EXAM 2
Math 212, 2017 Summer Term 2, Clark Bray.
You have 75 minutes.
No notes, no books, no calculators.
YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING. All answers must be simplified. All of the policies and guidelines on the class webpages are in effect on this exam.

Good luck!

Name ________________________

“I have adhered to the Duke Community Standard in completing this examination.”

Signature: ____________________________

1. __________

2. __________

3. __________

4. __________

5. __________

Total Score _________ (/100 points)
1. (25 pts) The differentiable function \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^1 \) represents temperature (°C) in terms of coordinates \( x \) and \( y \) (representing distances in miles, east and north (respectively) of a given fixed point) in the vicinity of that fixed point. The gradient of \( f \) at the point \((1, 3)\) is the vector \((5, 2)\).

(a) Compute \( \frac{\partial f}{\partial x} \) at \((1, 3)\).

\[
\nabla f(1, 3) = \left( \frac{\partial f}{\partial x}(1, 3), \frac{\partial f}{\partial y}(1, 3) \right) = \left( \frac{\partial f}{\partial x}(1, 3) = 5 \right)
\]

(b) Find the unit vector indicating the direction on the map at \((1, 3)\) in which the temperature is increasing the fastest.

\[
\nabla f \text{ points in the direction of fastest increase.}
\]

So \( \hat{\mu} = \frac{(5, 2)}{||(5, 2)||} = \frac{(5, 2)}{\sqrt{29}} \)

(c) Bob is walking along a trail in this vicinity, and at the moment he comes to the point \((1, 3)\) his velocity is \((2, -3)\) (in units of miles per hour). How fast is he experiencing temperature change at that moment?

\[
\frac{df}{dt} = D_v f(1, 3) = \nabla f(1, 3) \cdot \vec{v} = \left( \frac{5}{2}, \frac{2}{-3} \right) \cdot (2, -3) = 4
\]

(d) What is the rate of change of temperature with respect to distance traveled on the trail that Bob is walking on at \((1, 3)\)?

\[
\dot{w} = \frac{\vec{v}}{||\vec{v}||} = \frac{(2, -3)}{\sqrt{13}}
\]

\[
\frac{df}{ds} = D_w f(1, 3) = \nabla f(1, 3) \cdot \dot{w} = \left( \frac{5}{2}, \frac{2}{-3} \right) \left( \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right) = \frac{4}{\sqrt{13}}
\]

\[
\frac{df}{ds} / \frac{ds}{dt} = \frac{df}{dt} = \frac{4}{\sqrt{13}}
\]

Any of these 3 calculations!

(e) If Bob were to leave the trail at \((1, 3)\) and start moving in the direction in which the temperature is increasing the fastest, with a walking speed of 3, how fast would he experience temperature change at that moment?

\[
\frac{df}{dt} = D (\hat{\mu}) f(1, 3) = 3 \hat{\mu} \cdot \nabla f(1, 3) = 3 \frac{\nabla f(1, 3)}{||\nabla f(1, 3)||} = 3 \frac{5, 2}{\sqrt{29}}
\]

\[
3 \nabla f \cdot \hat{\mu} = 3 \left( \frac{5}{2}, \frac{2}{2} \right) \frac{1}{\sqrt{29}} = 3 \sqrt{29}
\]
2. (20 pts)

(a) Compute directly from the definition the directional derivative of \( g(x, y) = (x^2y, 3y^2) \) at the point (1, 2) with velocity (2, 3).

\[
\nabla = \hat{a} + \hat{v} = \left( \frac{1}{2} \right) + \left( \frac{2}{3} \right) = \left( \frac{1+2k}{2+3k} \right)
\]

\[
D_{(2,3)}g(1,2) = \frac{\partial}{\partial \hat{n}} \bigg|_{\hat{n}=0} g(\hat{a} + \hat{v}) = \frac{\partial}{\partial \hat{n}} \bigg|_{\hat{n}=0} g \left( \frac{1+2k}{2+3k} \right)
\]

\[
= \frac{\partial}{\partial \hat{n}} \bigg|_{\hat{n}=0} \left( \frac{(1+2k)}{3(2+3k)} \right) \left( 2+3k \right)
\]

\[
= \frac{\partial}{\partial \hat{n}} \bigg|_{\hat{n}=0} \left( \frac{(1+4k+14k^2)}{12+36k+27k^2} \right)
\]

\[
= \left( \frac{11}{36} \right)
\]

(b) Compute the same directional derivative using a Jacobian matrix.

\( g \) has all continuous partials, so it is continuously diff. and thus also differentiable.

\[
J_g = \begin{pmatrix} 2xy & x^2 \\ 0 & 6y \end{pmatrix} \quad J_g, (1,2) = \begin{pmatrix} 4 & 1 \\ 0 & 12 \end{pmatrix}
\]

\[
D_{(2,3)}g(1,2) = J_g, (1,2) \cdot (2,3)
\]

\[
= \begin{pmatrix} 4 & 1 \\ 0 & 12 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ 36 \end{pmatrix}
\]
3. (15 pts) The variable $w$ is a twice continuously differentiable function of $x$, $y$, and $z$, which themselves are functions of $\rho$, $\phi$, and $\theta$ by way of spherical coordinates in the usual way.

In a particular application we find ourselves interested in the expression

$$M = w_y \rho \sin \phi$$

Find an expression for $\frac{\partial M}{\partial \rho}$, simplified as much as possible.

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$\frac{\partial M}{\partial \rho} = \frac{\partial}{\partial \rho} (w_y \rho \sin \phi) = \left( \frac{\partial w_y}{\partial \rho} \right) (\rho \sin \phi) + (w_y) \left( \frac{\partial (\rho \sin \phi)}{\partial \rho} \right)$$

$$= \left( \frac{\partial w_y}{\partial x} \frac{\partial x}{\partial \rho} + \frac{\partial w_y}{\partial y} \frac{\partial y}{\partial \rho} + \frac{\partial w_y}{\partial z} \frac{\partial z}{\partial \rho} \right) (\rho \sin \phi) + w_y \sin \phi$$

$$= (w_{xy} \sin \phi \cos \theta + w_{xy} \sin \phi \sin \theta + w_{xz} \cos \phi) (\rho \sin \phi) + w_y \sin \phi$$

$$= w_{xy} \rho \sin^2 \phi \cos \theta + w_{xy} \rho \sin^2 \phi \sin \theta + w_{xz} \rho \sin \phi \cos \phi + w_y \rho \sin \phi$$

$$= \left( w_{xy} x + w_{xy} y + w_{xz} z + w_y \right) (\sin \phi)$$
4. (20 pts) \( R \) is the solid triangle with vertices at \((0,0), (2,0), (-2,4)\). Bacteria are distributed across \( R \) by way of the population density function \( \delta(x,y) = x + y \) (in millions of bacteria per unit area). Use an integral to compute the total number of bacteria in \( R \), and explain explicitly your motivation behind your choice of order of the differentials.

We slice with \( dy \) on the outside to avoid the corner at the origin.

\[
P = \iint_R \delta \, dA
\]

\[
= \int_0^4 \int_{-\frac{y}{2}}^{2-y} (x+y) \, dx \, dy
\]

\[
= \int_0^4 \left( \frac{1}{2} x^2 + xy \right)_{x=-\frac{y}{2}}^{x=2-y} \, dy
\]

\[
= \int_0^4 \left( \frac{1}{2} (2-y)^2 + (2-y)(y) - \frac{1}{2} \left(-\frac{y}{2}\right)^2 - \left(-\frac{y}{2}\right)(y) \right) \, dy
\]

\[
= \int_0^4 \frac{1}{2} \left(4-4y+y^2+2y-y^2 - \frac{1}{8} y^2 + \frac{1}{2} y^2\right) \, dy
\]

\[
= \int_0^4 2 - \frac{1}{8} y^2 \, dy
\]

\[
= \left[ 2y - \frac{1}{24} y^3 \right]_0^4
\]

\[
= 8 - \frac{8}{3} = \frac{16}{3}
\]
5. (20 pts) Express as an iterated integral (but do not evaluate!) the integral of the function $f(x, y, z) = xyz^3$ over the domain bounded by the surfaces $y = x^2 + z^2 - 3xz$ and $y = 1 - 3xz$.

Surfaces intersect when \[ x^2 + z^2 - 3xz = 1 - 3xz \]
\[ x^2 + z^2 = 1 \]

So this projects nicely into the $xz$-plane.

\[
\int_{-1}^{1} \int_{\sqrt{1-x^2}}^{1-3xz} \int_{x^2+z^2-3xz}^{1-3xz} (xyz^3) \, dy \, dx \, dz
\]