

EXAM 2

Math 103, Fall 2004, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name Key

ID number _____

1. _____ (/20 points)

2. _____ (/40 points)

“I have adhered to the Duke Community Standard in completing this examination.”

3. _____ (/20 points)

Signature: _____

4. _____ (/20 points)

Total _____ (/100 points)

1. Determine the values of the following limits, if they exist, making sure to justify your answers:

(a)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + xy}{x^2 + y^2}$$

$$\text{Along } x=0 : \lim \frac{x^2 + xy}{x^2 + y^2} = \lim \frac{0}{x^2 + y^2} = 0$$

$$\text{Along } y=0 : \lim \frac{x^2 + xy}{x^2 + y^2} = \lim \frac{x^2}{x^2} = \lim 1 = 1$$

Because these values are not equal, the given limit cannot exist.

(b)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4 + y^2}$$

Along any line $y=mx$:

$$\begin{aligned} \lim \frac{2x^2y}{x^4 + y^2} &= \lim \frac{2mx^3}{x^4 + m^2x^2} \leq \lim \frac{2mx^3}{m^2x^2} \\ &\leq \lim \frac{2x}{m} = 0 \end{aligned}$$

However — along the parabola $y=x^2$,

$$\lim \frac{2x^2y}{x^4 + y^2} = \lim \frac{2x^2x^2}{x^4 + (x^2)^2} = \lim \frac{2x^4}{2x^4} = \lim 1 = 1$$

Because these values are not equal, the given limit cannot exist.

2. Let $f(x, y, z) = x^2y + y^2z^3$, and $\vec{p}(t) = (t^2, t^3 + 1, t^6 - 2)$

(a) Compute ∇f .

$$\begin{aligned}\nabla f &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \\ &= (2xy, x^2 + 2yz^3, 3y^2z^2)\end{aligned}$$

(b) Compute $D_{\vec{u}}f(\vec{a})$ with $\vec{u} = (2, 3, 6)/7$ and $\vec{a} = (1, 2, -1)$.

f is a polynomial and thus is continuously differentiable; so

$$\begin{aligned}D_{\vec{u}}f(\vec{a}) &= \nabla f(\vec{a}) \cdot \vec{u} \\ &= (2 \cdot 1 \cdot 2, 1^2 + 2 \cdot 2 \cdot (-1)^3, 3 \cdot 2^2 \cdot (-1)^2) \cdot (2, 3, 6)/7 \\ &= (4, -3, 12) \cdot (2, 3, 6)/7 \\ &= \frac{8 + (-9) + 72}{7} \\ &= \frac{71}{7}\end{aligned}$$

(c) Compute $\vec{p}(1)$ and $\vec{p}'(1)$.

$$\vec{p}(1) = (1^2, 1^3+1, 1^6-2) = (1, 2, -1)$$

$$\vec{p}'(t) = (2t, 3t^2, 6t^5)$$

$$\vec{p}'(1) = (2, 3, 6)$$

(d) Use the chain rule to compute $\frac{d}{dt} f(\vec{p}(t))|_{t=1}$ (without referring to a directional derivative); then explain what your answer has to do with the directional derivative computed in part (b).

$$\begin{aligned}\frac{d}{dt} f(\vec{p}(t)) &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \\ &= \nabla f(\vec{p}(t)) \cdot \vec{p}'(t)\end{aligned}$$

$$\begin{aligned}\left. \frac{d}{dt} f(\vec{p}(t)) \right|_{t=1} &= \nabla f(\vec{p}(1)) \cdot \vec{p}'(1) \\ &= \nabla f(1, 2, -1) \cdot (2, 3, 6) \\ &= (4, -3, 12) \cdot (2, 3, 6) = 71\end{aligned}$$

Noting that $\vec{p}(1) = \vec{a}$, $\vec{p}'(1) = 7\vec{u}$, we could have computed $\frac{df}{dt}$ by observing $\frac{df}{dt} = \frac{df}{ds} \cdot \frac{ds}{dt} = D_{\vec{u}} f(\vec{a}) \cdot \|\vec{p}'(1)\|$

$$= \left(\frac{71}{7}\right) \cdot (7) = 71$$

Using the chain rule as indicated, we avoid having to compute $\|\vec{p}'(1)\|$ and then divide and multiply by it.

3. (a) Find a normal vector to the graph of the function

$$z = f(x, y) = x^4y^2 - y^3 + x$$

at the origin. (Hint: This graph can be thought of as a level set of some function $g : \mathbb{R}^3 \rightarrow \mathbb{R}^1$.)

Defining $g(x, y, z) = z - f(x, y) = z - x^4y^2 + y^3 - x$, we note the graph of f is the level set $g=0$ of g . And of course ∇g is \perp to any level set of g .

$$\nabla g = (-4x^3y^2 - 1, -2x^4y + 3y^2, 1)$$

$$\nabla g(0, 0, 0) = (-1, 0, 1)$$

So $(-1, 0, 1)$ is \perp the graph of f at $(0, 0, 0)$

- (b) What unit vector \vec{u} maximizes the directional derivative of f at the origin?

The unit vector $\vec{\mu}$ maximizing $D_{\vec{u}} f(\vec{0})$ is

$$\vec{\mu} = \frac{\nabla f(\vec{0})}{\|\nabla f(\vec{0})\|}$$

We have

$$\nabla f = (4x^3y^2 + 1, 2x^4y - 3y^2)$$

$$\nabla f(\vec{0}) = (1, 0)$$

$$\|\nabla f(\vec{0})\| = 1$$

$$\text{So } \vec{\mu} = \frac{(1, 0)}{1} = (1, 0)$$

4. Let D be the collection of points in \mathbb{R}^3 satisfying

$$4x^2 + 2y^2 + 5z^2 - 6xz + 4yz \leq \frac{3}{4}$$

What is the "highest" (greatest value of z) point in this set?

We want to maximize the function $f(x, y, z) = z$
on the set $g(x, y, z) = 4x^2 + 2y^2 + 5z^2 - 6xz + 4yz \leq \frac{3}{4}$

Interior: We need $\nabla f = \vec{0}$... but $\nabla f = (0, 0, 1) \neq \vec{0}$

So there are no critical points in the interior.

Boundary: Need $\nabla f = \lambda \nabla g$

$$\nabla f = (0, 0, 1)$$

$$\nabla g = (8x - 6z, 4y + 4z, 10z - 6x + 4y)$$

$$0 = \lambda \cdot (8x - 6z)$$

$$0 = \lambda \cdot (4y + 4z)$$

$$1 = \lambda \cdot (10z - 6x + 4y)$$

For a solution to exist, the third equation $\Rightarrow \lambda \neq 0$

Thus the first two equations become

$$\begin{aligned} 0 &= 8x - 6z \\ 0 &= 4y + 4z \end{aligned} \quad \left\{ \Rightarrow \begin{array}{l} x = \frac{3}{4}z \\ y = -z \end{array} \right.$$

(next page...)

We also know that, on the boundary,

$$4x^2 + 2y^2 + 5z^2 - 6xz + 4yz = \frac{3}{4}$$

Plugging in for x and y , we get

$$4\left(\frac{3}{4}z\right)^2 + 2(-z)^2 + 5z^2 - 6\left(\frac{3}{4}z\right)z + 4(-z)z = \frac{3}{4}$$

$$\frac{9}{4}z^2 + 2z^2 + 5z^2 - \frac{9}{2}z^2 - 4z^2 = \frac{3}{4}$$

$$\frac{3}{4}z^2 = \frac{3}{4}$$

$$z^2 = 1$$

$$z = \pm 1$$

This gives us two candidates to consider,

$$\left(-\frac{3}{4}, 1, -1\right) \text{ and } \left(\frac{3}{4}, -1, 1\right)$$

The first is the lowest point; the second is the highest point.