EXAM 3

Math 103, Fall 2004, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT

Good luck!

Name Key________________________

ID number________________________

1. ____________ (/40 points)

2. ____________ (/30 points)

3. ____________ (/10 points)

4. ____________ (/20 points)

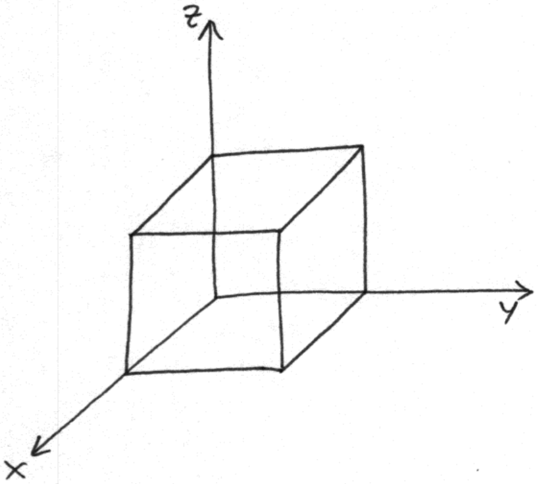
Total ____________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ________________________

1. For each of the items below, write down (but do not evaluate) a *single* explicit triple integral that could be evaluated directly to determine the quantity in question.

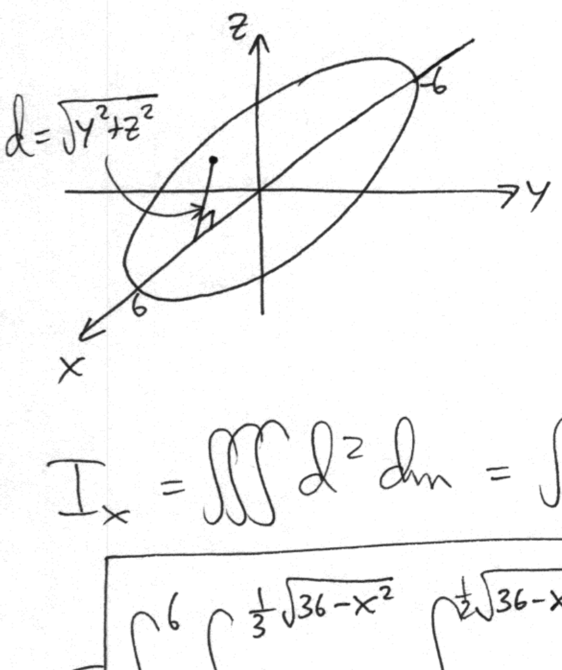
(a) The center of mass of the unit cube, with density given by $\delta(x, y, z) = x + y + z$.



$$\text{c.o.m.} = \iiint \vec{x}\left(\frac{dm}{m}\right) = \frac{1}{m}\iiint \vec{x}\,\delta\,dV$$

$$= \boxed{\frac{1}{m}\int_0^1\int_0^1\int_0^1 \vec{x}\,(x+y+z)\,dx\,dy\,dz}$$

(b) The moment of inertia about the x-axis of the region defined by $x^2 + 4y^2 + 9z^2 \leq 36$, where the density is given by $\delta(x, y, z) = z^2$. (For this question, use the differentials in the order $dy\,dz\,dx$.)



Slice dx first: $x \in [-6, 6]$

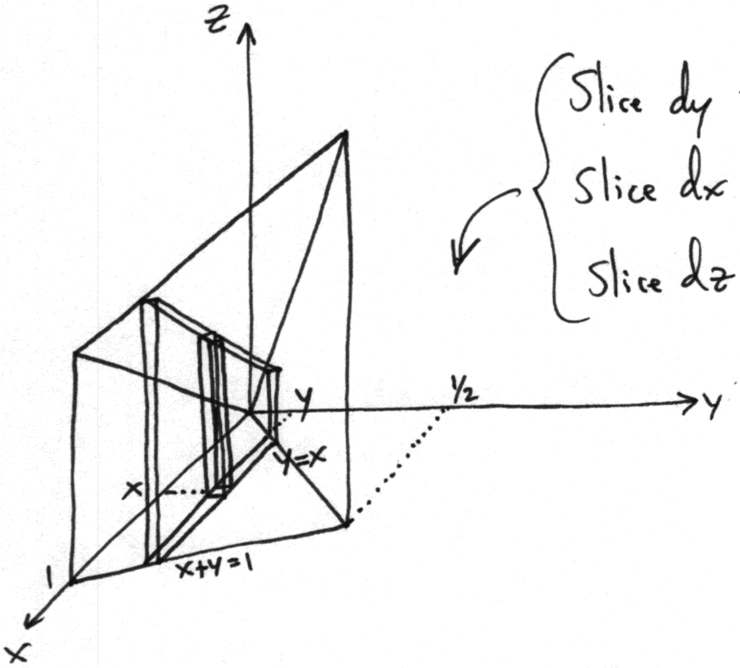
Slice dz second: $z \in \left[\frac{-1}{3}\sqrt{36-x^2},\ \frac{1}{3}\sqrt{36-x^2}\right]$

Slice dy third: $y \in \left[\frac{-1}{2}\sqrt{36-x^2-9z^2},\ \frac{1}{2}\sqrt{36-x^2-9z^2}\right]$

$$I_x = \iiint d^2\,dm = \iiint (y^2+z^2)\,\delta\,dV$$

$$= \boxed{\int_{-6}^{6}\int_{-\frac{1}{3}\sqrt{36-x^2}}^{\frac{1}{3}\sqrt{36-x^2}}\int_{-\frac{1}{2}\sqrt{36-x^2-9z^2}}^{\frac{1}{2}\sqrt{36-x^2-9z^2}} (y^2+z^2)(z^2)\,dy\,dz\,dx}$$

(c) The mass of the region bounded by the planes $y = 0$, $y = x$, $x + y = 1$, $z = x + 2y$, and $z = 0$, with density given by $\delta(x, y, z) = e^z$. *(Hint: For this question, there is only one ordering of the differentials that will allow this to be represented by a single integral.)*

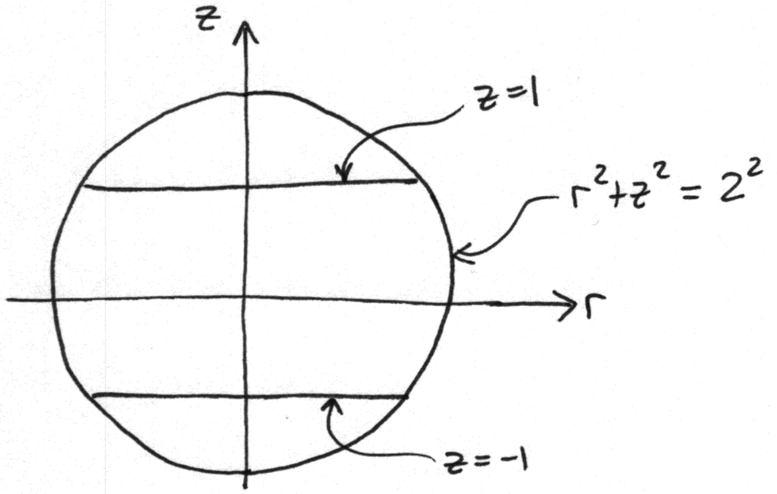


$$\begin{cases} \text{Slice } dy \text{ first: } & y \in [0, \tfrac{1}{2}] \\ \text{Slice } dx \text{ second: } & x \in [y, 1-y] \\ \text{Slice } dz \text{ third: } & z \in [0, x+2y] \end{cases}$$

$$m = \iiint dm = \iiint \delta \, dV = \iiint \delta \, dz \, dx \, dy$$

$$= \boxed{\int_0^{1/2} \int_y^{1-y} \int_0^{x+2y} e^z \, dz \, dx \, dy}$$

(d) The volume of the region inside the sphere $\rho = 2$ and between the planes $z = 1$ and $z = -1$. (For this question, you may choose any coordinate system EXCEPT for rectangular.)



Use cylindrical coordinates:

Slice dz first: $z \in [-1,1]$

Slice dr second: $r \in [-\sqrt{4-z^2}, \sqrt{4-z^2}]$

Slice dθ third: $\theta \in [0, 2\pi]$

$$V = \iiint dV = \iiint r \, d\theta \, dr \, dz$$

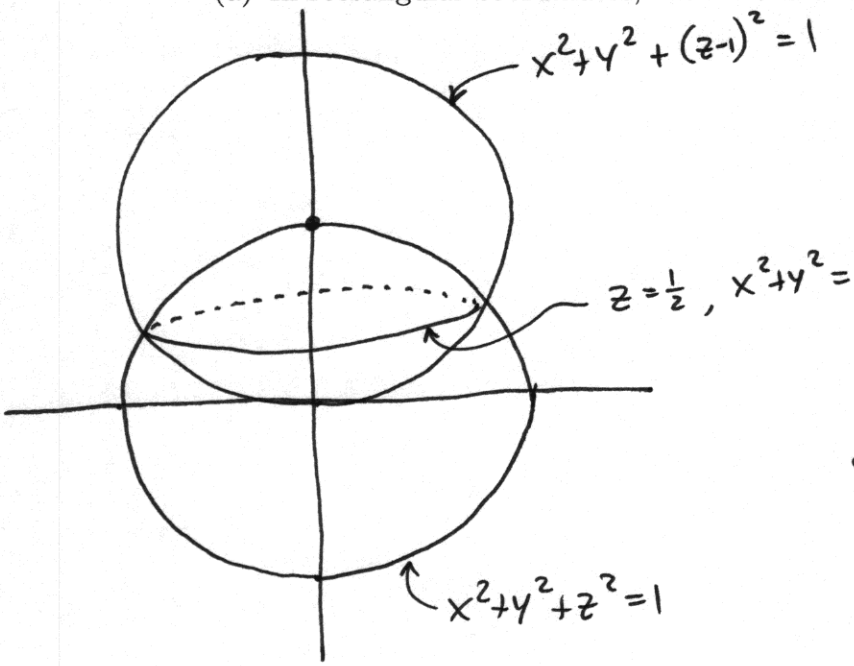
$$= \boxed{\int_{-1}^{1} \int_{0}^{\sqrt{4-z^2}} \int_{0}^{2\pi} r \, d\theta \, dr \, dz}$$

2. For each of the items below, write (but do not evaluate) integrals satisfying the given description that represent the same quantity as

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2}\int_{-\sqrt{3/4-x^2}}^{\sqrt{3/4-x^2}}\int_{1-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx$$

(Hint: The domain of the above integral is the intersection of two solid spheres, each with radius 1, and with centers at $(0,0,0)$ *and* $(0,0,1)$*.)*

(a) In rectangular coordinates, with the differentials in the order $dz\,dx\,dy$.



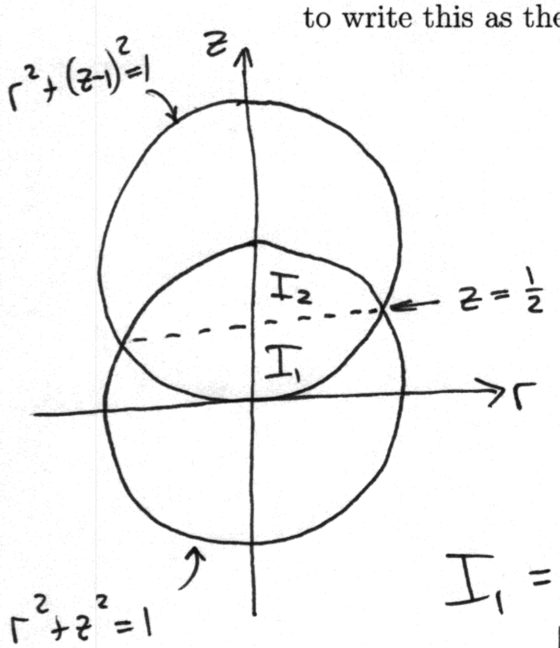
Slice dy 1st: $y \in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$

Slice dx 2nd: $x \in \left[-\sqrt{\frac{3}{4}-y^2}, \sqrt{\frac{3}{4}-y^2}\right]$

Slice dz 3rd: $z \in \left[1-\sqrt{1-x^2-y^2}, \sqrt{1-x^2-y^2}\right]$

$$\boxed{\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\int_{-\sqrt{\frac{3}{4}-y^2}}^{\sqrt{\frac{3}{4}-y^2}}\int_{1-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy}$$

(b) In cylindrical coordinates, with the differentials in the order $dr\,d\theta\,dz$. (You will have to write this as the sum of two integrals.)



$$\iiint_R = \underbrace{\int_0^{1/2} dz}_{I_1} + \underbrace{\int_{1/2}^{1} dz}_{I_2}$$

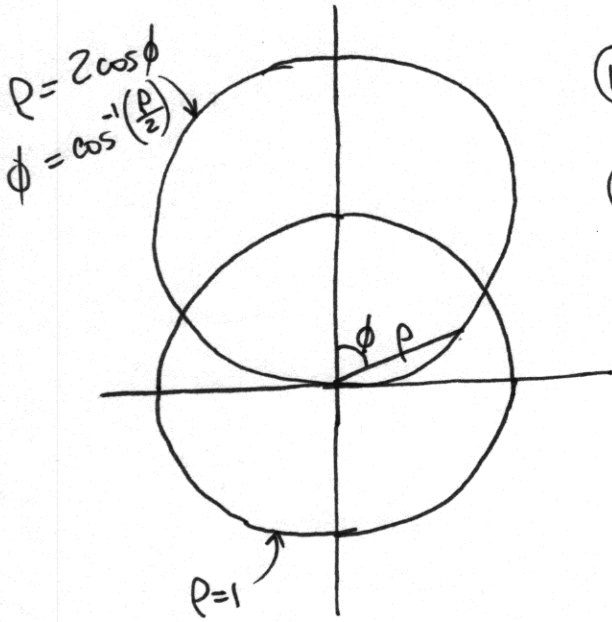
$$I_1 = \iiint f\,dV = \iiint f\,(r\,dr\,d\theta\,dz)$$

$$= \boxed{\int_0^{1/2}\int_0^{2\pi}\int_0^{\sqrt{1-(z-1)^2}} f(x,y,z)\,r\,dr\,d\theta\,dz}$$

$$I_2 = \iiint f\,dV = \iiint f\,(r\,dr\,d\theta\,dz)$$

$$= \boxed{\int_{1/2}^{1}\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} f(x,y,z)\,r\,dr\,d\theta\,dz}$$

(c) In spherical coordinates, with the differentials in the order $d\theta\, d\phi\, d\rho$. *(Hint: Recall that the spherical equation of the sphere of radius 1 through the origin with center on the z-axis is $\rho = 2\cos\phi$.)*



① Slice $d\rho$: $\rho \in [0,1]$

② Slice $d\phi$: $\phi \in [0, \arccos(\rho/2)]$

③ Slice $d\theta$: $\theta \in [0, 2\pi]$

$$\iiint f\, dV = \iiint f\, \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho$$

$$= \boxed{\int_0^1 \int_0^{\arccos(\rho/2)} \int_0^{2\pi} f(x,y,z)\, \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho}$$

3. The parallelopiped with one vertex at the origin defined by the three vectors

$$(a, b, c), \quad (d, e, f), \quad (g, h, i)$$

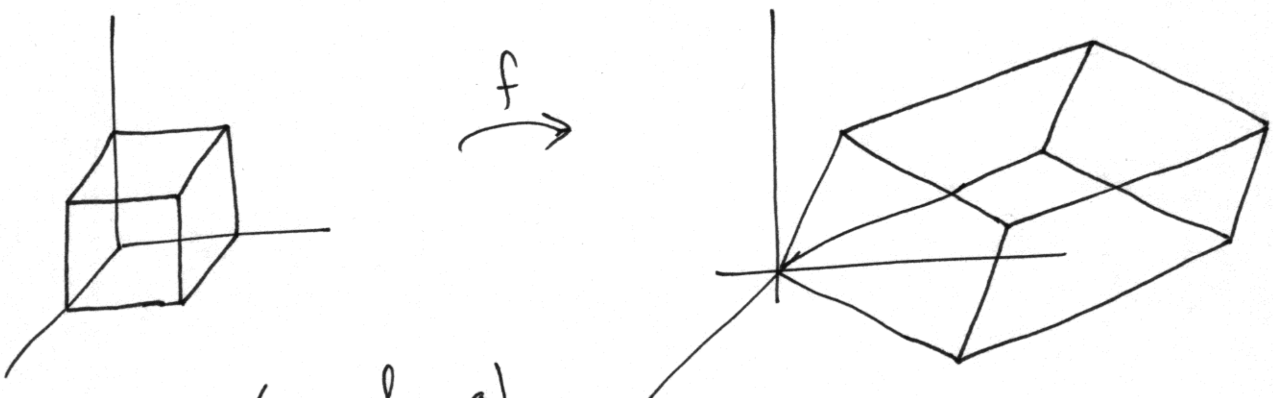
is the image of the unit cube by the function

$$f(u, v, w) = (au + dv + gw, bu + ev + hw, cu + fv + iw)$$

Use this information to compute the mass of the parallelopiped defined by the vectors

$$(2, 1, 0), \quad (0, 1, 2), \quad (-1, 0, 0)$$

with density given by $\delta(x, y, z) = z$.



$$J_f = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\det J_f = 2(0) - 0(0) + (-1)(2) = -2$$

$$|\det J_f| = 2$$

$$m = \iiint dm = \iiint \delta \, dV = \iiint z \, dx\,dy\,dz$$

$$= \iiint z(2)\, du\,dv\,dw = \int_0^1\int_0^1\int_0^1 2(cu + fv + iw)\, du\,dv\,dw$$

$$= \int_0^1\int_0^1\int_0^1 4v \, du\,dv\,dw = \int_0^1\int_0^1 4v\, dv\,dw = \int_0^1 2\, dw = \boxed{2}$$

4. Consider the torus parametrized by

$$\begin{aligned} x &= (3 + 2\cos u)\cos v \\ y &= (3 + 2\cos u)\sin v \\ z &= 2\sin u \end{aligned}$$

(a) Write down (but do not evaluate) an integral representing the surface area of this torus. *(Hint: If two vectors $\overrightarrow{p}$ and $\overrightarrow{q}$ are orthogonal, then $\|\overrightarrow{p} \times \overrightarrow{q}\| = \|\overrightarrow{p}\|\|\overrightarrow{q}\|$.)*

$\vec{r}_u = (-2\sin u \cos v,\ -2\sin u \sin v,\ 2\cos u)$

$\vec{r}_v = (-(3+2\cos u)\sin v,\ (3+2\cos u)\cos v,\ 0)$

Note: $\vec{r}_u \cdot \vec{r}_v = 0 \ldots$ So $\|\vec{r}_u \times \vec{r}_v\| = \|\vec{r}_u\|\,\|\vec{r}_v\|$

$\|\vec{r}_u\| = \sqrt{4\sin^2 u \cos^2 v + 4\sin^2 u \sin^2 v + 4\cos^2 u}$

$= 2$

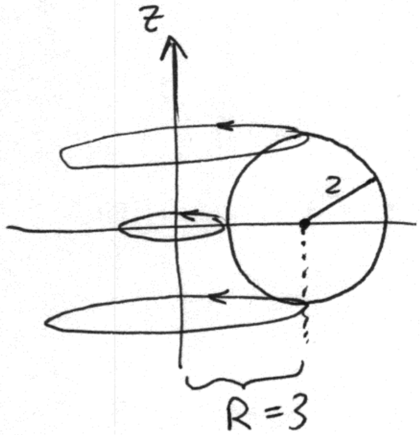
$\|\vec{r}_v\| = \sqrt{(3+2\cos u)^2 \sin^2 v + (3+2\cos u)^2 \cos^2 v}$

$= 3 + 2\cos u$

$S = \iint dS = \iint \|\vec{r}_u \times \vec{r}_v\|\, dA = \iint \|\vec{r}_u\|\,\|\vec{r}_v\|\, du\, dv$

$= \boxed{\int_0^{2\pi}\int_0^{2\pi} (6 + 4\cos u)\, du\, dv}$

(b) Without computing the integral above or any other integral, determine the surface area of the given torus.



By Pappus Thm,

$$S = (\text{circumfrence}) \cdot (\text{dist. trav. by centroid})$$

$$= (4\pi) \cdot (6\pi)$$

$$= \boxed{24\pi^2}$$