EXAM I

Math 103, Fall 2004, Clark Bray.

You have 50 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING
TO RECEIVE CREDIT

Good luck!

Name KEY

ID number____________________

1. __________ (/40 points)

2. __________ (/30 points)

3. __________ (/30 points)

Total __________ (/100 points)

"I have adhered to the Duke Community Standard in completing this examination."

Signature: ____________________

1. Consider the following three vectors:

$$\vec{u} = < 3, 4, 12 > \quad \vec{v} = < 2, 3, 6 > \quad \vec{w} = < 3, 6, 6 >$$

(a) Find the magnitude of each of these vectors.

$$\|\vec{u}\| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$$

$$\|\vec{v}\| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$$

$$\|\vec{w}\| = \sqrt{3^2 + 6^2 + 6^2} = \sqrt{81} = 9$$

(b) Find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$3 \cdot 2 + 4 \cdot 3 \quad 12 \cdot 6 = 13 \cdot 7 \cdot \cos\theta$$

$$\cos\theta = \frac{90}{91}$$

$$\theta = \arccos\left(\frac{90}{91}\right)$$

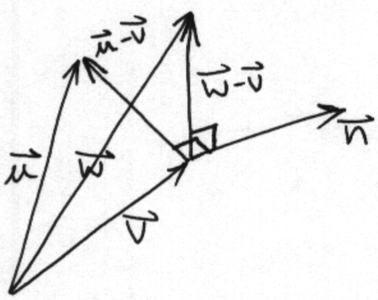
(c) Find the volume of the unique parallelepiped that is defined by $\vec{u}$, $\vec{v}$; and $\vec{w}$.

$$V = \left| \begin{vmatrix} 3 & 4 & 12 \\ 2 & 3 & 6 \\ 3 & 6 & 6 \end{vmatrix} \right| = \left| 3 \begin{vmatrix} 3 & 6 \\ 6 & 6 \end{vmatrix} - 4 \begin{vmatrix} 2 & 6 \\ 3 & 6 \end{vmatrix} + 12 \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} \right|$$

$$= \left| 3(-18) - 4(-6) + 12(3) \right| = |6| = 6$$

$$\Longrightarrow V = 6$$

(d) Find the equation of the unique plane that passes through the points $\vec{u}$, $\vec{v}$, and $\vec{w}$.



$$\vec{u} - \vec{v} = \langle 1,1,6 \rangle \qquad \vec{w} - \vec{v} = \langle 1,3,0 \rangle$$

$$\vec{n} = (\vec{u}-\vec{v}) \times (\vec{w}-\vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 6 \\ 1 & 3 & 0 \end{vmatrix}$$

$$\vec{n} = \langle -18, 6, 2 \rangle$$

Eqn: $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{v}$

$$-18x + 6y + 2z = \langle -18, 6, 2 \rangle \cdot \langle 2, 3, 6 \rangle$$

$$= -6$$

$$\boxed{-18x + 6y + 2z = -6}$$

$$\boxed{9x - 3y - z = 3}$$

2. Consider the curve parametrized by

$$< t, t^3 - t > \quad t \in [-2, 2]$$

(a) Show the process of rewriting the expression

$$s = \int ds$$

to arrive at an integral, in terms of t, that represents the length of this curve.

$$S = \int ds = \int \sqrt{dx^2 + dy^2} = \int_{-2}^{2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

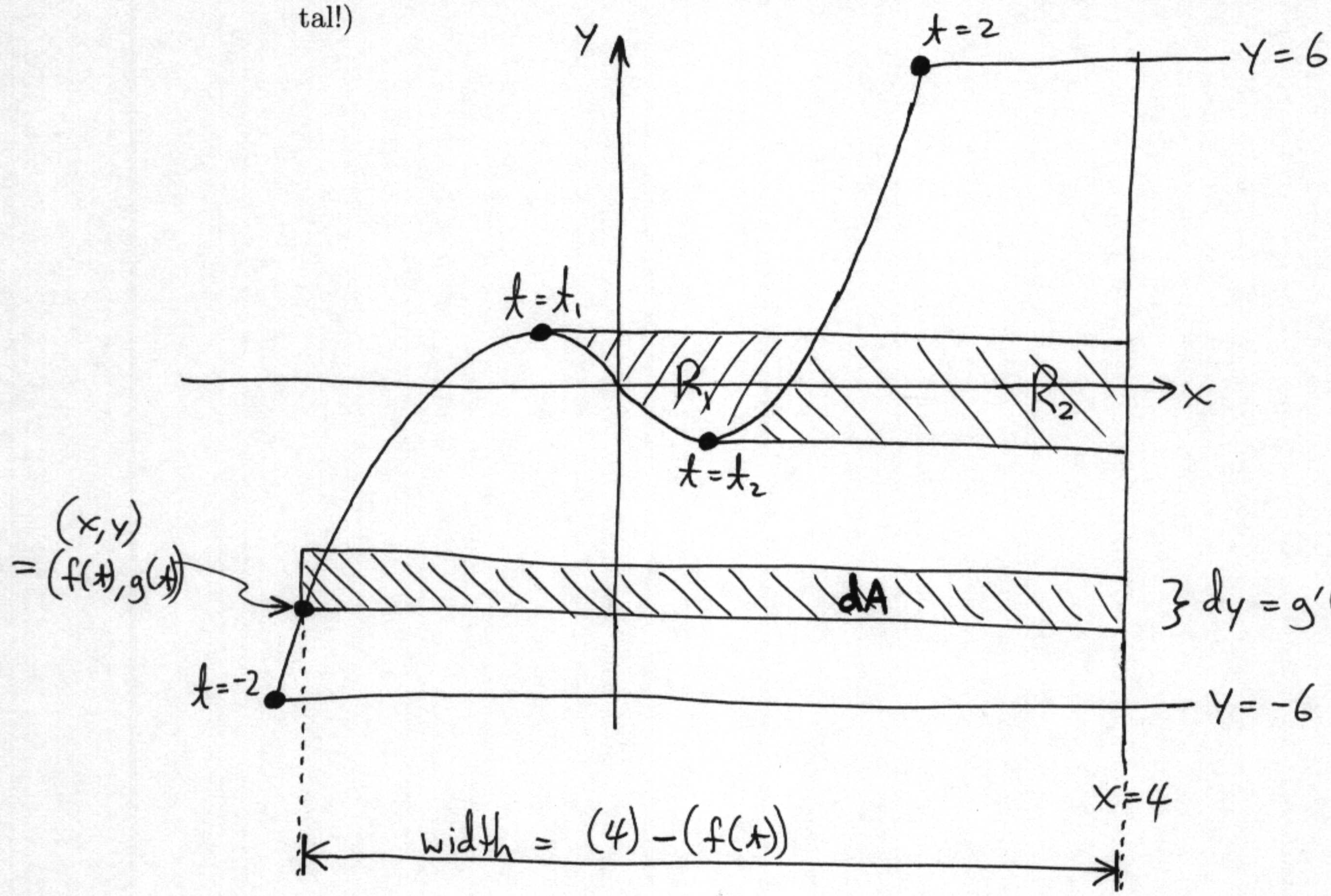
$$= \int_{-2}^{2} \sqrt{(x')^2 + (y')^2}\, dt = \int_{-2}^{2} \sqrt{(1)^2 + (3t^2-1)^2}\, dt$$

$$= \boxed{\int_{-2}^{2} \sqrt{9t^4 - 6t^2 + 2}\, dt}$$

(b) Write down a single integral, in terms of t, that represents the area between this curve and the lines

$$x = 4 \quad y = -6 \quad y = 6$$

Make sure to explain how you know that your integral represents the desired area. (Note: In order to get a single integral, your "slices" of area will have to be horizontal!)



We naively write $A = \int dA = \int (4 - f(t))\, dy = \int_{-2}^{2} (4 - f(t))\, g'(t)\,dt$

Of course, these slices of area overlap, and protrude outside of the desired area... But we get cancellations!

$\int_{-2}^{t_1}$ includes R_1, <u>positively</u> (since $g' > 0$)
$\int_{t_1}^{t_2}$ includes R_1, <u>negatively</u> (since $g' < 0$)
} these cancel; so in total, R_1 is not counted.

Similarly, R_2 is counted in $\int_{-2}^{t_1}$, $\int_{t_1}^{t_2}$, $\int_{t_2}^{2}$ — pos'ly, neg'ly, pos'ly (resp.) so in total, R_2 is counted pos'ly once.

3. The Enterprise is damaged, and its position sensors are not working; however, the computer still has complete control over the amount of force the engines put out.

(a) Spock proposes firing the engines in order to realize a (time-dependent) acceleration vector of

$$a(t) = < 6t, 6t - 24t^2, 4 >$$

over the time interval $[0, 10]$.

Assuming that the ship's initial position and velocity are the zero vectors, find an expression for the ship's position as a function of time.

$$\vec{v}' = < 6t, 6t - 24t^2, 4 >$$

$$\vec{v} = < 3t^2, 3t^2 - 8t^3, 4t > + \vec{c}_1$$

$$\vec{0} = \vec{v}(0) = < 0, 0, 0 > + \vec{c}_1 \implies \vec{c}_1 = \vec{0}$$

$$\vec{r}' = \vec{v} = < 3t^2, 3t^2 - 8t^3, 4t >$$

$$\vec{r} = < t^3, t^3 - 2t^4, 2t^2 > + \vec{c}_2$$

$$\vec{0} = \vec{r}(0) = < 0, 0, 0 > + \vec{c}_2 \implies \vec{c}_2 = \vec{0}$$

$$\boxed{\vec{r} = < t^3, t^3 - 2t^4, 2t^2 >}$$

(b) Following this path over the given time interval, will the ship cross the boundary of the Neutral Zone, with equation defined by

$$f(\vec{r}) = -10x - y + 10z = 5000$$

(Hint: What are the starting and ending values for $-10x - y + 10z$ as the ship moves from $t = 0$ to $t = 10$?)

$$f(\vec{r}(0)) = f(\vec{0}) = 0$$

$$f(\vec{r}(10)) = f(< 1{,}000, -19{,}000, 200 >) = -10{,}000 + 19{,}000 + 2{,}000$$

$$= 11{,}000$$

Since $f(\vec{r}(t))$ is continuous and $0 < 5{,}000 < 11{,}000$, the Enterprise must have crossed the boundary of the neutral zone.

(c) At the time $t = 1$, what are the normal and tangential components of the acceleration? What is the curvature of the path at that point?

$$a_T(1) = \frac{\vec{v}(1) \cdot \vec{a}(1)}{v(1)} = \frac{\langle 3,-5,4\rangle \cdot \langle 6,-18,4\rangle}{5\sqrt{2}}$$

$$= \frac{124}{5\sqrt{2}}$$

$$a_N(1) = \frac{\|\vec{v}(1) \times \vec{a}(1)\|}{v(1)} = \frac{\|\langle 52, 12, -24\rangle\|}{5\sqrt{2}}$$

$$= \frac{\sqrt{52^2 + 12^2 + 24^2}}{5\sqrt{2}} = \frac{\sqrt{2704 + 144 + 576}}{5\sqrt{2}}$$

$$= \frac{\sqrt{1712}}{5}$$

$$\kappa = \frac{a_N}{v^2} = \frac{\sqrt{1712}/5}{50} = \frac{\sqrt{856}}{125}$$