

EXAM 4

Math 103, Fall 2004, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

**YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT**

Good luck!

Name Key

ID number _____

1. _____ (/40 points)

2. _____ (/30 points)

"I have adhered to the Duke Community Standard in completing this examination."

3. _____ (/15 points)

Signature: _____

4. _____ (/15 points)

Total _____ (/100 points)

1. In each of the following cases, use any techniques from this course to compute

$$\int_C \vec{F} \cdot d\vec{r}$$

- (a) $\vec{F} = (y, x)$, C = the arc of the unit circle going counter-clockwise from $(1, 0)$ to $(0, 1)$.

$$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 0, \text{ so } \vec{F} = \nabla f \text{ for some } f.$$

By inspection, $f(x, y) = xy$. Then F.T.L.I. gives us

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) \\ &= f(0, 1) - f(1, 0) \\ &= \boxed{0} \end{aligned}$$

- (b) $\vec{F} = (x + y, y)$, C = the entire unit circle, oriented counter-clockwise.

$$\begin{aligned} \text{Green's Thm gives us } \oint \vec{F} \cdot d\vec{r} &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\ &= \iint_D (0 - 1) dx dy \\ &= (-1) (\text{area of disk } D) \\ &= (-1)(\pi(1)^2) = \boxed{-\pi} \end{aligned}$$

(c) $\vec{F} = (x+y, y)$, C = the arc of the unit circle going counter-clockwise from $(1, 0)$ to $(0, 1)$.

$$\vec{r}(t) = (x, y) = (\cos t, \sin t), t \in [0, \frac{\pi}{2}]$$

$$\vec{r}'(t) = (x', y') = (-\sin t, \cos t) = (-y, x)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} (x+y, y) \cdot (-y, x) dt = \int_0^{\pi/2} -y^2 dt$$

$$= \int -\sin^2 t dt = \int_0^{\pi/2} \frac{\cos 2t - 1}{2} dt = \boxed{\frac{-\pi}{4}}$$

(d) $\vec{F} = (2x, z, y)$, $\vec{r} = (t(t-1)e^{t^2} \cos t, t + t^2(t-1), (t-1)\sin t)$, $t \in [0, 1]$

Note: $D_x \vec{F} = (0-0, 0-0, 1-1) = \vec{0}$, so $\vec{F} = Df$

By inspection, $f = x^2 + yz$. Then by F.T.L.I.,

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= f(\vec{r}(1)) - f(\vec{r}(0)) \\ &= f(1, 0, 0) - f(0, 0, 0) \end{aligned}$$

$$= 1 - 0$$

$$= \boxed{1}$$

2. In each of the following cases, use any techniques from this course to compute

$$\iint_S \vec{F} \cdot \vec{n} dS$$

- (a) $\vec{F} = \nabla \times (x, 0, x^2yz^3)$, $S = \{x^2 + 4y^2 + 5z^2 = 21, z \geq -1\}$, oriented such that the normal vector is $(0, 0, 1)$ at the point where z is greatest.

By Stokes theorem,

$$\begin{aligned} \iint_S \vec{F} \cdot \vec{n} dS &= \oint_{\partial S} (\nabla \times (x, 0, x^2yz^3)) \cdot d\vec{r} \\ &= \oint_{\partial S} (x, 0, x^2yz^3) \cdot d\vec{r} \end{aligned}$$

∂S is $\{z = -1, x^2 + 4y^2 = 16\}$

parametrized by $\vec{r}(t) = (4\cos t, 2\sin t, -1)$

with $\vec{r}'(t) = (-4\sin t, 2\cos t, 0)$

So $dz = 0$, and thus

$$\oint_{\partial S} (x, 0, x^2yz^3) \cdot d\vec{r} = \oint_{\partial S} x dx + 0 dy + 0 dz$$

$$= 0$$

(b) $\vec{F} = (x+xy, y+yz, z+zx)$, S is the unit sphere.

$$\begin{aligned}\nabla \cdot \vec{F} &= (1+y) + (1+z) + (1+x) \\ &= 3 + x + y + z\end{aligned}$$

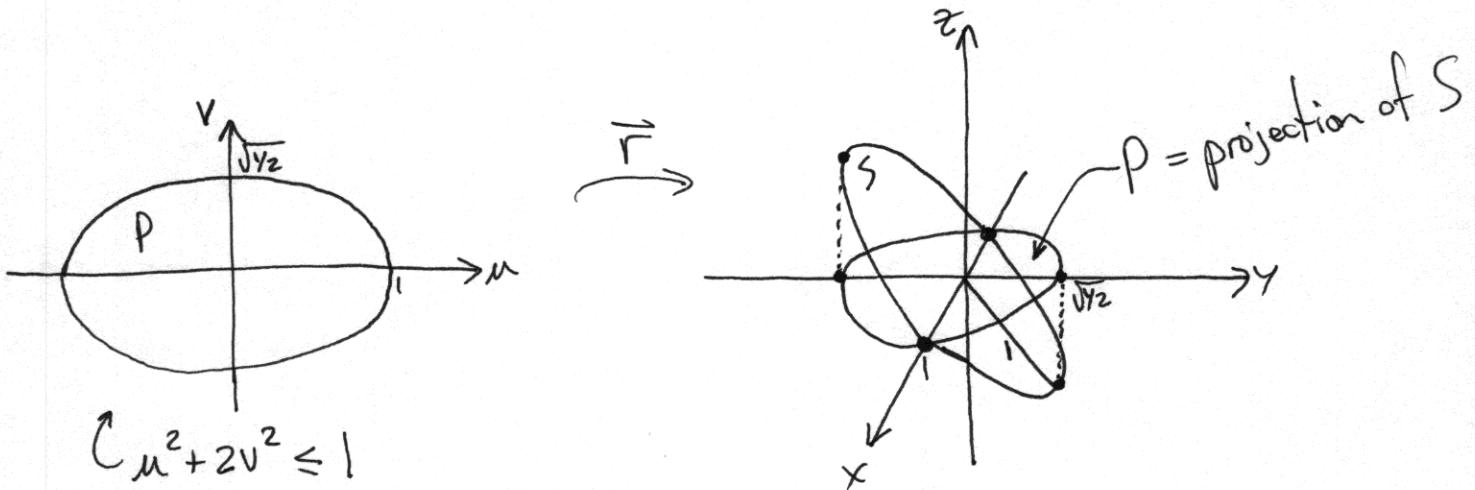
Gauss' Thm then gives us

$$\begin{aligned}\iiint_S \vec{F} \cdot \hat{n} dS &= \iiint_B \nabla \cdot \vec{F} dV \\ &= \iiint_B (3 + x + y + z) dV\end{aligned}$$

The terms x, y , and z are antisymmetric about the yz , zx , and xy planes (resp.), and the domain (the unit ball) is symmetric about each of those planes — so those terms integrate to zero. We are left with

$$\begin{aligned}\iiint_B 3 dV &= (3)(\text{vol. of unit ball}) \\ &= (3)\left(\frac{4}{3}\pi(1)^3\right) \\ &= \boxed{4\pi}\end{aligned}$$

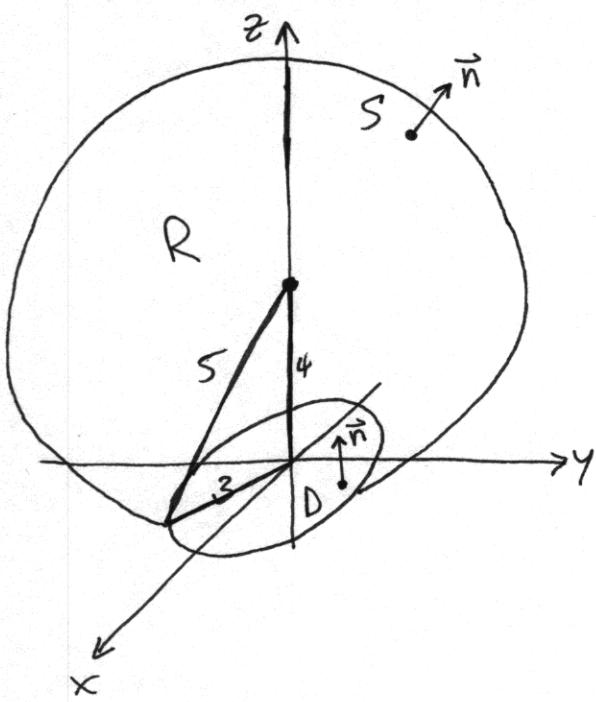
- (c) $\vec{F} = (0, y+1, 0)$, S is the disk of radius 1 in the plane $y+z=0$ with center at $(0,0,0)$, and upward normal.



$$\begin{aligned}\vec{F}(u, v) &= (u, v, -v) \\ \vec{r}_u &= (1, 0, 0) \\ \vec{r}_v &= (0, 1, -1) \\ \vec{N} &= (0, 1, 1)\end{aligned}$$

$$\begin{aligned}\iint_S \vec{F} \cdot \vec{n} dS &= \iint_P \vec{F} \cdot \vec{N} du dv \\ &= \iint_P (v+1) du dv \\ &= \int_{-1}^1 \int_{-\sqrt{\frac{1-u^2}{2}}}^{\sqrt{\frac{1-u^2}{2}}} (v+1) dv du \quad \text{this integrates to } 0 \text{ by symmetry} \\ &= \int_{-1}^1 2 \sqrt{\frac{1-u^2}{2}} du \\ &= \sqrt{2} \int_{-1}^1 \sqrt{1-u^2} du \\ &= (\sqrt{2}) (\text{area of half of a unit circle}) \\ &= (\sqrt{2}) \left(\frac{1}{2} \pi (1)^2 \right) \\ &= \boxed{\frac{\pi \sqrt{2}}{2}}\end{aligned}$$

3. Let S be the part of the sphere $x^2 + y^2 + (z - 4)^2 = 25$ that is above the xy -plane, oriented with the outward pointing normal from the sphere itself. Let $\vec{F} = (ze^{y^2}, xe^z, y)$. Determine the value of



$$\iint_S \vec{F} \cdot \vec{n} dS$$

$$\nabla \cdot \vec{F} = 0 + 0 + 0 = 0,$$

so we would like to apply
Gauss' theorem...

Define D to be the disk in
the xy -plane with the same boundary
as S , and let R be the region
cut off by S and D .

Gauss gives us

$$0 = \iiint_R \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \vec{n} dS - \iint_D \vec{F} \cdot \vec{n} dS$$

$$\text{So } \iint_S \vec{F} \cdot \vec{n} dS = \iint_D \vec{F} \cdot \vec{n} dS$$

On D , we clearly see $\vec{n} = (0, 0, 1)$, so this becomes

$$= \iint_D y dx dy$$

and this $\boxed{= 0}$ by symmetry.

4. Suppose that $\vec{F}(x, y, z) = (P(x, y), Q(x, y), 0)$, and that C is the closed curve parametrized by $\vec{r}(t) = (\cos^3(t), \cos^3(t), \sin^3(t))$, $t \in [0, 2\pi]$. Show that

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Stokes' theorem tells us

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$

where S has boundary C . Since we notice that C is entirely in the plane $x=y$, we can take S to be the surface in that plane bounded by C .

Therefore the normal vector \vec{n} of S is just

$$\vec{n} = \frac{(1, -1, 0)}{\sqrt{2}}$$

We compute $\nabla \times \vec{F}$ as

$$\nabla \times \vec{F} = \left(0-0, 0-0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\text{So } (\nabla \times \vec{F}) \cdot \vec{n} = 0$$

$$\Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS = 0$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$