EXAM 4
Math 103, Fall 2004, Clark Bray.
You have 50 minutes.
No notes, no books, no calculators.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT
Good luck!

Name ____________________________
ID number ____________________________

1. ____________ (/40 points)

2. ____________ (/30 points)  
“I have adhered to the Duke Community Standard in completing this examination.”

3. ____________ (/15 points)  
Signature: ____________________________

4. ____________ (/15 points)

Total ____________ (/100 points)
1. In each of the following cases, use any techniques from this course to compute 

\[ \oint_C \vec{F} \cdot d\vec{r} \]

(a) \( \vec{F} = (y, x) \), \( C \) = the arc of the unit circle going counter-clockwise from \((1, 0)\) to \((0, 1)\).

\[
\left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 0 \quad \text{so} \quad \vec{F} = \nabla f \quad \text{for some} \quad f.
\]

By inspection, \( f(xy) = xy \). Then F.T.L.I. gives us

\[
\oint_C \vec{F} \cdot d\vec{r} = \oint \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))
\]

\[
= f(0, 1) - f(1, 0)
\]

\[
= 0
\]

(b) \( \vec{F} = (x + y, y) \), \( C \) = the entire unit circle, oriented counter-clockwise.

Green's Thm gives us

\[
\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\text{area}
\]

\[
= \iint_D 0 \ d\text{area}
\]

\[
= (-1) \ (\text{area of disk } D)
\]

\[
= (-1) \left( \pi (1)^2 \right) = -\pi
\]
(c) \( \vec{F} = (x + y, y) \), \( C \) = the arc of the unit circle going counter-clockwise from \((1, 0)\) to \((0, 1)\).

\[
\vec{r}(t) = (x, y) = (\cos t, \sin t) \quad , \quad t \in \left[0, \frac{\pi}{2}\right]
\]

\[
\vec{r}'(t) = (x', y') = (-\sin t, \cos t) = (-y, x)
\]

\[
\begin{align*}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^{\frac{\pi}{2}} (x + y, y) \cdot (-y, x) \, dt \\
&= \int_0^{\frac{\pi}{2}} -y^2 \, dt \\
&= \int_0^{\frac{\pi}{2}} -\sin^2 t \, dt \\
&= \int_0^{\frac{\pi}{2}} \cos 2t \frac{-1}{2} \, dt \\
&= \frac{-11}{4}
\end{align*}
\]

(d) \( \vec{F} = (2x, z, y) \), \( \vec{r} = (t(t-1)e^t \cos t, t + t^2(t-1), (t - 1) \sin t) \), \( t \in [0, 1] \)

Note: \( \text{div} \vec{F} = (0-0, 0-0, 1-1) = 0 \), so \( \vec{F} = \text{Df} \)

By inspection, \( f = x^2 + yz \). Then by F.T.L.I.,

\[
\begin{align*}
\int_C \vec{F} \cdot d\vec{r} &= f(\vec{r}(1)) - f(\vec{r}(0)) \\
&= f(1, 0, 0) - f(0, 0, 0) \\
&= 1 - 0 \\
&= 1
\end{align*}
\]
2. In each of the following cases, use any techniques from this course to compute

$$\iint_S \vec{F} \cdot \hat{n} \, dS$$

(a) $$\vec{F} = \nabla \times (x, 0, x^2yz^3)$$, $$S = \{x^2 + 4y^2 + 5z^2 = 21, z \geq -1\}$$, oriented such that the normal vector is $$(0, 0, 1)$$ at the point where $$z$$ is greatest.

By Stokes' theorem,

$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iint_S (D_x(x, 0, x^2yz^3)) \cdot \hat{n} \, dS$$

$$= \oint_{\partial S} (x, 0, x^2yz^3) \cdot \, d\vec{r}$$

$$\partial S$$ is $$\{z = -1, x^2 + 4y^2 = 16\}$$

parametrized by $$\vec{r}(t) = (4\cos t, 2\sin t, -1)$$

with $$\vec{r}'(t) = (-4\sin t, 2\cos t, 0)$$

So $$dz = 0$$, and thus

$$\oint_{\partial S} (x, 0, x^2yz^3) \cdot \, d\vec{r} = \oint_{\partial S} x \, dx + 0 \, dy + 0 \, dz$$

$$= 0$$
(b) \( \vec{F} = (x+xy, y+yz, z+zx) \), \( S \) is the unit sphere.

\[
\text{D} \cdot \vec{F} = (1+y) + (1+z) + (1+x) = 3 + x + y + z
\]

Gauss' Theorem then gives us

\[
\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_B \text{D} \cdot \vec{F} \, dV
\]

\[
= \iiint_B (3 + x + y + z) \, dV
\]

The terms \( x, y, \) and \( z \) are antisymmetric about the \( yz, zx, \) and \( xy \) planes (resp.), and the domain (the unit ball) is symmetric about each of these planes — so those terms integrate to zero. We are left with

\[
\iiint_B 3 \, dV = (3) \, (\text{vol. of unit ball}) = (3) \left( \frac{4}{3} \pi \, (1)^3 \right) = \frac{4\pi}{3}
\]
(c) $\vec{F} = (0, y + 1, 0)$, $S$ is the disk of radius 1 in the plane $y + z = 0$ with center at $(0,0,0)$, and upward normal.

\[ C_{u^2 + 2v^2} \leq 1 \]

\[ \vec{r}(u, v) = (u, v, -v) \]

\[ \vec{r}_u = (1, 0, 0) \]
\[ \vec{r}_v = (0, 1, -1) \]
\[ \vec{N} = (0, 1, 1) \]

\[ \iint_S \vec{F} \cdot \vec{N} \, dS = \iint_{\bar{S}} \vec{F} \cdot \vec{N} \, du \, dv \]

\[ = \iint_{\bar{S}} (y + 1) \, du \, dv \]
\[ = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (v + 1) \, dv \, du \]

\[ = \int_{-1}^{1} 2 \sqrt{1-u^2} \, du \]
\[ = \sqrt{2} \int_{-1}^{1} \sqrt{1-u^2} \, du \]

\[ = (\sqrt{2}) \left( \text{area of half of a unit circle} \right) \]
\[ = (\sqrt{2}) \left( \frac{1}{2} \pi (1)^2 \right) \]
\[ = \frac{\pi \sqrt{2}}{2} \]
3. Let $S$ be the part of the sphere $x^2 + y^2 + (z - 4)^2 = 25$ that is above the $xy$-plane, oriented with the outward pointing normal from the sphere itself. Let $\vec{F} = (ze^y, xe^z, y)$. Determine the value of

\[ \iint_S \vec{F} \cdot \vec{n} \, dS \]

so we would like to apply Gauss' theorem...

Define $D$ to be the disk in the $xy$-plane with the same boundary as $S$, and let $R$ be the region cut off by $S$ and $D$.

Gauss gives us

\[ 0 = \iiint_R \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \cdot \vec{n} \, dS - \iint_D \vec{F} \cdot \vec{n} \, dS \]

So

\[ \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_D \vec{F} \cdot \vec{n} \, dS \]

On $D$, we clearly see $\vec{n} = (0, 0, 1)$, so this becomes

\[ = \iint_D y \, dxdy \]

and this \( = 0 \) by symmetry.
4. Suppose that $\vec{F}(x, y, z) = (P(x, y), Q(x, y), 0)$, and that $C$ is the closed curve parametrized by $\vec{r}(t) = (\cos^3(t), \cos^3(t), \sin^3(t)), \ t \in [0, 2\pi]$. Show that

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

Stokes' theorem tells us

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS$$

where $S$ has boundary $C$. Since we notice that $C$ is entirely in the plane $X=Y$, we can take $S$ to be the surface in that plane bounded by $C$.

Therefore the normal vector $\vec{n}$ of $S$ is just $\vec{n} = \left(\frac{1,-1,0}{\sqrt{2}}\right)$.

We compute $\nabla \times \vec{F}$ as

$$\nabla \times \vec{F} = \left(0-0, 0-0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

So $$(\nabla \times \vec{F}) \cdot \vec{n} = 0$$

$$\Rightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS = 0$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$$