EXAM 1

Math 103, Summer 2005 Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name Solutions

ID number________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

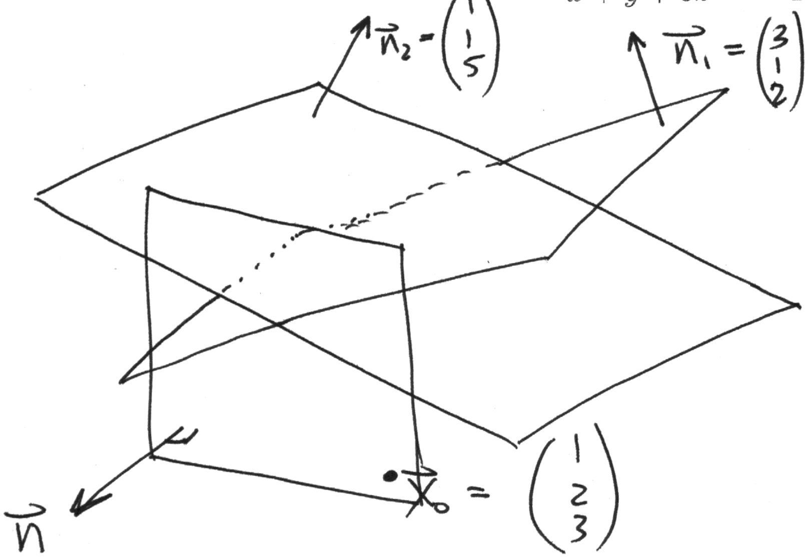
"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________

Total __________ (/100 points)

1. (a) Find an equation of the unique plane that contains the point $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and is perpendicular to each of the planes below:

$$\begin{aligned} 3x + y + 2z &= 2 \\ x + y + 5z &= 1 \end{aligned}$$



The normal vector must be $\perp$ to $\vec{n}_1$ and $\vec{n}_2$ ... so we choose

$$\vec{n} = \vec{n}_1 \times \vec{n}_2 = \det\begin{pmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & 1 & 2 \\ 1 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -13 \\ 2 \end{pmatrix}$$

So eqn is

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{x}_0$$

$$3x - 13y + 2z = \begin{pmatrix} 3 \\ -13 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\boxed{3x - 13y + 2z = -17}$$

(b) What is the function whose graph is the plane from part (a)?

$$3x - 13y + 2z = -17$$

$$2z = -17 - 3x + 13y$$

$$z = \frac{-17 - 3x + 13y}{2}$$

This is the graph of $\boxed{z = f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-17 - 3x + 13y}{2}}$.

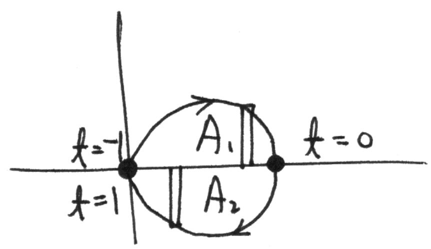
2. Consider the parametric curve in the plane defined by

$$\vec{r}(t) = \begin{bmatrix} 1/(1+t^2) \\ t^3 - t \end{bmatrix}$$

(a) Find the velocity vector as a function of t.

$$\text{velocity} = \vec{r}\,'(t) = \begin{pmatrix} -2t/(1+t^2)^2 \\ 3t^2 - 1 \end{pmatrix}$$

(b) Between $t = -1$ and $t = 1$, this parametric curve traces out (clockwise) a closed loop that starts and ends at the origin, and does not intersect itself anywhere else. Write down (but do not evaluate!) a single integral representing the area of that loop, and make sure to explain your reasoning.



height width

for $t \in [-1, 0]$, $x' > 0$, $y > 0$ so $\int y\,dx = \int y\,x'\,dt = A_1$

for $t \in [0, 1]$, $x' < 0$, $y < 0$ so $\int y\,dx = \int y\,x'\,dt$

$$= \int \underbrace{(-y)}_{\text{height}}\underbrace{(-x'\,dt)}_{\text{width}} = A_2$$

So,

$$A = A_1 + A_2 = \int_{-1}^{0} y x'\,dt + \int_{0}^{1} y x'\,dt$$

$$= \int_{-1}^{1} y x'\,dt = \boxed{\int_{-1}^{1} \frac{-2t\,(t^3 - t)}{(1+t^2)^2}\,dt}$$

3. Consider the function f and vectors $\vec{p}$ and $\vec{q}$ given by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x^2 + y^2 \end{bmatrix} \quad \text{and} \quad \vec{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{q} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

(a) Compute, directly from the definition, the value of $D_{\vec{q}} f(\vec{p})$.

$$D_{\vec{q}} f(\vec{p}) = \frac{d}{dt}\Big|_{t=0} f(\vec{p} + t\vec{q}) = \frac{d}{dt}\Big|_{t=0} f\begin{pmatrix} 1+3t \\ 2+4t \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} (1+3t)(2+4t) \\ (1+3t)^2 + (2+4t)^2 \end{pmatrix}$$

$$= \frac{d}{dt}\Big|_{t=0} \begin{pmatrix} 12t^2 + 10t + 2 \\ 25t^2 + 22t + 5 \end{pmatrix}$$

$$= \begin{pmatrix} 24t + 10 \\ 50t + 22 \end{pmatrix}\Bigg|_{t=0}$$

$$\boxed{= \begin{pmatrix} 10 \\ 22 \end{pmatrix}}$$

(b) Compute the value of $D_{\vec{q}} f(\vec{p})$ using gradient vectors.

$$D_{\vec{q}} f(\vec{p}) = \begin{pmatrix} D_{\vec{q}} f_1(\vec{p}) \\ D_{\vec{q}} f_2(\vec{p}) \end{pmatrix} = \begin{pmatrix} \nabla f_1(\vec{p}) \cdot \vec{q} \\ \nabla f_2(\vec{p}) \cdot \vec{q} \end{pmatrix}$$

$$\nabla f_1 = \begin{pmatrix} y \\ x \end{pmatrix} \qquad \nabla f_1(\vec{p}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\nabla f_2 = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \nabla f_2(\vec{p}) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$D_{\vec{q}} f(\vec{p}) = \begin{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{pmatrix} = \boxed{\begin{pmatrix} 10 \\ 22 \end{pmatrix}}$$

(c) At the point $\vec{p}$, in what unit vector direction is the second component of f increasing the fastest?

f_2 increases fastest at $\vec{p}$ in the direction of $\nabla f_2(\vec{p})$

$$= \frac{\nabla f_2(\vec{p})}{\|\nabla f_2(\vec{p})\|} = \frac{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}{\left\| \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\|} = \boxed{\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}}$$

4. We have a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, and all we know about it is that

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}3\\4\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

(a) Find the matrix that represents T.

$$\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$T\begin{pmatrix}1\\1\end{pmatrix} + T\begin{pmatrix}1\\-1\end{pmatrix} = T\begin{pmatrix}2\\0\end{pmatrix}$$

$$\Rightarrow T(\vec{e}_1) = \frac{\begin{pmatrix}3\\4\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix}}{2} = \begin{pmatrix}2\\2\end{pmatrix}$$

$$\begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$

$$T\begin{pmatrix}1\\1\end{pmatrix} - T\begin{pmatrix}1\\-1\end{pmatrix} = T\begin{pmatrix}0\\2\end{pmatrix}$$

$$\hookrightarrow T(\vec{e}_2) = \frac{\begin{pmatrix}3\\4\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}}{2} = \begin{pmatrix}1\\2\end{pmatrix}$$

These are columns of the matrix A, so $A = \begin{pmatrix}2 & 1\\2 & 2\end{pmatrix}$

(b) Find the matrix that represents $T \circ T$.

The matrix for $T \circ T$ is AA:

$$\begin{pmatrix}2 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\2 & 2\end{pmatrix} = \boxed{\begin{pmatrix}6 & 4\\8 & 6\end{pmatrix}}$$

5. What is the value, if it exists, of the limit below?

$$\lim_{\vec{x}\to\vec{0}} \frac{x^3y - xy^3}{(x^2+y^2)^2}$$

Let's compute the value of this limit along the line $y = mx$:

$$\lim_{x\to 0} \frac{(x^3)(mx) - (x)(mx)^3}{(x^2+(mx)^2)^2}$$

$$= \lim_{x\to 0} \frac{x^4(m-m^3)}{x^4(1+m^2)^2}$$

$$= \frac{m-m^3}{(1+m^2)^2}$$

So, along the line $y=0$, we have $m=0 \Rightarrow \lim = 0$

But along the line $y=2x$, we have $m=2 \Rightarrow \lim = \frac{-6}{25}$

These are not equal — so the given limit can not exist.