GENERAL RULES

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

No calculators.

All answers must be reasonably simplified.

All of the policies and guidelines on the class webpages are in effect on this exam.

WRITING RULES

Use black pen only. You may use a pencil for initial sketches of diagrams, but the final sketch must be drawn over in black pen and you must wipe all erasure residue from the paper.

DUKE COMMUNITY STANDARD STATEMENT

“I have adhered to the Duke Community Standard in completing this examination.”

Signature: __________________________
1. (20 pts) A solar cell is in the shape of a rectangle with edge vectors \((2, 2, 0)\) and \((1, -1, 1)\), and one corner at the origin. (Of course it is facing toward the sky, not toward the ground.)

(a) Find the equation of the plane containing the solar cell.

\[
\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -4 \end{pmatrix}
\]

Choose \(\hat{n} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\), to point toward sky and to make simpler. Then the equation is

\[
\hat{n} \cdot \vec{x} = \hat{n} \cdot \vec{x}_0, \text{ or } -x + y + 2z = 0.
\]

(b) The “efficiency of the orientation” is the cosine of the angle between the “orientation vector” of the cell (unit vector perpendicular to the cell in the direction it is facing) and the direction of the sun. Compute the efficiency of this cell’s orientation at the moment when the sun is in the direction of \(\theta = \pi/2\) and \(\phi = \pi/4\).

Direction of sun is

\[
\vec{v} = \begin{pmatrix} \sin \phi \cos \theta \\ \sin \phi \sin \theta \\ \cos \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}.
\]

Then the efficiency is

\[
\cos \alpha = \frac{\vec{v} \cdot \hat{n}}{\|\vec{v}\| \|\hat{n}\|} = \frac{3/\sqrt{2}}{1 \cdot \sqrt{6}} = \frac{\sqrt{3}}{2}.
\]
(extra space for questions from other side)
2. (15 pts) Find the spherical equation of the sphere of radius $\sqrt{2}$ centered at $(1, 1, 0)$.

\[
(x-1)^2 + (y-1)^2 + z^2 = (\sqrt{2})^2
\]

\[
x^2 + y^2 + z^2 - 2x + 1 - 2y + 1 = 2
\]

\[
\rho^2 - 2\rho \sin \phi \cos \theta - 2\rho \sin \phi \sin \theta = 0
\]

\[
\rho = 2 \sin \phi (\cos \theta + \sin \theta)
\]

3. (5 pts) Find the rectangular equation of the surface with spherical equation $\rho = \csc \phi \sec \theta$.

\[
\iff \rho \sin \phi \cos \theta = 1
\]

\[
\iff x = 1
\]
4. (5 pts) The function \( m : \mathbb{R}^3 \to \mathbb{R}^1 \) is given by \( m(x, y, z) = 2x + 3y - z \), and the function \( p : \mathbb{R}^2 \to \mathbb{R}^1 \) is given by \( p(x, y) = 2x + 3y \). There is a relationship between these functions involving their graphs and/or level sets; express that relationship in a single brief sentence. (Be especially careful with your words – this question is about the proper use of these terms!)

\[ m^{-1}(0) \text{ has equation } 2x + 3y - z = 0 \]

The graph of \( p \) has equation \( z = 2x + 3y \), which is equivalent.

So: "The graph of \( p \) is a level set of \( m \)."

5. (15 pts) A child standing at \((0, 4)\) in the \( xy \)-plane has a ball attached to a string, and she is swinging the ball over her head counterclockwise in a horizontal circle of radius 3 feet, making one complete revolution per second. At the moment \((t = 0)\) when the ball is at \((3, 4)\), she starts running in the \( x \)-direction with speed 5 feet per second, while continuing to swing the ball. Find the position and the velocity of the ball as functions of time.

\[ \mathbf{C}(t) = \text{child location} \]
\[ \mathbf{S}(t) = \text{string vector} \]
\[ \mathbf{B}(t) = \text{ball location} \]

\[ \mathbf{C}(t) = (5t, 4) \]
\[ \mathbf{S}(t) = (3 \cos k t, 3 \sin k t) \]

\[ \mathbf{B}(t) = \mathbf{C} + \mathbf{S} = (5t, 4) + (3 \cos 2\pi t, 3 \sin 2\pi t) \]

\[ \mathbf{V} = \mathbf{B}' = \begin{pmatrix} 5 - 6\pi \sin 2\pi t \\ 6\pi \cos 2\pi t \end{pmatrix} \]
(extra space for questions from other side)
6. (5 pts) Bob is told that $T$ is a linear transformation, and that $T(\vec{e}_1) = (1, 2, 3)$, $T(\vec{e}_2) = (4, 5, 6)$, and $T(\vec{e}_3) = (7, 8, 9)$. He then computes $T(\vec{e}_1 + \vec{e}_3)$ as

$$
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
4 \\
10 \\
16
\end{pmatrix}
$$

and also as $T(\vec{e}_1) + T(\vec{e}_3) = \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix}$

Can you help Bob resolve this seeming contradiction?

Bob’s matrix is wrong. The images of the standard basis vectors should be the columns of the matrix, not the rows.

7. (15 pts) Reflection over the line $L$ in the $xy$-plane defined by the polar angle $\theta$ can be achieved by first rotating clockwise by the angle $\theta$, then reflecting over the $x$-axis, and then rotating counterclockwise by the angle $\theta$. Use this information to find the matrix representing reflection over the line $L$.

$T_1$, clockwise rotation by $\theta$:

$T_1(\vec{e}_2) = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix}$

$T_1(\vec{e}_1) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

$A_1 = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

$T_2$, reflection over $x$-axis:

$T_2(\vec{e}_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$T_2(\vec{e}_2) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T_3$, cc-wise rotation by $\theta$:

$T_3(\vec{e}_2) = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}$

$T_3(\vec{e}_1) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$

$A_3 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$
\[ A = A_3 A_2 A_1 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]

\[ = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \]

\[ = \begin{pmatrix} \cos^2 \theta - \sin^2 \theta & 2\sin \theta \cos \theta \\ 2\sin \theta \cos \theta & \sin^2 \theta - \cos^2 \theta \end{pmatrix} \]

\[ = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \]
8. (20 pts) We have $f : \mathbb{R}^2 \to \mathbb{R}^1$, and $D_v f(\vec{a}) = 3$, where $\vec{v} = (2, 3)$. We consider here a particle at $\vec{a}$ moving with velocity $\vec{v}$.

(a) Compute $\frac{df}{dt}$, the rate of change of $f$ with respect to time $t$.

$$\frac{df}{dt} = D_v f(\vec{v}) = 3$$

(b) Compute $\frac{df}{ds}$, the rate of change of $f$ with respect to distance traveled $s$.

$$\frac{df}{ds} = \frac{df}{dt} \frac{ds}{dt} = \frac{3}{\|\vec{v}\|} = \frac{3}{\sqrt{13}}$$

(c) Suppose the particle were moving the same direction, but with speed equal to 5; what would be the rate of change of $f$ with respect to time $t$?

$$\frac{df}{dt} = \frac{df}{ds} \frac{ds}{dt} \underbrace{\frac{3}{\sqrt{13}} \cdot 5}_{\text{same as above}} = \frac{15}{\sqrt{13}}$$
(Scratch space. Nothing on this page will be graded!)
(Scratch space. Nothing on this page will be graded!)
(Scratch space. Nothing on this page will be graded!)