 2)\,dt = \int_{-2}^{2} (3t^5 - 5t^3 + 18t^2 + 2t - 12)\,dt$$

$$= \frac{1}{2}t^6 - \frac{5}{4}t^4 + 6t^3 + t^2 - 12t\Big]_{-2}^{2} =$$

$$= (32 - 20 + 48 + 4 - 24) - (32 - 20 - 48 + 4 + 24)$$

$$= 48$$

3. Compute the following:

(a) $\frac{\partial}{\partial z}((x^2+1)^{x-y}z^{xy})$ $= (x^2+1)^{x-y}\left(xy\, z^{xy-1}\right)$

(b) The slope of the tangent line to the cross-section of the graph of $f(x,y) = x^2+y^2$ by the vertical plane defined by the line $2x+3y=6$, at the point $(3,0)$.

The slope is $D_{\vec{\mu}} f(\vec{a})$, with $\vec{a} = \begin{pmatrix}3\\0\end{pmatrix}$, $\vec{v} = \begin{pmatrix}3\\-2\end{pmatrix}$, $\vec{\mu} = \frac{\vec{v}}{\|\vec{v}\|}$

since $2x+3y=6$ is parametrized by $\begin{pmatrix}3\\0\end{pmatrix} + t\begin{pmatrix}3\\-2\end{pmatrix}$.

First we compute $D_{\vec{v}} f(\vec{a})$:

$$= \frac{d}{dt}\Big|_{t=0} f(\vec{a}+t\vec{v}) = \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix}3+3t\\-2t\end{pmatrix} = \frac{d}{dt}\Big|_{t=0}\left((3+3t)^2+(-2t)^2\right)$$

$$= \frac{d}{dt}\Big|_{t=0}\left(9+18t+13t^2\right) = (18+26t)\Big|_{t=0} = 18.$$

Then $D_{\vec{\mu}} f(\vec{a}) = \frac{1}{\|\vec{v}\|} D_{\vec{v}} f(\vec{a}) = \frac{1}{\sqrt{3^2+(-2)^2}}\cdot 18$

$$= \frac{18}{\sqrt{13}}$$

(c) The sun causes a point (x, y, z) to cast a shadow on the xy-plane at the point $(x - z, y - 2z, 0)$. Find the instantaneous velocity of the shadow of a point that is at $(1, 2, 4)$ and moving with velocity $(3, 2, 6)$.

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-z \\ y-2z \\ 0 \end{pmatrix}$$

velocity of the shadow $= \frac{df}{dt} = D_{\vec{v}} f(\vec{a})$ where $\frac{d\vec{x}}{dt} = \vec{v}$, at $\vec{x} = \vec{a}$. Then we compute

$$D_{\vec{v}} f(\vec{a}) = \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 1+3t \\ 2+2t \\ 4+6t \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} (1+3t) - (4+6t) \\ (2+2t) - 2(4+6t) \\ 0 \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} -3 - 3t \\ -6 - 10t \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ -10 \\ 0 \end{pmatrix}$$

4. The linear transformation T has

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_3) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Compute $(T \circ T)\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right)$

We can write the matrix A with $T(\vec{x}) = A\vec{x}$ as

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}$$

Then $(T \circ T)(\vec{x}) = (AA)\vec{x}$, so we compute

$$AA = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 4 \\ 15 & 0 & 9 \end{pmatrix}$$

and then

$$(T \circ T)\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 3 & 1 & 4 \\ 15 & 0 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 33 \end{pmatrix}$$