EXAM 2
Math 103, Summer 2009 Term 2, Clark Bray.

You have 75 minutes.

No notes, no books, no calculators.

YOU MUST SHOW ALL WORK AND EXPLAIN ALL REASONING TO RECEIVE CREDIT. CLARITY WILL BE CONSIDERED IN GRADING.

Good luck!

Name: Solutions

ID number:

1. _________

2. _________

3. _________

4. _________

5. _________

6. _________

7. _________

8. _________

9. _________

Total Score _________ (/100 points)

10. _________

"I have adhered to the Duke Community Standard in completing this examination."

Signature: __________________________
1. \(10 \text{ pts}\) Compute the acceleration vector for the parametric curve defined by \(\vec{x}(t) = (\cos(4t), e^{t^2}, t^2e^t)\).

\[
\vec{x} = \begin{pmatrix}
\cos(4t) \\
e^{t^2} \\
t^2e^t
\end{pmatrix}, \quad \vec{x}' = \begin{pmatrix}
-4\sin(4t) \\
2t^2e^{t^2} \\
2te^{t^2} + t^2e^t
\end{pmatrix}, \quad \vec{x}'' = \begin{pmatrix}
-16\cos(4t) \\
(2t^2e^{t^2}) + (2t)(2te^{t^2}) \\
(2t)(e^{t^2}) + (2t + t^2)(e^t)
\end{pmatrix}
\]

\[
\frac{\vec{a}}{a} = \frac{\vec{x}''}{\vec{x}'} = \begin{pmatrix}
-16\cos(4t) \\
(2t^2e^{t^2}) + (2t)(2te^{t^2}) \\
(2t)(e^{t^2}) + (2t + t^2)(e^t)
\end{pmatrix} = \begin{pmatrix}
-16\cos(4t) \\
(2 + 4t^2)e^{t^2} \\
(2 + 4t + t^2)e^t
\end{pmatrix}
\]

2. \(10 \text{ pts}\) Compute \(D_v f(\vec{a})\) directly from the definition, where \(f(x, y) = xy^2 + e^x\), \(\vec{a} = (3, 1)\), and \(\vec{v} = (3, 4)\).

\[
D_v f(\vec{a}) = \left. \frac{\partial f}{\partial \vec{x}} \right|_{\vec{x} = \vec{a}} = \left. \frac{\partial}{\partial \vec{x}} f(\vec{a} + \delta \vec{v}) \right|_{\delta = 0} \cdot \vec{v} = \left. \frac{\partial}{\partial \vec{x}} f(3 + 3\delta, 1 + 4\delta) \right|_{\delta = 0} \cdot \vec{v}
\]

\[
= \left. \frac{\partial}{\partial \vec{x}} \right|_{\delta = 0} \left( (3 + 3\delta)(1 + 4\delta)^2 + e^{3 + 3\delta} \right)
\]

\[
= \left. \frac{\partial}{\partial \vec{x}} \right|_{\delta = 0} \left( 3 + 27\delta + 72\delta^2 + 48\delta^3 + e^{3 + 3\delta} \right)
\]

\[
= \left. \left( 27 + 144\delta + 144\delta^2 + 3e^{3 + 3\delta} \right) \right|_{\delta = 0}
\]

\[
= 27 + 3e^3
\]
3. (10 pts) Use the chain rule to compute the Jacobian matrix of the composition \( f \circ g \) in terms of the variables \( x \) and \( y \) WITHOUT computing the composition function itself. The function \( f \) is defined by \( f(x, y) = (x^2y - y^2, 3xy^4) \) and the function \( g \) is defined by \( g(x, y) = (xy, x - y) \).

\[
\mathbf{J}_f = \begin{pmatrix}
2uv & u^2 - 2v \\
3v^4 & 12uv^3
\end{pmatrix}
\]

\[
\mathbf{J}_g = \begin{pmatrix}
y \\
x
\end{pmatrix}
\begin{pmatrix}
y \\
x - y
\end{pmatrix}
\]

\[
\mathbf{J}_{f \circ g} = \mathbf{J}_f \mathbf{J}_g = \begin{pmatrix}
2xy^2(x - y) + x^2y^2 - 2(x - y) & 2x^2y(x - y) - x^2y^2 + 2(x - y) \\
3y(x - y)^4 + 12xy(x - y)^3 & 3x(x - y)^4 - 12xy(x - y)^3
\end{pmatrix}
\]

4. (10 pts) Near the point \((1, 2, 1)\) on the surface defined by the equation \(3x^2y^3 - xyz^2 = 22\), explain how you know that \(x\) can be viewed locally as a function of \(y\) and \(z\). Compute \(\frac{\partial x}{\partial z}\) at this point.

\[
F(x, y, z) = 3x^2y^3 - xyz^2
\]

\[
\frac{\partial F}{\partial x} = 6xy^3 - yz^2 \quad \frac{\partial F}{\partial x}\bigg|_{(1,2,1)} = 48 - 2 = 46 \neq 0
\]

Since \(\frac{\partial F}{\partial x} \neq 0\), \(x\) is locally a fn of \(y, z\).

So

\[
6xy^3 \frac{\partial x}{\partial z} - (\frac{\partial x}{\partial z})(yz^2) - x(2yz) = 0
\]

\[
\frac{\partial x}{\partial z} = \frac{2xymz}{6xy^3 - yz^2}
\]

\[
\frac{\partial x}{\partial z}\bigg|_{(1,2,1)} = \frac{4}{46} = \frac{2}{23}
\]
5. (10 pts) Points on Normal Hill have altitude given by the function

\[ h(x, y) = 200 - 10(x + y)^2 - 30y^2 \]

where \( x \) and \( y \) represent the horizontal distances east and north (respectively) from a fixed point of reference. Bob is standing on the hill at the point where \( x = 1 \) and \( y = 1 \). How steep is it where Bob is standing?

\[
\nabla h = \begin{pmatrix}
-20(x+y) \\
-20(x+y) - 60y
\end{pmatrix}
\]

\[ \nabla h \bigg|_{(1,1)} = \begin{pmatrix}
-40 \\
-100
\end{pmatrix} \]

\[ \| \nabla h \| = 10 \sqrt{116} \]

6. (10 pts) The region \( D \) in the \( xy \)-plane is bounded by the curves \( y = x^2 \), \( y = (x - 2)^2 \), and \( x + 1 = 0 \). The concentration of bacteria at nearby points on the \( xy \)-plane is given by \( C(x, y) = 1000 - x^2y \). Compute the total number of bacteria in the region \( D \).

\[
B = \iint_D C \, dA = \iint_D (1000 - x^2y) \, dA
\]

\[
= \int_{-1}^{1} \int_{x^2}^{(x-2)^2} (1000 - x^2y) \, dy \, dx
\]

\[
= \int_{-1}^{1} (1000y - \frac{1}{2}x^2y^2) \bigg|_{y=x^2}^{y=(x-2)^2} \, dx
\]

\[
= \int_{-1}^{1} (1000(x-2)^2 - \frac{1}{2}x^2(x-2)^4) - (1000x^2 - \frac{1}{2}x^2x^2) \, dx
\]

\[
= \int_{-1}^{1} 10000 - 4000x - \frac{1}{2}x^2(8x^3 - 12x^2 - 32x + 16) \, dx
\]

\[
= \int_{-1}^{1} 10000 - 4000x - \frac{1}{2}x^2 \frac{8}{3}x^3 - \frac{12}{5}x^5 \, dx
\]

\[
= \left[ 10000x - \frac{4}{3}x^2 - \frac{8}{5}x^5 - \frac{8}{3}x^3 \right]_{-1}^{1} = 8000 - \frac{1}{5} - \frac{24}{5} - \frac{16}{3}
\]
7. (10 pts) Write down a nested integral (but do not evaluate it) that represents the triple integral of the function \( f(x, y, z) = x^2yz \) over the region bounded by the surfaces \( z = 0 \), \( z = 4 - x^2 \), \( y + x^2 + z^2 = 0 \), and \( y = e^x \).

The projection to the \( xz \)-plane is:

So \( x \in [-2, 2] \)

\( z \in [0, 4-x^2] \)

For any \( x, z \),

\( (y = -x^2 - z^2 < 0) \) is \( < (y = e^x > 0) \)

So \( \iiint_R f \, dV = \int_{-2}^{2} \int_{0}^{4-x^2} \int_{-x^2-z^2}^{e^x} x^2 y z \, dy \, dz \, dx \)

8. (10 pts) Compute the integral \( \iint_R y^3 e^x \, dx \, dy \), where \( R \) is the region bounded by the four curves \( y - e^x = -1 \), \( y - e^x = -2 \), \( y + e^x = 1 \), and \( y + e^x = 2 \).

These curves are reflections of each other through the \( x \)-axis, so \( R \) is symmetric through the \( x \)-axis.

\( f(x, y) = (y)^3 e^x = -y^3 e^x = -f(x, y) \), so \( f \) has odd symmetry through the \( x \)-axis.

\( \Rightarrow \iint_R = 0 \) by symmetry.
9. (10 pts) Write down a nested integral in spherical coordinates (but do not evaluate it) that represents the triple integral of the function \( f(x, y, z) = x - y - z \) over the region defined by \((x-5)^2 + y^2 + z^2 \leq 25\), and \(x, y, z \geq 0\).

\[
(x-5)^2 + y^2 + z^2 = 25
\]
\[
x^2 - 10x + 25 + y^2 + z^2 = 25
\]
\[
\rho^2 - 10 \rho \sin \phi \cos \theta = 0
\]

\[
\Rightarrow \quad \rho = 10 \sin \phi \cos \theta
\]

Then

\[
\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{10 \sin \phi \cos \theta} (\rho \sin (\cos \theta - \sin \theta) - \rho \cos \theta) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta
\]

10. (10 pts) Compute the mass of the wire bent into the shape of the curve \( y = x^2 \) between \( x = 0 \) and \( x = 1 \), where the density is given by \( \delta(x, y) = 24xy \).

\[
m = \int_c \delta \, ds = \int_c \delta \, \| \mathbf{r}' \| \, dt
\]
\[
x = \frac{t}{\sqrt{2}}, \quad \frac{t}{\sqrt{2}} \in [0, 1] \quad \frac{x'}{t} = 1, \quad \frac{y'}{t} = 2
\]

\[
m = \int_0^1 24\left(\frac{t}{\sqrt{2}}\right)\left(\frac{t}{\sqrt{2}}\right)^2 \sqrt{1 + \left(\frac{2t}{\sqrt{2}}\right)^2} \, dt
\]
\[
= \int_0^1 \left(\frac{t^2}{2}\right) \left(24 + 1 + 4t^2\right) \, dt
\]
\[
f' = 2t, \quad \quad g = 2\left(1 + 4t^2\right)^{3/2}
\]
\[
= 2t^2 \left(1 + 4t^2\right)^{3/2} \int_0^1 24t \left(1 + 4t^2\right)^{1/2} \, dt
\]
\[
m = 1 + 4t^2
\]
\[
\frac{dm}{dt} = 8t \, dt
\]

\[
= (2t^2(1 + 4t^2)^{3/2} - \frac{1}{2} \cdot \frac{2}{3} (1 + 4t^2)^{3/2})
\]
\[
= (2.5 \cdot 1 - \frac{1}{3} \cdot 5 \cdot \frac{1}{3}) - (\frac{1}{5})
\]
\[
= 5\sqrt{5} - \frac{1}{5}
\]