The Remainder Theorem And The Factor Theorem

**Remainder Theorem:** If \( p(x) \) is any polynomial, then the remainder after division by \((x - a)\) is exactly \( p(a) \).

**Proof:** Dividing \( p(x) \) by \((x - a)\), we get a quotient \( q(x) \) and a remainder \( R \), which must be a constant. We can then write \( p(x) \) as

\[
p(x) = (x - a) \cdot q(x) + R
\]

We now simply plug in \( x = a \) in the above equation.

\[
\begin{align*}
p(a) &= (a - a) \cdot q(x) + R \\
p(a) &= R
\end{align*}
\]

and have the desired result.

\[
\square
\]

**Factor Theorem:** The value \( a \) is a root of the polynomial \( p(x) \) if and only if \((x - a)\) is a factor of \( p(x) \).

**Proof:**

1. \((\implies)\) Assume that \( a \) is a root of the polynomial \( p(x) \). This means that \( p(a) = 0 \).

   By the remainder theorem, we conclude that the remainder after division by \((x - a)\) must be zero. So,

   \[
   \begin{align*}
p(x) &= (x - a) \cdot q(x) + 0 \\
p(x) &= (x - a) \cdot q(x)
\end{align*}
   \]

   Therefore, \((x - a)\) is a factor of \( p(x) \).

2. \((\iff)\) Now we assume that \((x - a)\) is a factor of \( p(x) \). Then

   \[
   \begin{align*}
p(x) &= (x - a) \cdot q(x) \\
p(a) &= (a - a) \cdot q(x) = 0
\end{align*}
   \]

   So, \( a \) is a root of \( p(x) \).

\[
\square
\]

(over)
These theorems are very useful when dealing with polynomials.

**Example:** Evaluate

\[
\int \frac{x^2 + x}{x^3 - 1} \, dx
\]

In order to apply partial fractions, we need to factor the denominator

\[p(x) = x^3 - 1\]

completely.

We of course immediately notice that 1 is a root of this polynomial. By the Factor Theorem, we can then conclude that \((x - 1)\) must be a factor. Dividing, we get

\[p(x) = (x - 1)(x^2 + x + 1)\]

Now we need to try to factor the remaining quadratic term. As we did above, we try to find factors by finding roots. However, this quadratic does not have any roots, because it is always positive; we can see this most easily by completing the square:

\[x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}\]

Since it has no roots, the Factor Theorem then tells us that it can have no linear factors; and since it is quadratic, any factor must be linear. So, it can therefore have no factors at all. So it is irreducible.

So, to evaluate the antiderivative above, we would first apply the partial fractions technique to

\[\frac{x^2 + x}{(x - 1)(x^2 + x + 1)}\]