Remarks on Spinors in Low Dimension

0. Introduction. The purpose of these notes is to study the orbit structure of the groups $\mathrm{Spin}(p,q)$ acting on their spinor spaces for certain values of $n = p{+}q$, in particular, the values

$$(p,q) = (8,0),\ (9,0),\ (10,0),\ \text{and } (10,1).$$

though it will turn out in the end that there are a few interesting things to say about the cases $(p,q) = (10,2)$ and $(9,1)$, as well.

1. The Octonions. Let $\mathbb{O}$ denote the ring of octonions. Elements of $\mathbb{O}$ will be denoted by bold letters, such as $\mathbf{x}$, $\mathbf{y}$, etc. Thus, $\mathbb{O}$ is the unique $\mathbb{R}$-algebra of dimension 8 with unit $\mathbf{1} \in \mathbb{O}$ endowed with a positive definite inner product $\langle , \rangle$ satisfying $\langle \mathbf{xy}, \mathbf{xy}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle\,\langle \mathbf{y}, \mathbf{y}\rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{O}$. As usual, the norm of an element $\mathbf{x} \in \mathbb{O}$ is denoted $|\mathbf{x}|$ and defined as the square root of $\langle \mathbf{x}, \mathbf{x}\rangle$. Left and right multiplication by $\mathbf{x} \in \mathbb{O}$ define maps $L_{\mathbf{x}}\,, R_{\mathbf{x}} : \mathbb{O} \to \mathbb{O}$ that are isometries when $|\mathbf{x}| = 1$.

The conjugate of $\mathbf{x} \in \mathbb{O}$, denoted $\overline{\mathbf{x}}$, is defined to be $\overline{\mathbf{x}} = 2\langle \mathbf{x}, \mathbf{1}\rangle\,\mathbf{1} - \mathbf{x}$. When a symbol is needed, the map of conjugation will be denoted $C : \mathbb{O} \to \mathbb{O}$. The identity $\mathbf{x}\,\overline{\mathbf{x}} = |\mathbf{x}|^2$ holds, as well as the conjugation identity $\overline{\mathbf{xy}} = \overline{\mathbf{y}}\,\overline{\mathbf{x}}$. In particular, this implies the useful identities $C\,L_{\mathbf{x}}\,C = R_{\overline{\mathbf{x}}}$ and $C\,R_{\mathbf{x}}\,C = L_{\overline{\mathbf{x}}}$.

The algebra $\mathbb{O}$ is not commutative or associative. However, any subalgebra of $\mathbb{O}$ that is generated by two elements is associative. It follows that $\mathbf{x}\,\bigl(\overline{\mathbf{x}}\mathbf{y}\bigr) = |\mathbf{x}|^2\,\mathbf{y}$ and that $(\mathbf{xy})\mathbf{x} = \mathbf{x}(\mathbf{yx})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{O}$. Thus, $R_{\mathbf{x}}\,L_{\mathbf{x}} = L_{\mathbf{x}}\,R_{\mathbf{x}}$ (though, of course, $R_{\mathbf{x}}\,L_{\mathbf{y}} \neq L_{\mathbf{y}}\,R_{\mathbf{x}}$ in general). In particular, the expression $\mathbf{xyx}$ is unambiguously defined. In addition, there are the *Moufang Identities*

$$\begin{aligned}(\mathbf{xyx})\mathbf{z} &= \mathbf{x}\bigl(\mathbf{y}(\mathbf{xz})\bigr),\\ \mathbf{z}(\mathbf{xyx}) &= \bigl((\mathbf{zx})\mathbf{y}\bigr)\mathbf{x},\\ \mathbf{x}(\mathbf{yz})\mathbf{x} &= (\mathbf{xy})(\mathbf{zx}),\end{aligned}$$

which will be useful below. (See, for example, *Spinors and Calibrations*, by F. Reese Harvey, for proofs.)

2. $\mathrm{Spin}(8)$**.** For $\mathbf{x} \in \mathbb{O}$, define the linear map $m_{\mathbf{x}} : \mathbb{O} \oplus \mathbb{O} \to \mathbb{O} \oplus \mathbb{O}$ by the formula

$$m_{\mathbf{x}} = \begin{bmatrix} 0 & C\,R_{\mathbf{x}} \\ -C\,L_{\mathbf{x}} & 0 \end{bmatrix}.$$

By the above identities, it follows that $(m_{\mathbf{x}})^2 = -|\mathbf{x}|^2$ and hence this map induces a representation on the vector space $\mathbb{O} \oplus \mathbb{O}$ of the Clifford algebra generated by $\mathbb{O}$ with its standard quadratic form. This Clifford algebra is known to be isomorphic to $M_{16}(\mathbb{R})$, the algebra of 16-by-16 matrices with real entries, so this representation must be faithful. By dimension count, this establishes the isomorphism $\mathrm{C}\ell\bigl(\mathbb{O}, \langle , \rangle\bigr) = \mathrm{End}_{\mathbb{R}}\bigl(\mathbb{O} \oplus \mathbb{O}\bigr)$.

The group $\mathrm{Spin}(8) \subset \mathrm{GL}_{\mathbb{R}}(\mathbb{O} \oplus \mathbb{O})$ is defined as the subgroup generated by products of the form $m_{\mathbf{x}}\,m_{\mathbf{y}}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{O}$ satisfy $|\mathbf{x}| = |\mathbf{y}| = 1$. Such endomorphisms preserve the splitting of $\mathbb{O} \oplus \mathbb{O}$ into the two given summands since

$$m_{\mathbf{x}}\,m_{\mathbf{y}} = \begin{bmatrix} -L_{\overline{\mathbf{x}}}\,L_{\mathbf{y}} & 0 \\ 0 & -R_{\overline{\mathbf{x}}}\,R_{\mathbf{y}} \end{bmatrix}.$$

In fact, setting $\mathbf{x} = -\mathbf{1}$ in this formula shows that endomorphisms of the form

$$\begin{bmatrix} L_{\mathbf{u}} & 0 \\ 0 & R_{\mathbf{u}} \end{bmatrix}, \qquad \text{with } |\mathbf{u}| = 1$$

lie in Spin(8). In fact, they generate Spin(8), since $m_{\mathbf{x}}\, m_{\mathbf{y}}$ is clearly a product of two of these when $|\mathbf{x}| = |\mathbf{y}| = 1$.

Fixing an identification $\mathbb{O} \simeq \mathbb{R}^8$, this defines an embedding $\mathrm{Spin}(8) \subset \mathrm{SO}(8) \times \mathrm{SO}(8)$, and the projections onto either of the factors is a group homomorphism. Since neither of these projections is trivial, since the Lie algebra $\mathfrak{so}(8)$ is simple, and since SO(8) is connected, it follows that each of these projections is a surjective homomorphism. Since Spin(8) is simply connected and since the fundamental group of SO(8) is $\mathbb{Z}_2$, it follows that that each of these homomorphisms is a non-trivial double cover of SO(8). Moreover, it follows that the subsets $\{\ L_{\mathbf{u}} \mid |\mathbf{u}| = 1\ \}$ and $\{\ R_{\mathbf{u}} \mid |\mathbf{u}| = 1\ \}$ of SO(8) each suffice to generate SO(8).

Let $H \subset \big(\mathrm{SO}(8)\big)^3$ be the set of triples $(g_1, g_2, g_3) \in \big(\mathrm{SO}(8)\big)^3$ for which

$$g_2(\mathbf{x}\mathbf{y}) = g_1(\mathbf{x})\, g_3(\mathbf{y})$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{O}$. The set H is closed and is evidently closed under multiplication and inverse. Hence it is a compact Lie group.

By the third Moufang identity, H contains the subset

$$\Sigma = \{\ (L_{\mathbf{u}},\ L_{\mathbf{u}}\, R_{\mathbf{u}},\ R_{\mathbf{u}}) \mid |\mathbf{u}| = 1\}\,.$$

Let $K \subset H$ be the subgroup generated by Σ, and for $i = 1, 2, 3$, let $\rho_i : H \to \mathrm{SO}(8)$ be the homomorphism that is projection onto the i-th factor. Since $\rho_1(K)$ contains $\{\ L_{\mathbf{u}} \mid |\mathbf{u}| = 1\ \}$, it follows that $\rho_1(K) = \mathrm{SO}(8)$, so *a fortiori* $\rho_1(H) = \mathrm{SO}(8)$. Similarly, $\rho_3(H) = \mathrm{SO}(8)$.

The kernel of ρ_1 consists of elements (I_8, g_2, g_3) that satisfy $g_2(\mathbf{x}\mathbf{y}) = \mathbf{x}\, g_3(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{O}$. Setting $\mathbf{x} = \mathbf{1}$ in this equation yields $g_2 = g_3$, so that $g_2(\mathbf{x}\mathbf{y}) = \mathbf{x}\, g_2(\mathbf{y})$. Setting $\mathbf{y} = \mathbf{1}$ in this equation yields $g_2(\mathbf{x}) = \mathbf{x}\, g_2(\mathbf{1})$, i.e., $g_2 = R_{\mathbf{u}}$ for $\mathbf{u} = g_2(\mathbf{1})$. Thus, the elements in the kernel of ρ_1 are of the form $(1, R_{\mathbf{u}}, R_{\mathbf{u}})$ for some $\mathbf{u}$ with $|\mathbf{u}| = 1$. However, any such $\mathbf{u}$ would, by definition, satisfy $(\mathbf{x}\mathbf{y})\mathbf{u} = \mathbf{x}(\mathbf{y}\mathbf{u})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{O}$, which is impossible unless $\mathbf{u} = \pm\mathbf{1}$. Thus, the kernel of ρ_1 is $\big\{(I_8, \pm I_8, \pm I_8)\big\} \simeq \mathbb{Z}_2$, so that ρ_1 is a 2-to-1 homomorphism of H onto SO(8). Similarly, ρ_3 is a 2-to-1 homomorphism of H onto SO(8), with kernel $\big\{(\pm I_8, \pm I_8, I_8)\big\}$. Thus, H is either connected and isomorphic to Spin(8) or else disconnected, with two components.

Now K is a connected subgroup of H and the kernel of ρ_1 intersected with K is either trivial or $\mathbb{Z}_2$. Moreover, the product homomorphism $\rho_1{\times}\rho_3 : K \to \mathrm{SO}(8){\times}\mathrm{SO}(8)$ maps the generator $\Sigma \subset K$ into generators of $\mathrm{Spin}(8) \subset \mathrm{SO}(8){\times}\mathrm{SO}(8)$. It follows that $\rho_1{\times}\rho_3(K) = \mathrm{Spin}(8)$ and hence that ρ_1 and ρ_3 must be non-trivial double covers of Spin(8) when restricted to K. In particular, it follows that K must be all of H and, moreover, that the homomorphism $\rho_1{\times}\rho_3 : H \to \mathrm{Spin}(8)$ must be an isomorphism. It also follows that the homomorphism $\rho_2 : H \to \mathrm{SO}(8)$ must be a double cover of SO(8) as well.

Henceforth, H will be identified with Spin(8) via the isomorphism $\rho_1 \times \rho_3$. Note that the center of H consists of the elements $(\varepsilon_1 I_8, \varepsilon_2 I_8, \varepsilon_3 I_8)$ where ${\varepsilon_i}^2 = \varepsilon_1\varepsilon_2\varepsilon_3 = 1$ and is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Triality. For $(g_1, g_2, g_3) \in H$, the identity $g_2(\mathbf{xy}) = g_1(\mathbf{x})\, g_3(\mathbf{y})$ can be conjugated, giving

$$Cg_2C(\mathbf{xy}) = \overline{g_2(\overline{\mathbf{y}}\,\overline{\mathbf{x}})} = \overline{g_1(\overline{\mathbf{y}})\, g_3(\overline{\mathbf{x}})} = \overline{g_3(\overline{\mathbf{x}})}\,\overline{g_1(\overline{\mathbf{y}})}.$$

This implies that $\bigl(Cg_3C, Cg_2C, Cg_1C\bigr)$ also lies in H. Also, replacing $\mathbf{x}$ by $\mathbf{z}\overline{\mathbf{y}}$ in the original formula and multiplying on the right by $\overline{g_3(\mathbf{y})}$ shows that

$$g_2(\mathbf{z})\overline{g_3(\mathbf{y})} = g_1(\mathbf{z}\overline{\mathbf{y}}),$$

implying that $\bigl(g_2, g_1, Cg_3C\bigr)$ lies in H as well. In fact, the two maps $\alpha, \beta : H \to H$ defined by

$$\alpha(g_1, g_2, g_3) = \bigl(Cg_3C, Cg_2C, Cg_1C\bigr), \quad \text{and} \qquad \beta(g_1, g_2, g_3) = \bigl(g_2, g_1, Cg_3C\bigr)$$

are outer automorphisms (since they act nontrivially on the center of H) and generate a group of automorphisms isomorphic to S_3, the symmetric group on three letters. The automorphism $\tau = \alpha\beta$ is known as the triality automorphism.

Notation. To emphasize the group action, denote $\mathbb{O} \simeq \mathbb{R}^8$ by V_i when regarding it as a representation space of Spin(8) via the representation ρ_i. Thus, octonion multiplication induces a Spin(8)-equivariant projection

$$V_1 \otimes V_3 \longrightarrow V_2\,.$$

In the standard notation, it is traditional to identify V_1 with $\mathbb{S}_-$ and V_3 with $\mathbb{S}_+$ and to refer to V_2 as the 'vector representation'. Let $\rho_i' : \mathfrak{spin}(8) \to \mathfrak{so}(8)$ denote the corresponding Lie algebra homomorphisms, which are, in fact, isomorphisms. For simplicity of notation, for any $a \in \mathfrak{spin}(8)$, the element $\rho_i'(a) \in \mathfrak{so}(8)$ will be denoted by a_i when no confusion can arise.

Orbit structure. Let SO(Im$\mathbb{O}$) $\simeq$ SO(7) denote the subgroup of SO($\mathbb{O}$) $\simeq$ SO(8) that leaves $\mathbf{1} \in \mathbb{O}$ fixed, and let $K_i \subset H$ be the preimage of SO(Im$\mathbb{O}$) under the homomorphism $\rho_i : H \to \mathrm{SO}(\mathbb{O})$. Then K_i is a non-trivial double cover of SO(Im$\mathbb{O}$) and hence is isomorphic to Spin(7). Note, in particular that K_1 contains $(I_8, -I_8, -I_8)$ and hence $\rho_3(K_1) \subset \mathrm{SO}(8)$ contains $-I_8$. This implies that $\rho_3 : K_1 \to \mathrm{SO}(V_3)$ is a faithful representation of Spin(7) and hence K_1 acts transitively on the unit sphere in V_3.

In particular, it follows that $\mathrm{Spin}(8) \subset \mathrm{SO}(V_1) \times \mathrm{SO}(V_3)$ acts transitively on the product of the unit spheres in V_1 and V_3. Consequently, it follows that the quadratic polynomials

$$q_1(\mathbf{x}, \mathbf{y}) = |\mathbf{x}|^2 \qquad \text{and} \qquad q_2(\mathbf{x}, \mathbf{y}) = |\mathbf{y}|^2$$

generate the ring of Spin(8)-invariant polynomials on $\mathbb{O} \oplus \mathbb{O}$ and that every point of this space lies on the Spin(8)-orbit of a unique element of the form $(a\,\mathbf{1}, b\,\mathbf{1})$ for some pair of real numbers $a, b \geq 0$. For $ab \neq 0$, the stabilizer of such an element is the 14-dimensional simple group G_2, and this group acts transitively on the unit sphere in $\mathrm{Im}\mathbb{O}$.

2. Spin(9)**.** For $(r, \mathbf{x}) \in \mathbb{R} \oplus \mathbb{O}$, define a $\mathbb{C}$-linear map $m_{(r,\mathbf{x})} : \mathbb{C} \otimes \mathbb{O}^2 \to \mathbb{C} \otimes \mathbb{O}^2$ by the formula

$$m_{(r,\mathbf{x})} = i \begin{bmatrix} r\, I_8 & C\, R_{\mathbf{x}} \\ C\, L_{\mathbf{x}} & -r\, I_8 \end{bmatrix} .$$

Since $(m_{(r,\mathbf{x})})^2$ is $-\left(r^2 + |\mathbf{x}|^2\right)$ times the identity map, this defines a $\mathbb{C}$-linear representation on $\mathbb{C} \otimes \mathbb{O}^2$ of the Clifford algebra generated by $\mathbb{R} \oplus \mathbb{O}$ endowed with its direct sum inner product. Since this Clifford algebra is known to be isomorphic to $M_{16}(\mathbb{C})$, it follows, for dimension reasons, that this representation is one-to-one and onto, establishing the isomorphism $\mathrm{C}\ell(\mathbb{R} \oplus \mathbb{O}, \langle , \rangle) = \mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes \mathbb{O}^2)$.

As usual, Spin(9) is the subgroup generated by the products of the form $m_{(r,\mathbf{x})} m_{(s,\mathbf{y})}$ where $r^2 + |\mathbf{x}|^2 = s^2 + |\mathbf{y}|^2 = 1$. Note that these products have real coefficients, and so actually lie in $\mathrm{GL}_{\mathbb{R}}(\mathbb{O}^2) \simeq \mathrm{GL}(16, \mathbb{R})$. In fact, these products are themselves seen to be products of the products of the special form

$$p_{(r,\mathbf{x})} = m_{(-1,\mathbf{0})} m_{(r,\mathbf{x})} = \begin{bmatrix} r\, I_8 & C\, R_{\mathbf{x}} \\ -C\, L_{\mathbf{x}} & r\, I_8 \end{bmatrix} , \qquad \text{where } r^2 + |\mathbf{x}|^2 = 1,$$

so these latter matrices suffice to generate Spin(9). By the results of the previous section, products of an even number of the $p_{(0,\mathbf{u})}$ with $|\mathbf{u}| = 1$ generate $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$.

Since the linear transformations of the form $p_{(r,\mathbf{x})}$ preserve the quadratic form

$$q(x, y) = |\mathbf{x}|^2 + |\mathbf{y}|^2,$$

it follows that Spin(9) is a subgroup of $\mathrm{SO}(\mathbb{O}^2) = \mathrm{SO}(16)$.

The Lie algebra. Since Spin(9) contains Spin(8), the containment $\mathfrak{spin}(8) \subset \mathfrak{spin}(9)$ yields the containment

$$\left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_3 \end{pmatrix} \middle| \; a \in \mathfrak{spin}(8) \right\} \subset \mathfrak{spin}(9) .$$

Moreover, since Spin(9) contains the 8-sphere consisting of the $p_{(r,\mathbf{x})}$ with $r^2 + |\mathbf{x}|^2 = 1$, its Lie algebra must contain the tangent space to this 8-sphere at $(r, \mathbf{x}) = (1, \mathbf{0})$, i.e.,

$$\left\{ \begin{pmatrix} 0 & C\, R_{\mathbf{x}} \\ -C\, L_{\mathbf{x}} & 0 \end{pmatrix} \middle| \; \mathbf{x} \in \mathbb{O} \right\} \subset \mathfrak{spin}(9) .$$

By dimension count, this implies the equality

$$\mathfrak{spin}(9) = \left\{ \begin{pmatrix} a_1 & C\, R_{\mathbf{x}} \\ -C\, L_{\mathbf{x}} & a_3 \end{pmatrix} \middle| \; \mathbf{x} \in \mathbb{O}, \; a \in \mathfrak{spin}(8) \right\} .$$

Let $\rho : \mathrm{Spin}(9) \to \mathrm{SO}(\mathbb{R}\oplus\mathbb{O}) \simeq \mathrm{SO}(9)$ be the homomorphism for which the induced map on Lie algebras is

$$\rho'\left(\begin{pmatrix} a_1 & C\,R_{\mathbf{z}} \\ -C\,L_{\mathbf{z}} & a_3 \end{pmatrix}\right) = \begin{pmatrix} 0 & 2\,\overline{\mathbf{z}}^* \\ -2\,\overline{\mathbf{z}} & a_2 \end{pmatrix}.$$

where $\mathbf{x}^* : \mathbb{O} \to \mathbb{R}$ is just $\mathbf{x}^*(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y}\rangle$. (The triality constructions imply that ρ' is, indeed, a Lie algebra homormophism. Note that, when restricted to Spin(8), this becomes the homomorphism $\rho_2 : \mathrm{Spin}(8) \to \mathrm{SO}(\mathbb{O}) = \mathrm{SO}(8)$.) Then ρ is a double cover of SO(9).

Define the squaring map $\sigma : \mathbb{O}^2 \to \mathbb{R}\oplus\mathbb{O}$ by

$$\sigma\left(\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right) = \begin{pmatrix} |\mathbf{x}|^2 - |\mathbf{y}|^2 \\ 2\,\mathbf{x}\,\mathbf{y} \end{pmatrix}.$$

A short calculation using the Moufang Identities shows that σ is equivariant with respect to ρ, i.e., that $\sigma\big(g\mathbf{v}\big) = \rho(g)\big(\sigma(\mathbf{v})\big)$ for all $\mathbf{v} \in \mathbb{O}^2$ and all $g \in \mathrm{Spin}(9)$. This will be useful below.

Orbit structure and stabilizer. Every point of $\mathbb{O}^2$ lies on an Spin(8)-orbit of an element of the form $(a\,\mathbf{1}, b\,\mathbf{1})$, for some pair of real numbers $a, b \geq 0$. Thus, the orbits of Spin(9) on the unit sphere in $\mathbb{O}^2$ are unions of the Spin(8)-orbits of the elements $(\cos\theta\,\mathbf{1}, \sin\theta\,\mathbf{1})$. Now, calculation yields

$$p_{(\cos\phi, \sin\phi\,\mathbf{1})}\begin{pmatrix} \cos\theta\,\mathbf{1} \\ \sin\theta\,\mathbf{1} \end{pmatrix} = \begin{pmatrix} \cos(\theta{-}\phi)\,\mathbf{1} \\ \sin(\theta{-}\phi)\,\mathbf{1} \end{pmatrix}.$$

Since all of the elements $(\cos\theta\,\mathbf{1}, \sin\theta\,\mathbf{1})$ lie on a single Spin(9)-orbit, it follows that Spin(9) acts transitively on the unit sphere in $\mathbb{O}^2$ and, consequently, that the quadratic form q generates the ring of Spin(9)-invariant polynomials on $\mathbb{O}^2$.

Since the orbit of $(\mathbf{1}, \mathbf{0}) \in \mathbb{O}^2$ is the 15-sphere and since Spin(9) is connected and simply connected, it follows that the Spin(9)-stabilizer of this element must be connected, simply connected, and of dimension 21. Since $K_1 \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ lies in this stabilizer and has dimension 21, it follows that K_1 must be equal to this stabilizer.

For use in the next two sections, it will be useful to understand the orbits of Spin(9) acting on $\mathbb{O}^2 \oplus \mathbb{O}^2$ and to understand the ring of Spin(9)-invariant polynomials on this vector space of real dimension 32. The first observation is that the generic orbit has codimension 4. This can be seen as follows: Since Spin(9) acts transitively on the unit sphere in $\mathbb{O}^2$, every element lies on the Spin(9) orbit of an element of the form

$$\left(\begin{pmatrix} a\,\mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right),$$

where $a \geq 0$. Assuming $a > 0$, the stabilizer in Spin(9) of this first component is $K_1 \simeq \mathrm{Spin}(7)$ and this acts transitively on the unit sphere in the second $\mathbb{O}$-summand of $\mathbb{O}^2$, so that an element of the above form lies on the orbit of an element of the form

$$\left(\begin{pmatrix} a\,\mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{x} \\ b\,\mathbf{1} \end{pmatrix}\right),$$

where $b \geq 0$. Assuming $b > 0$, the stabilizer in K_1 of $\mathbf{1}$ in this second $\mathbb{O}$-summand is G_2, which acts transitively on the unit sphere in $\mathrm{Im}\mathbb{O}$ in the first $\mathbb{O}$-summand. This implies that an element of the above form lies on the orbit of an element of the form

$$\mathbf{z} = \left(\begin{pmatrix} a\,\mathbf{1} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} c\,\mathbf{1} + d\,\mathbf{u} \\ b\,\mathbf{1} \end{pmatrix} \right),$$

for some $c, d \geq 0$ and $\mathbf{u} \in \mathrm{Im}\mathbb{O}$ some fixed unit imaginary octonion. Thus, the generic Spin(9)-orbit has codimension at most 4. It is still possible that two elements of the above form with distinct values of $a, b, c, d > 0$ might lie on the same Spin(9)-orbit, but this will be ruled out directly.

To see that these latter elements lie on distinct Spin(9)-orbits, it will be sufficient to construct Spin(9)-invariant polynomials on $\mathbb{O}^2 \oplus \mathbb{O}^2$ that separate these elements. To do so, write the typical element of $\mathbb{O}^2 \oplus \mathbb{O}^2$ in the form

$$(\mathbf{v}_1, \mathbf{v}_2) = \left(\begin{pmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \end{pmatrix}, \begin{pmatrix} \mathbf{x}_2 \\ \mathbf{y}_2 \end{pmatrix} \right),$$

and first consider the quadratic polynomials

$$\begin{aligned} q_{2,0} &= |\mathbf{x}_1|^2 + |\mathbf{y}_1|^2 \\ q_{1,1} &= \mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{y}_1 \cdot \mathbf{y}_2 \\ q_{0,2} &= |\mathbf{x}_2|^2 + |\mathbf{y}_2|^2 \end{aligned}$$

These polynomials are manifestly Spin(9)-invariant and satisfy

$$q_{2,0}(\mathbf{z}) = a^2, \qquad q_{1,1}(\mathbf{z}) = ac, \qquad q_{0,2}(\mathbf{z}) = b^2 + c^2 + d^2.$$

Evidently, these three polynomials span the vector space of Spin(9)-invariant quadratic polynomials on $\mathbb{O}^2 \oplus \mathbb{O}^2$.

Since Spin(9) contains -1 times the identity, there are no Spin(9)-invariant cubic polynomials. A representation-theoretic argument shows that the Spin(9)-invariant quartic polynomials on $\mathbb{O}^2 \oplus \mathbb{O}^2$ form a vector space of dimension 7. Six of these are accounted for by quadratic polynomials in $q_{2,0}$, $q_{1,1}$, and $q_{0,2}$, while a seventh can be constructed as follows. Define

$$q_{2,2} = \sigma(\mathbf{v}_1) \cdot \sigma(\mathbf{v}_2) = \big(|\mathbf{x}_1|^2 - |\mathbf{y}_1|^2\big)\big(|\mathbf{x}_2|^2 - |\mathbf{y}_2|^2\big) + 4\,(\mathbf{x}_1\mathbf{y}_1) \cdot (\mathbf{x}_2\mathbf{y}_2).$$

Using the Spin(9)-equivariance of the squaring map σ, it follows that $q_{2,2}$ is indeed invariant under Spin(9). Note that

$$q_{2,2}(\mathbf{z}) = a^2(c^2 + d^2 - b^2),$$

so that knowledge of $\big(q_{2,0}(\mathbf{z}), q_{1,1}(\mathbf{z}), q_{0,2}(\mathbf{z}), q_{2,2}(\mathbf{z})\big)$ suffices to recover $a, b, c, d > 0$ when these numbers are all non-zero. It now follows that the simultaneous level sets of these four polynomials are exactly the Spin(9)-orbits on $\mathbb{O}^2 \oplus \mathbb{O}^2$. (It seems likely that these polynomials generate the ring of Spin(9)-invariant polynomials on $\mathbb{O}^2 \oplus \mathbb{O}^2$, but such a result will not be needed, so this problem will not be discussed further.)

3. Spin(10)**.** Rather than construct the Clifford representation for an inner product on a vector space of dimension 10, it is convenient to use the fact that Spin(10) already appears as a subgroup of $C\ell(\mathbb{R}\oplus\mathbb{O},\langle,\rangle) = \mathrm{End}_{\mathbb{C}}(\mathbb{C}\otimes\mathbb{O}^2)$. In fact, by the discussion in the last section, Spin(10) is the connected subgroup of this latter algebra whose Lie algebra is

$$\mathfrak{spin}(10) = \left\{ \begin{pmatrix} a_1 + ir\,I_8 & C\,R_{\mathbf{x}} + i\,C\,R_{\mathbf{y}} \\ -C\,L_{\mathbf{x}} + i\,C\,L_{\mathbf{y}} & a_3 - ir\,I_8 \end{pmatrix} \;\middle|\; r\in\mathbb{R},\ \mathbf{x},\mathbf{y}\in\mathbb{O},\ a\in\mathfrak{spin}(8) \right\}.$$

Note that $\mathfrak{spin}(10)$ appears as a subspace of $\mathfrak{su}(16)$, so that Spin(10) acts on $\mathbb{C}^{16} = \mathbb{C}\otimes\mathbb{O}^2$ preserving the complex structure and the quadratic form

$$q = q_{2,0} + q_{0,2} = |\mathbf{x}_1|^2 + |\mathbf{y}_1|^2 + |\mathbf{x}_2|^2 + |\mathbf{y}_2|^2\,,$$

where, now, the typical element of $\mathbb{C}\otimes\mathbb{O}^2$ will be written as

$$\mathbf{z} = \begin{pmatrix} \mathbf{x}_1 + i\,\mathbf{x}_2 \\ \mathbf{y}_1 + i\,\mathbf{y}_2 \end{pmatrix}.$$

Note that, because there are no connected Lie groups that lie properly between Spin(9) and Spin(10), it follows that Spin(10) is generated by Spin(9) and the circle subgroup

$$\mathbf{T} = \left\{ \begin{pmatrix} e^{ir}\,I_8 & 0 \\ 0 & e^{-ir}\,I_8 \end{pmatrix} \;\middle|\; r\in\mathbb{R}/2\pi\mathbb{Z} \right\},$$

which lies in Spin(10), but does not lie in Spin(9). In particular, a polynomial on $\mathbb{C}\otimes\mathbb{O}^2$ is Spin(10)-invariant if and only if it is both Spin(9)-invariant and $\mathbf{T}$-invariant.

Invariant polynomials. Among the quadratic polynomials that are Spin(9)-invariant, only the multiples of $q = q_{2,0} + q_{0,2}$ are also $\mathbf{T}$-invariant. Thus, q spans the space of Spin(10)-invariant quadratic forms on $\mathbb{C}\otimes\mathbb{O}^2$. In particular, this implies that the action of Spin(10) on $\mathbb{C}\otimes\mathbb{O}^2$ is irreducible (even as a real vector space).

Among the quartic polynomials that are Spin(9)-invariant, a short calculation shows that only linear combinations of q^2 and

$$\begin{aligned} p &= \tfrac{1}{2}\left(q_{2,2} + q_{2,0}\,q_{0,2} - 2\,{q_{1,1}}^2\right) \\ &= |\mathbf{x}_1|^2|\mathbf{x}_2|^2 + |\mathbf{y}_1|^2|\mathbf{y}_2|^2 - (\mathbf{x}_1\cdot\mathbf{x}_2 + \mathbf{y}_1\cdot\mathbf{y}_2)^2 + 2\,(\mathbf{x}_1\mathbf{y}_1)\cdot(\mathbf{x}_2\mathbf{y}_2) \\ &= |\mathbf{x}_1\wedge\mathbf{x}_2|^2 + |\mathbf{y}_1\wedge\mathbf{y}_2|^2 - 2\,(\mathbf{x}_1\cdot\mathbf{x}_2)\,(\mathbf{y}_1\cdot\mathbf{y}_2) + 2\,(\mathbf{x}_1\mathbf{y}_1)\cdot(\mathbf{x}_2\mathbf{y}_2). \end{aligned}$$

are invariant under the action of $\mathbf{T}$. Thus, it follows that q^2 and p span the space of Spin(10)-invariant quartics. (Note the interesting feature that, in the latter expression for p, only the final term makes use of octonion multiplication operations.)

Orbits and stabilizers. Let $M \subset \mathbb{C}\otimes\mathbb{O}^2$ be the Spin(10)-orbit of $\mathbf{z}_0 = (\mathbf{1}+i\,\mathbf{0}, \mathbf{0}+i\,\mathbf{0})$. The tangent space to M at $\mathbf{z}_0$ is the set of vectors of the form

$$\begin{pmatrix} a_1 + ir\,I_8 & C\,R_{\mathbf{x}} + i\,C\,R_{\mathbf{y}} \\ -C\,L_{\mathbf{x}} + i\,C\,L_{\mathbf{y}} & a_3 - ir\,I_8 \end{pmatrix} \begin{pmatrix} \mathbf{1}+i\,\mathbf{0} \\ \mathbf{0}+i\,\mathbf{0} \end{pmatrix} = \begin{pmatrix} a_1\mathbf{1} + i\,r\mathbf{1} \\ -\overline{\mathbf{x}} + i\,\overline{\mathbf{y}} \end{pmatrix}.$$

and the Lie algebra of the Spin(10)-stabilizer of $\mathbf{z}_0$ is defined by the equations $a_1\mathbf{1} = r = \mathbf{x} = \mathbf{y} = 0$. Thus, the identity component of the stabilizer of $\mathbf{z}_0$ is $K_1 \simeq \mathrm{Spin}(7)$ and the full stabilizer must lie in the normalizer of K_1 in Spin(10). Evidently, the normalizer of K_1 in Spin(10) is $K_1\cdot\mathbf{T}$. Since only the identity in the subgroup $\mathbf{T}$ stabilizes $\mathbf{z}_0$, the full stabilizer of $\mathbf{z}_0$ is K_1. Thus, M is diffeomorphic to Spin(10)/Spin(7), which is a smooth manifold of dimension $45-21 = 24$ that is 2-connected, i.e., $\pi_0(M) = \pi_1(M) = \pi_2(M) = 0$.

The normal space to M at $\mathbf{z}_0$ is the orthogonal direct sum of the line $\mathbb{R}\mathbf{z}_0$ (which is normal to the unit sphere in $\mathbb{C}\otimes\mathbb{O}^2$) and the subspace of dimension 7

$$N_{\mathbf{z}_0} = \left\{ \begin{pmatrix} 0+i\,\mathbf{x} \\ \mathbf{0}+i\,\mathbf{0} \end{pmatrix} \;\middle|\; \mathbf{x}\in\mathrm{Im}\mathbb{O} \right\}.$$

The stabilizer K_1 acts as SO(7) on this subspace. In particular, it acts transitively on the unit sphere in $N_{\mathbf{z}_0}$, and hence it acts transitively on the space of geodesics in the unit 31-sphere that meet M orthogonally at $\mathbf{z}_0$. Since M is itself a Spin(10)-orbit, it follows that Spin(10) must act transitively on the the normal tube of any radius about M in the unit 31-sphere. Since, for generic radii, these normal tubes are hypersurfaces, it follows that the generic Spin(10)-orbit in the 31-sphere must be a hypersurface of dimension 30. Using the fact that such a hypersurface is an S^6-bundle over M, the long exact sequence in homotopy implies that these hypersurface orbits are also 2-connected, which implies that the Spin(10)-stabilizer of any point on such a hypersurface must be both connected and simply connected.

Now, the full group Spin(10) must act transitively on the space of geodesics in the unit 31-sphere that meet M orthogonally at any point while every point of the unit 31-sphere lies on some geodesic that meets M orthogonally. Thus, fixing some $\mathbf{u}\in\mathrm{Im}\mathbb{O}$ with $|\mathbf{u}| = 1$, it follows that every element of the 31-sphere lies on the Spin(10)-orbit of an element of the form

$$\mathbf{z}_\theta = \begin{pmatrix} \cos\theta + i\,\sin\theta\,\mathbf{u} \\ \mathbf{0}+i\,\mathbf{0} \end{pmatrix}.$$

Note that $p(\mathbf{z}_\theta) = \cos^2\theta\,\sin^2\theta = \frac{1}{4}\sin^2(2\theta)$, so it follows that for $0 \le \theta \le \pi/4$, the elements $\mathbf{z}_\theta$ lie on distinct orbits, and that $0 \le p \le \frac{1}{4}$, with the endpoints of this interval being the only critical values of p. While $M = p^{-1}(0)$ is one critical orbit, the other critical orbit is $M^* = p^{-1}(\frac{1}{4})$, and consists of the points of the 31-sphere that are at geodesic distance $\sqrt{2}/2$ from M. It follows from this that M^* is also connected and is a single orbit of Spin(10). In particular, the simultaneous level sets of q and p are exactly the Spin(10)-orbits in $\mathbb{C}\otimes\mathbb{O}^2$.

For $0 < \theta < \pi/4$, the nearest point on M to $\mathbf{z}_\theta$ is $\mathbf{z}_0$, so the Spin(10)-stabilizer of $\mathbf{z}_\theta$ is a subgroup of K_1 that has already been seen to be both connected and simply connected.

Also, the orbit of $\mathbf{z}_\theta$ is a 6-sphere bundle over M. By dimension count, this stabilizer must be of dimension 15 and must contain the stabilizer in K_1 of $\mathbf{1}$ and $\mathbf{u}$, which is Spin(6). Thus, the stabilizer of such a $\mathbf{z}_\theta$ is exactly $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$. In particular, the stabilizer of any point of the 31-sphere not on M or M^* must be a conjugate of SU(4).

Now, the tangent space to M^* at $\mathbf{z}_{\pi/4}$ is the set of vectors of the form

$$\begin{pmatrix} a_1 + ir\, I_8 & C\,R_{\mathbf{x}} + i\,C\,R_{\mathbf{y}} \\ -C\,L_{\mathbf{x}} + i\,C\,L_{\mathbf{y}} & a_3 - ir\, I_8 \end{pmatrix} \begin{pmatrix} \mathbf{1} + i\,\mathbf{u} \\ \mathbf{0} + i\,\mathbf{0} \end{pmatrix} = \begin{pmatrix} (a_1\mathbf{1} - r\,\mathbf{u}) + i\,(a_1\mathbf{u} + r\mathbf{1}) \\ -\overline{(\mathbf{x} + \mathbf{y}\mathbf{u})} + i\,\overline{(\mathbf{y} - \mathbf{x}\mathbf{u})} \end{pmatrix}.$$

Thus, the Lie algebra of the stabilizer G of $\mathbf{z}_{\pi/4}$ is defined by the relations $a_1\mathbf{1} - r\,\mathbf{u} = a_1\mathbf{u} + r\mathbf{1} = \mathbf{y} - \mathbf{x}\mathbf{u} = \mathbf{0}$. (Remember that $\mathbf{u}^2 = -\mathbf{1}$.) It follows that $a_1 \in \mathfrak{so}(8)$ must belong to the stabilizer of the 2-plane spanned by $\{\mathbf{1}, \mathbf{u}\}$, so that a_1 lies in $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$. Conversely, if a_1 lies in this subspace, then there exists a unique $r \in \mathbb{R}$ so that $a_1\mathbf{1} - r\,\mathbf{u} = a_1\mathbf{u} + r\mathbf{1} = 0$. From the matrix representation, it is clear that the maximal torus in $\mathfrak{so}(2) \oplus \mathfrak{so}(6)$ (which has rank 4) is a maximal torus in the full stabilizer algebra, which has dimension 24. The root pattern is evident from the matrix representation, implying that the stabilizer algebra is isomorphic to $\mathfrak{su}(5)$.

Now M^* has dimension 21 and is the base of a fibration whose total space is one of the hypersurface orbits and whose fiber is a 9-sphere. The 2-connectivity of the hypersurface orbits implies, by the long exact sequence in homotopy, that M^* is also 2-connected, which implies that $M^* = \mathrm{Spin}(10)/G$ where G is both connected and simply connected. Since its Lie algebra is $\mathfrak{su}(5)$, it follows that G is isomorphic to SU(5).

4. Spin(10, 1)**.** To construct the spinor representation of Spin(10, 1), it will be easiest to construct the Lie algebra representation by extending the Lie algebra representation of Spin(10) that was constructed in §3. It is convenient to identify $\mathbb{C} \otimes \mathbb{O}^2$ with $\mathbb{O}^4$ explicitly via the identification

$$\mathbf{z} = \begin{pmatrix} \mathbf{x}_1 + i\,\mathbf{x}_2 \\ \mathbf{y}_1 + i\,\mathbf{y}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{y}_1 \\ \mathbf{x}_2 \\ \mathbf{y}_2 \end{pmatrix}.$$

Via this identification, $\mathfrak{spin}(10)$ becomes the subspace

$$\mathfrak{spin}(10) = \left\{ \begin{pmatrix} a_1 & C\,R_{\mathbf{x}} & -r\,I_8 & -C\,R_{\mathbf{y}} \\ -C\,L_{\mathbf{x}} & a_3 & -C\,L_{\mathbf{y}} & r\,I_8 \\ r\,I_8 & C\,R_{\mathbf{y}} & a_1 & C\,R_{\mathbf{x}} \\ C\,L_{\mathbf{y}} & -r\,I_8 & -C\,L_{\mathbf{x}} & a_3 \end{pmatrix} \middle|\ r \in \mathbb{R},\ \mathbf{x}, \mathbf{y} \in \mathbb{O},\ a \in \mathfrak{spin}(8) \right\}.$$

Consider the one-parameter subgroup $\mathbf{R} \subset \mathrm{SL}_{\mathbb{R}}(\mathbb{O}^4)$ defined by

$$\mathbf{R} = \left\{ \begin{pmatrix} t\,I_{16} & 0 \\ 0 & t^{-1}\,I_{16} \end{pmatrix} \middle|\ t \in \mathbb{R}^+ \right\}.$$

It has a Lie algebra $\mathfrak{r} \subset \mathfrak{sl}(\mathbb{O}^4)$. Evidently, the the subspace $[\mathfrak{spin}(10), \mathfrak{r}]$ consists of matrices of the form

$$\begin{pmatrix} 0_8 & 0_8 & r\,I_8 & C\,R_{\mathbf{y}} \\ 0_8 & 0_8 & C\,L_{\mathbf{y}} & -r\,I_8 \\ r\,I_8 & C\,R_{\mathbf{y}} & 0_8 & 0_8 \\ C\,L_{\mathbf{y}} & -r\,I_8 & 0_8 & 0_8 \end{pmatrix}, \qquad r \in \mathbb{R},\ \mathbf{y} \in \mathbb{O}\,.$$

Let $\mathfrak{g} = \mathfrak{spin}(10) \oplus \mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}]$. Explicitly,

$$\mathfrak{g} = \left\{ \begin{pmatrix} a_1 + x\,I_8 & C\,R_{\mathbf{x}} & y\,I_8 & C\,R_{\mathbf{y}} \\ -C\,L_{\mathbf{x}} & a_3 + x\,I_8 & C\,L_{\mathbf{y}} & -y\,I_8 \\ z\,I_8 & C\,R_{\mathbf{z}} & a_1 - x\,I_8 & C\,R_{\mathbf{x}} \\ C\,L_{\mathbf{z}} & -z\,I_8 & -C\,L_{\mathbf{x}} & a_3 - x\,I_8 \end{pmatrix} \;\middle|\; \begin{matrix} x, y, z \in \mathbb{R}, \\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{O}, \\ a \in \mathfrak{spin}(8) \end{matrix} \right\}.$$

Compuation using the Moufang Identities shows that $\mathfrak{g}$ is closed under Lie bracket and hence is a Lie algebra of (real) dimension 55 that contains $\mathfrak{spin}(10)$. The induced representation of Spin(10) on $\mathfrak{g}/\mathfrak{spin}(10)$ evidently restricts to Spin(9) to preserve the splitting corresponding to the sum $\mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}] \simeq \mathbb{R} \oplus \mathbb{R}^9$ and acts as the standard (irreducible) representation on the $\mathbb{R}^9$ summand. It follows that Spin(10) must act via its standard (irreducible, ten dimensional) representation on $\mathfrak{g}/\mathfrak{spin}(10)$. Since the trace of the square of a non-zero element in the subspace $\mathfrak{r} \oplus [\mathfrak{spin}(10), \mathfrak{r}]$ is positive, $\mathfrak{g}$ is semisimple of non-compact type. It follows that $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(10,1)$ and hence is the Lie algebra of a representation of Spin(10, 1). This representation must be faithful since it is faithful on the maximal compact subgroup Spin(10).

Thus, define Spin(10, 1) to be the (connected) subgroup of $\mathrm{SL}_{\mathbb{R}}(\mathbb{O}^4)$ that is generated by Spin(10) and the subgroup $\mathbf{R}$. Its Lie algebra $\mathfrak{g}$ will henceforth be written as $\mathfrak{spin}(10,1)$.

Invariant Polynomials and Orbits. Consider the Spin(10)-invariant polynomial

$$p = |\mathbf{x}_1|^2|\mathbf{x}_2|^2 + |\mathbf{y}_1|^2|\mathbf{y}_2|^2 - (\mathbf{x}_1 \cdot \mathbf{x}_2 + \mathbf{y}_1 \cdot \mathbf{y}_2)^2 + 2\,(\mathbf{x}_1\mathbf{y}_1) \cdot (\mathbf{x}_2\mathbf{y}_2)\,,$$

Evidently, p is invariant under $\mathbf{R}$ and is therefore invariant under Spin(10, 1). In particular, it follows that the orbits of Spin(10, 1) on $\mathbb{O}^4 \simeq \mathbb{R}^{32}$ must lie in the level sets of p.

Also from the previous section, it is known that every element of $\mathbb{O}^4$ lies on the Spin(10)-orbit of exactly one of the elements

$$\mathbf{z}_{a,b} = \begin{pmatrix} a\,\mathbf{1} \\ \mathbf{0} \\ b\,\mathbf{u} \\ \mathbf{0} \end{pmatrix} \qquad \text{where } 0 \le b \le a.$$

and where $\mathbf{u} \in \mathrm{Im}\mathbb{O}$ is a fixed unit imaginary octonion. However, all of the elements of the form

$$\begin{pmatrix} at\,\mathbf{1} \\ \mathbf{0} \\ (b/t)\,\mathbf{u} \\ \mathbf{0} \end{pmatrix} \qquad (\text{where } 0 < t)$$

lie on the same $\mathbf{R}$-orbit and, hence, on the same $\mathrm{Spin}(10,1)$-orbit. Since $p(\mathbf{z}_{a,b}) = a^2b^2$, it now follows that each of the nonzero level sets of p is a single $\mathrm{Spin}(10,1)$-orbit while the zero level set is the union of the origin and a single $\mathrm{Spin}(10,1)$-orbit, say, the orbit of $\mathbf{z}_{1,0}$. Moreover, it follows that p generates the ring of $\mathrm{Spin}(10,1)$-invariant polynomials on $\mathbb{O}^4$.

Stabilizers. Multiplication by positive scalars acts transitively on the non-zero level sets of p, so they are all diffeomorphic. In fact, each such level set is contractible to the $\mathrm{Spin}(10)$-invariant locus where q reaches its minimum on this level set and this is a manifold of dimension 21 that is diffeomorphic to M^*. In particular, it follows that each of the non-zero level sets of p is 2-connected, so that the stabilizer in $\mathrm{Spin}(10,1)$ of a point on such a level set must be connected and simply connected.

If $\mathbf{z} \in \mathbb{O}^4$ has $p(\mathbf{z}) \neq 0$, then the $\mathrm{Spin}(10,1)$-orbit of $\mathbf{z}$ has dimension 31 and so its stabilizer in $\mathrm{Spin}(10,1)$ must be of dimension $55-31 = 24$. Moreover all of these stabilizers must be conjugate in $\mathrm{Spin}(10,1)$. Since the $\mathrm{Spin}(10)$-stabilizer of the point $\mathbf{z}_{1,1}$ is already known to be $\mathrm{SU}(5)$, which has dimension 24, it follows that this must be the $\mathrm{Spin}(10,1)$-stabilizer as well.

The $\mathrm{Spin}(10,1)$-orbit consisting of nonzero vectors in the zero locus of p is just the deleted cone on M, and so has dimension 25. Since it is contractible to M, it is 2-connected, so that the stabilizer in $\mathrm{Spin}(10,1)$ of a point in this orbit must be connected and simply connected and of dimension $55-25 = 30$. In fact, the Lie algebra of this stabilizer is just

$$\left\{ \begin{pmatrix} a_1 & 0 & y\,I_8 & C\,R_{\mathbf{y}} \\ 0 & a_3 & C\,L_{\mathbf{y}} & -y\,I_8 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_3 \end{pmatrix} \;\middle|\; \begin{matrix} y \in \mathbb{R}, \\ \mathbf{y} \in \mathbb{O}, \\ a \in \mathfrak{k}_1 \end{matrix} \right\}$$

where $\mathfrak{k}_1$ is the Lie algebra of $K_1 \subset \mathrm{Spin}(8)$. Thus, the stabilizer is a semi-direct product of $\mathrm{Spin}(7)$ with a copy of $\mathbb{R}^9$.

5. $\mathrm{Spin}(10,2)$**.** It might be tempting to conjecture that $\mathrm{Spin}(10,1)$ could be defined directly as the stabilizer of p. However, this is not the case, as the stabilizer of p is larger. One can see this directly by looking at the alternative expression

$$p = |\mathbf{x}_1 \wedge \mathbf{x}_2|^2 + |\mathbf{y}_1 \wedge \mathbf{y}_2|^2 - 2\,(\mathbf{x}_1 \cdot \mathbf{x}_2)\,(\mathbf{y}_1 \cdot \mathbf{y}_2) + 2\,(\mathbf{x}_1\mathbf{y}_1) \cdot (\mathbf{x}_2\mathbf{y}_2)\ .$$

which makes it evident that p is invariant under the 6-dimensional Lie group

$$G = \left\{ \begin{pmatrix} a\,I_8 & 0 & b\,I_8 & 0 \\ 0 & a'\,I_8 & 0 & b'\,I_8 \\ c\,I_8 & 0 & d\,I_8 & 0 \\ 0 & c'\,I_8 & 0 & d'\,I_8 \end{pmatrix} \;\middle|\; ad{-}bc = a'd'{-}b'c' = \pm 1 \right\}.$$

Since G does not lie in $\mathrm{Spin}(10,1)$, the invariance group of p must be properly larger than $\mathrm{Spin}(10,1)$.

In particular, consider the G-subgroup $\mathbf{R}' \simeq \mathbb{R}^+$ consisting of matrices of the form

$$\begin{pmatrix} t\,I_8 & 0 & 0 & 0 \\ 0 & t^{-1}\,I_8 & 0 & 0 \\ 0 & 0 & t^{-1}\,I_8 & 0 \\ 0 & 0 & 0 & t\,I_8 \end{pmatrix} \qquad \text{where } t > 0,$$

which is not a subgroup of $\mathrm{Spin}(10,1)$. Let $\mathfrak{r}'$ denote its Lie algebra. Calculation shows that

$$\mathfrak{r}' \oplus [\mathfrak{spin}(10,1), \mathfrak{r}'] = \left\{ \begin{pmatrix} w\,I_8 & C\,R_{\mathbf{w}} & u\,I_8 & 0 \\ C\,L_{\mathbf{w}} & -w\,I_8 & 0 & u\,I_8 \\ v\,I_8 & 0 & -w\,I_8 & -C\,R_{\mathbf{w}} \\ 0 & v\,I_8 & -C\,L_{\mathbf{w}} & w\,I_8 \end{pmatrix} \,\middle|\, \begin{matrix} u,v,w \in \mathbb{R}, \\ \mathbf{w} \in \mathbb{O} \end{matrix} \right\}.$$

and that the sum $\mathfrak{spin}(10,1) \oplus \mathfrak{r}' \oplus [\mathfrak{spin}(10,1), \mathfrak{r}']$ is closed under Lie bracket. Thus, this defines a Lie algebra of dimension 66 that lies in the stabilizer of p.

The details of further analysis will be omitted, but by using arguments similar to those used in previous sections, one sees that this algebra is isomorphic to $\mathfrak{so}(10,2)$ and that the connected Lie subgroup of $\mathrm{GL}_{\mathbb{R}}(\mathbb{O}^4)$ whose Lie algebra is this one is simply connected, so that it this group is $\mathrm{Spin}(10,2)$. Henceforth, this algebra will be denoted $\mathfrak{spin}(10,2)$. Thus,

$$\mathfrak{spin}(10,2) = \left\{ \begin{pmatrix} a_1 + x\,I_8 & C\,R_{\mathbf{w}} & y\,I_8 & C\,R_{\mathbf{y}} \\ C\,L_{\mathbf{x}} & a_3 + w\,I_8 & C\,L_{\mathbf{y}} & u\,I_8 \\ z\,I_8 & C\,R_{\mathbf{z}} & a_1 - x\,I_8 & -C\,R_{\mathbf{x}} \\ C\,L_{\mathbf{z}} & v\,I_8 & -C\,L_{\mathbf{w}} & a_3 - w\,I_8 \end{pmatrix} \,\middle|\, \begin{matrix} u,v,w,x,y,z \in \mathbb{R}, \\ \mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{O}, \\ a \in \mathfrak{spin}(8) \end{matrix} \right\}.$$

Moreover, representation theoretic methods show that the only connected proper subgroup of $\mathrm{SL}_{\mathbb{R}}(\mathbb{O}^4)$ that properly contains $\mathrm{Spin}(10,2)$ is $\mathrm{Sp}(16,\mathbb{R})$, the symplectic group preserving the symplectic 2-form Ω defined by

$$\Omega = d\mathbf{x}_1 \,\hat{\cdot}\, d\mathbf{x}_2 + d\mathbf{y}_1 \,\hat{\cdot}\, d\mathbf{y}_2\,.$$

Of course, $\mathrm{Sp}(16,\mathbb{R})$ does not stabilize any nonzero polynomials. It follows that $\mathrm{Spin}(10,2)$ is the identity component of the stabilizer of p, and hence that the stabilizer of p must lie in the normalizer of $\mathrm{Spin}(10,2)$ in $\mathrm{GL}_{\mathbb{R}}(\mathbb{O}^4)$. However, this normalizer is just $\mathbb{R}^+\!\cdot I_{32} \times \mathrm{Spin}(10,2)$ and the only element in $\mathbb{R}^+\!\cdot I_{32}$ that stabilizes p is the identity element. It follows that $\mathrm{Spin}(10,2)$ is the stabilizer of p.

6. $\mathrm{Spin}(9,1)$**.** As a final note, inspection reveals that the subalgebra

$$\mathfrak{spin}(9,1) = \left\{ \begin{pmatrix} a_1 + x\,I_8 & C\,R_{\mathbf{w}} \\ C\,L_{\mathbf{x}} & a_3 - x\,I_8 \end{pmatrix} \,\middle|\, \begin{matrix} x \in \mathbb{R}, \\ \mathbf{w},\mathbf{x} \in \mathbb{O}, \\ a \in \mathfrak{spin}(8) \end{matrix} \right\} \subset \mathfrak{sl}(16,\mathbb{R})\,,$$

which contains $\mathfrak{spin}(9)$, is actually the Lie algebra of a faithful representation of $\mathrm{Spin}(9,1)$ on $\mathbb{R}^{16} \simeq \mathbb{O}^2$. This action of $\mathrm{Spin}(9,1)$ has the interesting feature that it has only two orbits: The origin and the set of all non-zero vectors. This follows because the compact group $\mathrm{Spin}(9) \subset \mathrm{Spin}(9,1)$ already acts transitively on the unit spheres, but the larger group does not even preserve the quadratic form.