

EXAM 2

Math 103, Summer 2005 Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name _____

ID number _____

1. _____ (/20 points)

2. _____ (/20 points)

3. _____ (/20 points)

4. _____ (/20 points)

5. _____ (/20 points)

"I have adhered to the Duke Community
Standard in completing this
examination."

Signature: _____

Total _____ (/100 points)

1. Consider the function $f(x, y) = x + y^2$, and the point $\vec{a} = (1, 2)$. Of course the partial derivatives of f at \vec{a} are continuous, so f is continuously differentiable at \vec{a} and therefore also differentiable at \vec{a} . However in this problem, we will ignore that observation and prove directly from the definition that f is differentiable at \vec{a} .

- (a) Without assuming f is differentiable, compute $D_{\vec{a}}f(\vec{a})$ in terms of the components of $\vec{v} = (v_1, v_2)$ and then show that there is a linear transformation T (in other words, a matrix A) such that at this point \vec{a} ,

$$D_{\vec{a}}f(\vec{a}) = T(\vec{v}) = A\vec{v}$$

$$\begin{aligned} D_{\vec{v}}f(\vec{a}) &= \left. \frac{d}{dt} \right|_{t=0} f(\vec{a} + t\vec{v}) \\ &= \left. \frac{d}{dt} \right|_{t=0} f \begin{pmatrix} 1 + tv_1 \\ 2 + tv_2 \end{pmatrix} \\ &= \left. \frac{d}{dt} \right|_{t=0} \left((1 + tv_1) + (2 + tv_2)^2 \right) \\ &= \left. \frac{d}{dt} \right|_{t=0} \left(5 + (v_1 + 4v_2)t + (4v_2^2)t^2 \right) \\ &= v_1 + 4v_2 \\ &= \underbrace{\begin{pmatrix} 1 & 4 \end{pmatrix}}_A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = T(\vec{v}) \end{aligned}$$

- (b) Show (by direct computation of the limit) that for the appropriately chosen linear transformation, the limit of the relative error does equal zero, and thus that f is indeed differentiable.

$$\lim_{\vec{x} \rightarrow \vec{a}} \frac{f(\vec{x}) - f(\vec{a}) - T(\vec{x} - \vec{a})}{\|\vec{x} - \vec{a}\|}$$

$$= \lim_{\vec{x} \rightarrow \vec{a}} \frac{x+y^2 - 5 - (1 \ 4) \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}}{\left\| \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\|}$$

$$= \lim_{\vec{x} \rightarrow \vec{a}} \frac{x+y^2 - 5 - (x-1) - 4(y-2)}{\left\| \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \right\|}$$

$$= \lim_{\vec{x} \rightarrow \vec{a}} \frac{(y-2)^2}{\left\| \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \right\|} \geq 0$$

$$\leq \lim_{\vec{x} \rightarrow \vec{a}} \frac{\left\| \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \right\|^2}{\left\| \begin{pmatrix} x-1 \\ y-2 \end{pmatrix} \right\|} = \lim_{\vec{x} \rightarrow \vec{a}} \|\vec{x} - \vec{a}\| = 0$$

So the limit is zero, and thus f is differentiable at \vec{a} .

2. Suppose we have $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, and suppose we know the following:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5e^{xy} \\ x - y \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad J_{g, \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$

Compute $J_{f \circ g, \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$

$$J_{f \circ g, \begin{pmatrix} 1 \\ 2 \end{pmatrix}} = J_{f, g\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)} J_{g, \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$= J_{f, \begin{pmatrix} 1 \\ 0 \end{pmatrix}} J_{g, \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

$$J_f = \begin{pmatrix} 5ye^{xy} & 5xe^{xy} \\ 1 & -1 \end{pmatrix}$$

$$J_{f, \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 0 & 5 \\ 1 & -1 \end{pmatrix}$$

$$J_{f \circ g, \begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 0 & 5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 7 \\ 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 15 \\ 1 & 4 \end{pmatrix}$$

3. We have $f : (D \subset \mathbb{R}^3) \rightarrow \mathbb{R}^1$ with $f(x, y, z) = (x + y + z)^2$ and $D = \{x^2 + 4y^2 + 9z^2 \leq 1\}$. Find the point or points in the domain, if any exist, that attain the absolute maximum value of f on the domain D .

Interior: $\nabla f = \begin{pmatrix} 2(x+y+z) \\ 2(x+y+z) \\ 2(x+y+z) \end{pmatrix}$

∇f exists $\forall \vec{x}$; $\nabla f = \vec{0}$ only when $x+y+z=0$,

where we have $f(\vec{x}) = (x+y+z)^2 = (0)^2 = 0$.

Boundary: Let $g(\vec{x}) = x^2 + 4y^2 + 9z^2 - 1$, then $D = \{g \leq 0\}$

$\nabla g = \begin{pmatrix} 2x \\ 8y \\ 18z \end{pmatrix}$

$\nabla g = \vec{0}$ only at $\vec{x} = \vec{0}$, not on $g=0$.

$\nabla f = \lambda \nabla g$

$2(x+y+z) = \lambda(2x)$

$2(x+y+z) = \lambda(8y)$

$2(x+y+z) = \lambda(18z)$

$\Rightarrow 2x = 8y = 18z$

$\Rightarrow x = 9z, y = \frac{1}{4}z$

$\Rightarrow (9z)^2 + 4\left(\frac{1}{4}z\right)^2 + 9z^2 = 1$

$\Rightarrow \frac{441}{4}z^2 = 1 \Rightarrow z = \pm \frac{2}{21}$

(over)

Candidates :

$$\{(x+y+z)=0\} \Rightarrow f=0$$

$$\begin{pmatrix} 6/7 \\ 3/14 \\ 2/21 \end{pmatrix}$$

$$\Rightarrow f = \left(\frac{6}{7}\right)^2 + \left(\frac{3}{14}\right)^2 + \left(\frac{2}{21}\right)^2$$

$$\begin{pmatrix} -6/7 \\ -3/14 \\ -2/21 \end{pmatrix}$$

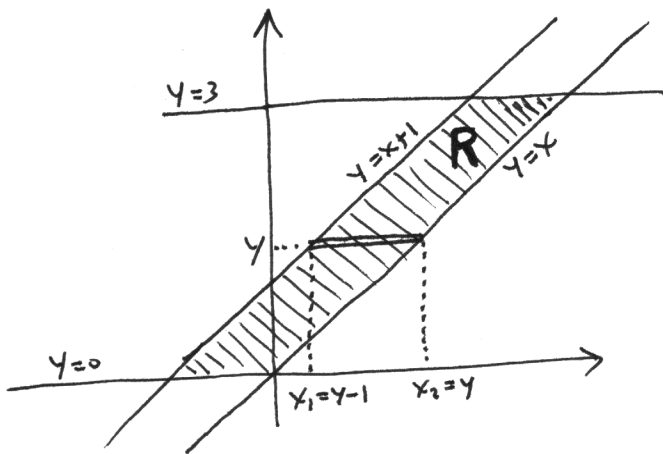
$$\Rightarrow f = \left(-\frac{6}{7}\right)^2 + \left(-\frac{3}{14}\right)^2 + \left(-\frac{2}{21}\right)^2$$

these are
equal

So, the abs. max is attained at the two points

$$\pm \begin{pmatrix} 6/7 \\ 3/14 \\ 2/21 \end{pmatrix}$$

4. Let R be the region in the xy -plane bounded by the lines $y = x$, $y = x + 1$, $y = 0$, and $y = 3$. Find the volume that is above R and below the graph of the function $f(x, y) = y + xy + 10$.



Outside \int : $y \in [0, 3]$.

for a given y ,

inside \int : $x \in [y-1, y]$

$$V = \int_0^3 \int_{y-1}^y (y + xy + 10) \, dx \, dy$$

$$= \int_0^3 \left(xy + \frac{1}{2}x^2y + 10x \right) \Big|_{x=y-1}^{x=y} dy$$

$$= \int_0^3 \left(\cancel{y^2} + \frac{1}{2}\cancel{y^3} + 10\cancel{y} \right) - \left(\cancel{y^2 - y} + \frac{1}{2}(\cancel{y^3 - 2y^2 - y}) + 10\cancel{y - 10} \right) dy$$

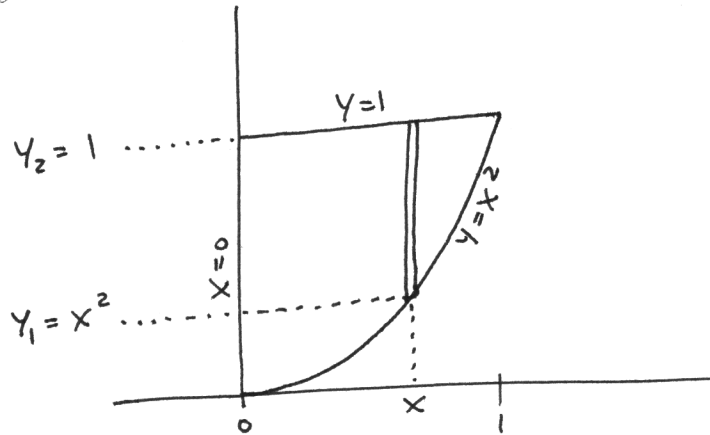
$$= \int_0^3 \left(y^2 + \frac{3}{2}y + 10 \right) dy$$

$$= \left(\frac{1}{3}y^3 + \frac{3}{4}y^2 + 10y \right) \Big|_0^3$$

$$= \left(9 + \frac{27}{4} + 30 \right) - (0)$$

$$= 45\frac{3}{4} = \boxed{\frac{183}{4}}$$

5. Find the centroid of the region in the first quadrant of the xy -plane, bounded by $y = x^2$, $x = 0$, and $y = 1$.



$$A = \int_0^1 (1 - x^2) dx$$

$$= \frac{2}{3}$$

$$\bar{x} = \frac{1}{A} \iint x \, dA$$

$$= \frac{1}{\frac{2}{3}} \int_0^1 \int_{x^2}^1 x \, dy \, dx$$

$$= \frac{3}{2} \int_0^1 (xy) \Big|_{y=x^2}^{y=1} dx$$

$$= \frac{3}{2} \int_0^1 (x - x^3) dx$$

$$= \frac{3}{2} \left(\frac{1}{2} x^2 - \frac{1}{4} x^4 \right) \Big|_0^1$$

$$= \frac{3}{2} \left(\frac{1}{4} \right)$$

$$= \frac{3}{8}$$

$$\bar{y} = \frac{1}{A} \iint y \, dA$$

$$= \frac{1}{\frac{2}{3}} \int_0^1 \int_{x^2}^1 y \, dy \, dx$$

$$= \frac{3}{2} \int_0^1 \left(\frac{1}{2} y^2 \right) \Big|_{y=x^2}^{y=1} dx$$

$$= \frac{3}{2} \int_0^1 \left(\frac{1}{2} - \frac{1}{2} x^4 \right) dx$$

$$= \frac{3}{2} \left(\frac{1}{2} x - \frac{1}{10} x^5 \right) \Big|_{x=0}^{x=1}$$

$$= \frac{3}{2} \left(\frac{2}{5} \right)$$

$$= \frac{3}{5}$$

$$\text{centroid} = (\bar{x}, \bar{y}) = \left(\frac{3}{8}, \frac{3}{5} \right)$$