EXAM 2
Math 212, 2016 Summer Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.
All answers must be simplified. All of the policies and guidelines on the class webpages are in effect on this exam.

Good luck!

Name ______________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________

“I have adhered to the Duke Community Standard in completing this examination.”

Signature: ____________________

Total Score ____________ (/100 points)
1. (15 pts) Bob is in a field at the location (2, 1), and he knows that the density of flowers (in blooms per square meter) near where he is in the field is given by \( \delta(x, y) = xy + x^2 \).

(a) If he were to walk in the direction of the vector \((3, 4)\), what would be the rate of change per unit distance travelled of the density function \(\delta\)?

\[
\nabla \delta = \left( \frac{y + 2x}{x} \right) \implies \nabla \delta(2, 1) = \left( \frac{5}{2} \right). \quad \vec{v} = \left( \frac{3}{4} \right), \quad \vec{\mu} = \left( \frac{3}{4} \right) / 5
\]

\[
\frac{d\delta}{dx} = D_{\vec{\mu}} \delta(2, 1) = \nabla \delta(2, 1) \cdot \vec{\mu} = \left( \frac{5}{2} \right) \left( \frac{3}{4} \right) / 5 = \frac{23}{5}
\]

(b) Which direction should he walk if he wants \(\delta\) to increase as quickly as possible?

\[
\vec{\mu} = \frac{\nabla \delta}{\|\nabla \delta\|} = \left( \frac{5}{2} \right) / \sqrt{29}
\]

(c) In terms of the rate of change of flower density, by what proportion is the direction from part (b) better than the direction from part (a)?

In (b), \(\frac{d\delta}{dx} = \|\nabla \delta\| = \sqrt{29}\)

\[
\text{So proportion is} \quad \frac{\sqrt{29}}{23/5} = \frac{5\sqrt{29}}{23}
\]
2. (15 pts) The function \( f \) is defined by
\[
f(x, y) = \begin{cases} 
xe^{(y^2/x^2)} & \text{if } x \neq 0 \\
0 & \text{if } x = 0
\end{cases}
\]

(a) Show that \( f \) is directionally differentiable at \((0, 0)\) by finding a general expression for \( D_\overrightarrow{v} f(0, 0) \) in terms of \( \overrightarrow{v} \).
\[
D_\overrightarrow{v} f(0, 0) = \left. \frac{d}{dt} \right|_{t=0} f(\overrightarrow{0} + t\overrightarrow{v}) = \left. \frac{d}{dt} \right|_{t=0} f\left(\frac{tv_1}{tv_2}\right)
\]
valid for all values of \( t \), with \( v_1 \neq 0 \)

\[
v_1 e^{(v_2^2/v_1^2)}
\]

\[
D_{\overrightarrow{0}} f(0, 0) = \left. \frac{d}{dt} \right|_{t=0} f(\overrightarrow{0} + t\overrightarrow{v}) = \left. \frac{d}{dt} \right|_{t=0} f\left(\frac{0}{tv_2}\right)
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} (0) = 0
\]

(b) Is \( f \) differentiable at \((0, 0)\)? Explain your reasoning.

Using (a):
\[
D_{(1,1)} f(0, 0) = 1 \cdot e^{(1^2/1^2)} = e
\]
\[
D_{(1,-1)} f(0, 0) = 1 \cdot e^{(-1^2/-1^2)} = e
\]
\[
D_{(2,0)} f(0, 0) = 2 \cdot e^{(0^2/2^2)} = 2
\]

\( f \) is not directional linear because \( D_{(1,1)} + D_{(1,-1)} \neq D_{(2,0)} \).

Therefore \( f \) cannot be differentiable at \((0,0)\).
3. (15 pts) The following is given about various functions and variables:

\[ f(r, s, t) = (x, y, z) \quad g(x, y, z) = (a, b, c) \quad f(1, 5, 3) = (2, 4, 6) \]

\[
J_f(1,5,3) = \begin{pmatrix}
1 & 5 & -1 \\
-1 & 1 & 3 \\
0 & 0 & 2 \\
\end{pmatrix} \quad J_g(2,4,6) = \begin{pmatrix}
2 & -1 & 3 \\
3 & 6 & -1 \\
4 & -2 & 2 \\
\end{pmatrix}
\]

Suppose that the variable \( c \) is set constant. Does this allow for interpreting \( t \) as a function of \( r \) and \( s \) near the point \((r, s, t) = (1, 5, 3)\)?

\[
f: \begin{cases}
{r} & \rightarrow x \\
{s} & \rightarrow y \\
{t} & \rightarrow c
\end{cases}
\]

\[
C(r, s, t) = \text{constant}, \quad \text{check } \frac{\partial c}{\partial x}.
\]

\[
\begin{pmatrix}
2 & -1 & 3 \\
3 & 6 & -1 \\
4 & -2 & 2 \\
\end{pmatrix} \begin{pmatrix}
1 & 5 & -1 \\
-1 & 1 & 3 \\
0 & 0 & 2 \\
\end{pmatrix} = \begin{pmatrix}
\ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
\ddots & \ddots & \ddots \\
\end{pmatrix} \quad \Rightarrow \frac{\partial c}{\partial x} = -6 \neq 0.
\]

\[
\Rightarrow f \text{ is locally a fn of } r, s.
\]

4. (15 pts) The region \( R \) in the \( xy \)-plane is bounded by the \( x \)-axis and the curves \( y = x^3 \) and \( x + y = 2 \). Compute the integral over \( R \) of the function \( f(x, y) = x + y \).

\[
\int_0^1 \int_{3\sqrt{y}}^{2-y} x + y \, dx \, dy
\]

\[
= \int_0^1 \left[ \frac{1}{2} x^2 + xy \right]_{x = 2-y}^{x = 3\sqrt{y}} \, dy
\]

\[
= \int_0^1 \left( \frac{1}{2} (2-y)^2 + (2-y)y \right) - \left( \frac{1}{2} y^{2/3} + y^{4/3} \right) \, dy
\]

\[
= \int_0^1 -\frac{1}{2} y^2 + 2 - \frac{1}{2} y^{2/3} - y^{4/3} \, dy
\]

\[
= \left[ -\frac{1}{6} y^3 + 2y - \frac{3}{10} y^{5/3} - \frac{3}{7} y^{7/3} \right]_0^1 = \frac{-70 + 240 - 126 - 180}{420} = \frac{464}{420} = \frac{116}{105}
\]
5. (15 pts) The domain $D$ in $\mathbb{R}^3$ is described by $x^2 + y^2 + z^2 \leq 4$, $z \leq 1 - x^2$, and $z \geq 0$. Set up, but do not evaluate, a triple nested integral in rectangular coordinates that represents the $y$-coordinate of the centroid of $D$. (You may leave the volume of this domain $D$ as $V$.)

\[
\overline{y} = \frac{1}{V} \iiint_D y \, dV
\]

\[
= \frac{1}{V} \int_{-1}^{1} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-z^2}}^{\sqrt{4-x^2-z^2}} y \, dy \, dz \, dx
\]

[Diagram of the domain $D$ with projection to the $xz$-plane and a formula for $z = 1 - x^2$.]

6. (10 pts) Your friend Bob says that the value of the integral from question 5. (above) is zero, by symmetry. Is he right or wrong? Explain fully.

- $D$ is symmetric through $xz$-plane.
- Reflection through $xz$-plane is $R(x,y,z) = (x, -y, z)$
- The integrand $f(x,y,z) = y$ is odd through this plane because
  \[
f(R(x,y,z)) = f(x,-y,z) = -y = -f(x,y,z)
\]
- So Bob is right, integral is zero by symmetry.
7. (15 pts) The sphere $S$ has radius 1 and center at $(0, 0, 1)$; the cone $C$ has equation $z = \sqrt{x^2 + y^2}$. The domain $T$ in $\mathbb{R}^3$ is the region inside of $S$ and above $C$. Set up, but do not evaluate, a triple nested integral in spherical coordinates representing the mass in $T$ given that the density in that region is given by $\delta = \delta(x, y, z)$.

\[ z = \sqrt{x^2 + y^2} = \rho \]
\[ x^2 + y^2 + (z-1)^2 = 1 \]
\[ \rho^2 - 2z + 1 = 1 \]
\[ \rho^2 = 2(\rho \cos \phi) \]
\[ \rho = 2 \cos \phi \]

\[ M = \iiint_T \delta \, dV \]
\[ = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{2 \cos \phi} \delta(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \]