# EXAM 2

Math 103, Summer 2005 Term 1, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name Solutions

ID number____________________

1. __________ (/20 points)

2. __________ (/20 points)

3. __________ (/20 points)

4. __________ (/20 points)

5. __________ (/20 points)

Total __________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. Compute the general Jacobian matrix $J_f$ for each of the functions below:

(a) $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2y \\ y^3 \\ y^2 - e^x \end{bmatrix}$

$$J_f = \begin{pmatrix} 2xy & x^2 \\ 0 & 3y^2 \\ -e^x & 2y \end{pmatrix}$$

(b) $f\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \begin{bmatrix} u^2 - vw \\ uvw - e^{uvw} \end{bmatrix}$

$$J_f = \begin{pmatrix} 2u & -w & -v \\ vw - vwe^{uvw} & uw - uwe^{uvw} & uv - uve^{uvw} \end{pmatrix}$$

2. Find the point (or points) in the unit disk in $\mathbb{R}^2$ that attains the maximum value on that domain of

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - x + y^2 - y$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2 - 1 \leq 0$$

<u>Interior</u>: $Df = \begin{pmatrix} 2x-1 \\ 2y-1 \end{pmatrix}$; $Df = \vec{0} \Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}}$

<u>Boundary</u>: $Dg = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$; $Df = \lambda Dg \Rightarrow$

$$\left.\begin{aligned} 2x-1 &= \lambda(2x) \\ 2y-1 &= \lambda(2y) \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda &= 1 - \tfrac{1}{2x} \\ \lambda &= 1 - \tfrac{1}{2y} \end{aligned}$$

$$\Rightarrow x = y$$

$$\Rightarrow x^2 + x^2 = 1 \Rightarrow x = \pm\sqrt{1/2}$$

$$\Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{pmatrix} \text{ or } \begin{pmatrix} -\sqrt{1/2} \\ -\sqrt{1/2} \end{pmatrix}}$$

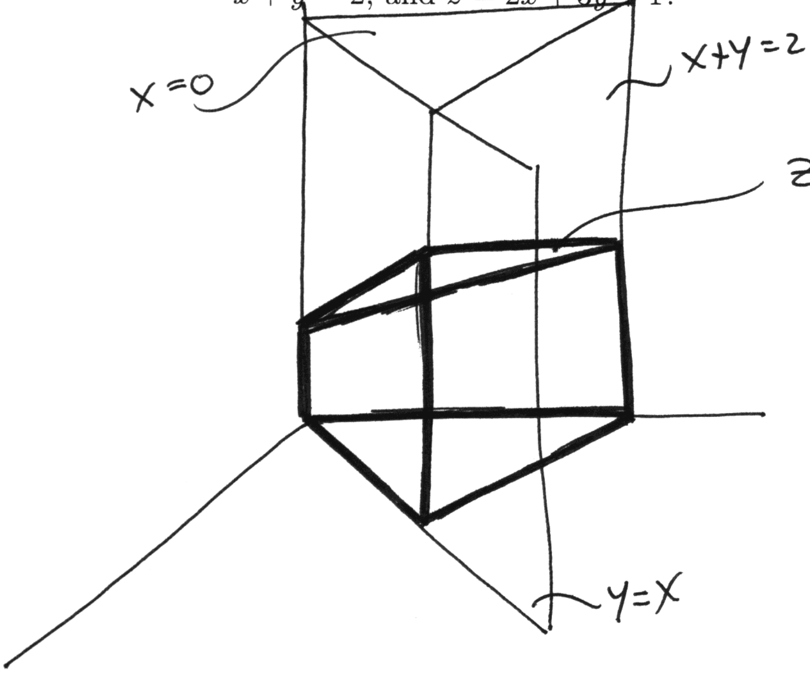
<u>Check values</u>:

$$f\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = -1/2$$

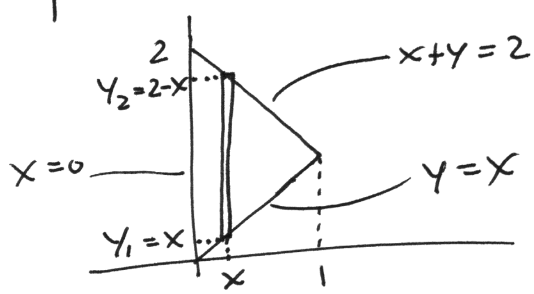
$$f\begin{pmatrix} \sqrt{1/2} \\ \sqrt{1/2} \end{pmatrix} = 1 - \sqrt{2}$$

$$f\begin{pmatrix} -\sqrt{1/2} \\ -\sqrt{1/2} \end{pmatrix} = 1 + \sqrt{2} \longleftarrow \boxed{\text{abs. max at } \begin{pmatrix} -\sqrt{1/2} \\ -\sqrt{1/2} \end{pmatrix}}$$

3. What is the volume in $\mathbb{R}^3$ above the $xy$-plane, bounded by the planes $x = 0$, $y = x$, $x + y = 2$, and $z = 2x + 3y + 1$?




This is the volume under $z = 2x+3y+1$, above the domain in the $xy$ plane pictured:




$x$ : ranges from 0 to 1

$y$ : ranges from $x$ to $2-x$

$$\text{Volume} = \int_0^1 \int_x^{2-x} (2x+3y+1)\,dy\,dx$$

$$= \int_0^1 \left( 2xy + \tfrac{3}{2}y^2 + y \Big]_{y=x}^{y=2-x} \right) dx$$

$$= \int_0^1 \left( 2x(2-x) + \tfrac{3}{2}(2-x)^2 + (2-x) - 2x^2 - \tfrac{3}{2}x^2 - x \right) dx$$

$$= \int_0^1 -2x^2 + 4x + \cancel{\tfrac{3}{2}x^2} - 6x + 6 + 2 - x - 2x^2 - \cancel{\tfrac{3}{2}x^2} - x \Big) dx$$

$$= \int_0^1 -4x^2 - 4x + 8\,dx = \left( -\tfrac{4}{3}x^3 - 2x^2 + 8x \right]_0^1$$

$$= \boxed{\frac{14}{3}}$$

4. Suppose that for a given function $f : \mathbb{R}^2 \to \mathbb{R}^3$ and a given point $\vec{a} \in \mathbb{R}^2$, we know that for any vector $\vec{v} = (v_1, v_2, v_3)$,

$$D_{\vec{v}} f(\vec{a}) = \begin{bmatrix} v_1 + 2v_3 \\ v_2^3/\|\vec{v}\|^2 \\ 7v_2 \end{bmatrix}$$

Explain, with specific examples to support your reasoning, how you know that this function cannot be differentiable at this point.

$$D_{\begin{pmatrix}1\\0\\0\end{pmatrix}} f(\vec{a}) = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad D_{\begin{pmatrix}0\\1\\0\end{pmatrix}} f(\vec{a}) = \begin{pmatrix}0\\1\\7\end{pmatrix}; \quad D_{\begin{pmatrix}1\\1\\0\end{pmatrix}} f(\vec{a}) = \begin{pmatrix}1\\1/2\\7\end{pmatrix}$$

If $f$ were differentiable, then we could rewrite these as

$$D_{f,\vec{a}}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}; \quad D_{f,\vec{a}}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\7\end{pmatrix}; \quad D_{f,\vec{a}}\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\1/2\\7\end{pmatrix}$$

But this contradicts the requirement that $D_{f,\vec{a}}$ is linear, since these do not satisfy

$$D_{f,\vec{a}}\begin{pmatrix}1\\0\\0\end{pmatrix} + D_{f,\vec{a}}\begin{pmatrix}0\\1\\0\end{pmatrix} = D_{f,\vec{a}}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = D_{f,\vec{a}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

So, $f$ cannot be differentiable at $\vec{a}$.

5. Suppose we have differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^3$, with

$$f\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \quad \text{and} \quad g\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right)$$

and suppose we know the following value of $f$, and three vector derivatives:

$$f(\vec{a}) = \vec{b} \qquad D_{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} g\left(\vec{b}\right) = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

$$D_{\begin{bmatrix} 2 \\ 5 \end{bmatrix}} f(\vec{a}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad D_{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} g\left(\vec{b}\right) = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

Use derivative transformations and the chain rule to compute

$$2\frac{\partial w}{\partial s}(\vec{a}) + 5\frac{\partial w}{\partial t}(\vec{a})$$

*(Hint: Rephrase the given vector derivatives in terms of derivative transformations. Then compute* $D_{(g\circ f),\vec{a}}\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right)$ *by first using the chain rule, and then the linearity of* $D_{g,\vec{b}}$*; then relate this to the quantity you are supposed to compute.)*

Rephrase: $D_{f,\vec{a}}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ; $D_{g,\vec{b}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}$ ; $D_{g,\vec{b}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$

As suggested, we compute:

$$D_{(g\circ f),\vec{a}}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = D_{g,f(\vec{a})}\left(D_{f,\vec{a}}\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right) = D_{g,\vec{b}}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = D_{g,\vec{b}}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$$

(chain rule)

$$= D_{g,\vec{b}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + D_{g,\vec{b}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

(linearity)

Also, $D_{(g\circ f),\vec{a}}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = J_{g\circ f,\vec{a}}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$

$$= \left.\begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{pmatrix}\right|_{\vec{a}} \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

So, $2\frac{\partial w}{\partial s}(\vec{a}) + 5\frac{\partial w}{\partial t}(\vec{a})$ is the third component of $D_{(g\circ f),\vec{a}}$, and so by previous computation, it equals $\boxed{3}$