Einstein's Greatest Ideas in Context, from the Ancient Greeks to Today

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The Mystery of the Cosmos

"There is geometry in the humming of the strings, there is music in the spacing of the spheres."

attributed to Pythagoras (570 - 495 BC)

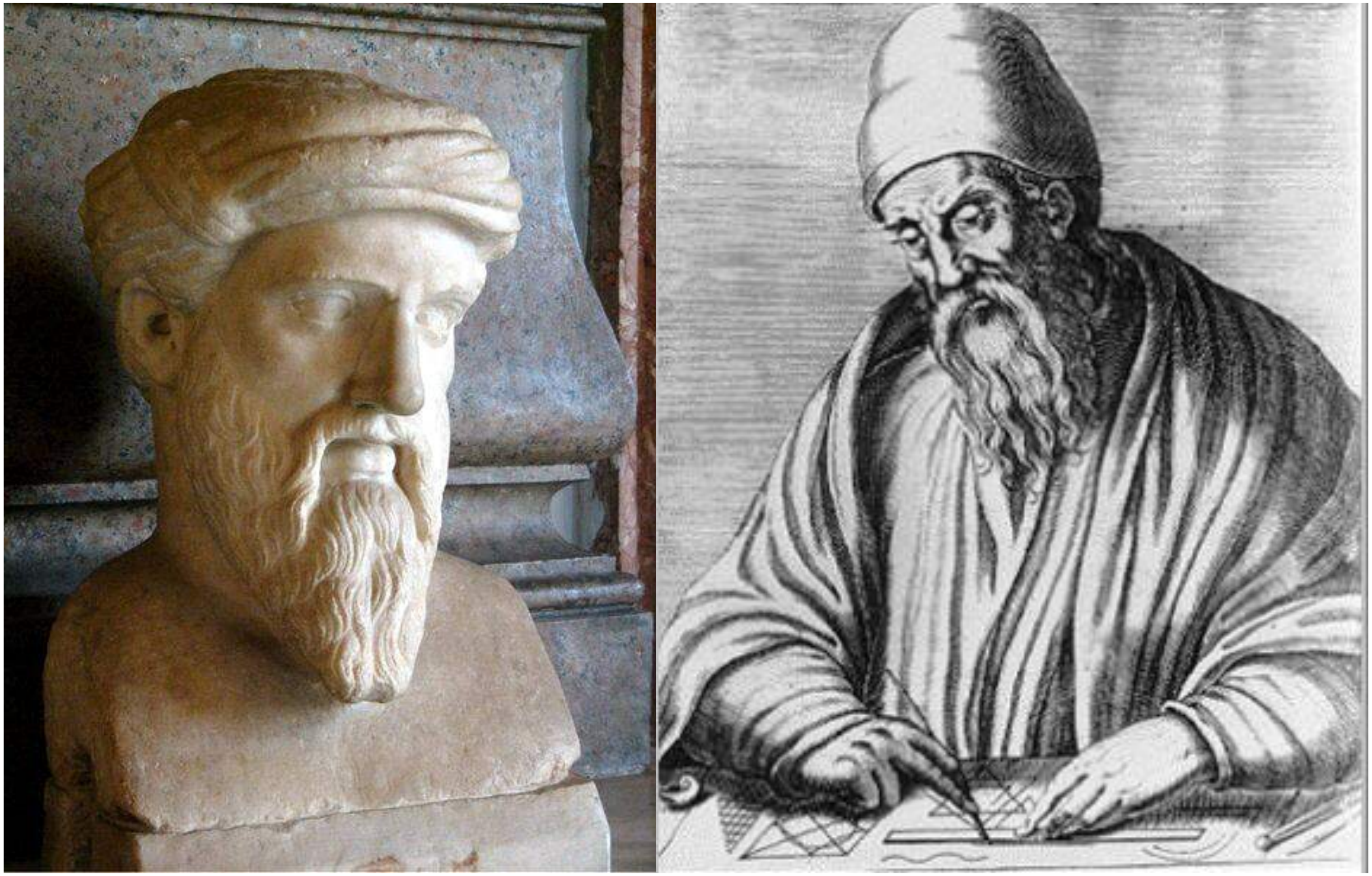
"The laws of nature are but the mathematical thoughts of God."

Euclid (323 - 283 BC)

"The Cosmos is all that is or ever was or ever will be. Our feeblest contemplations of the Cosmos stir us - there is a tingling in the spine, a catch in the voice, a faint sensation, as if a distant memory, of falling from a height. We know we are approaching the greatest of mysteries."

Carl Sagan (1934 - 1996)

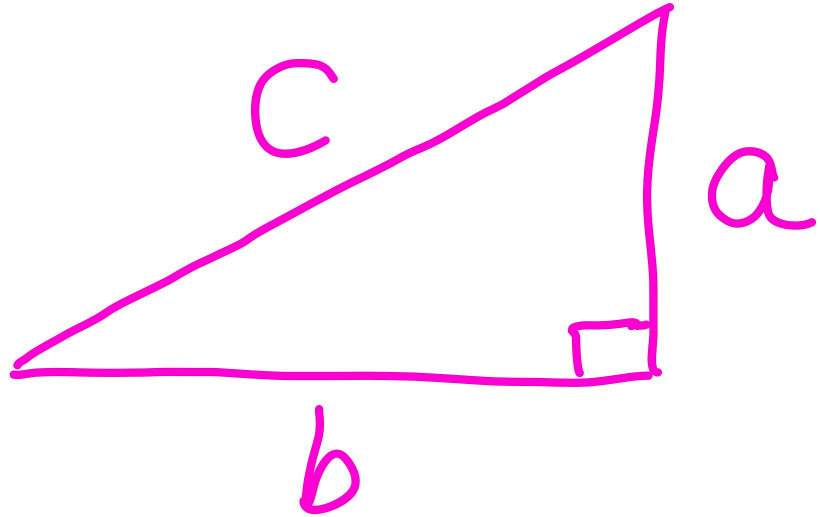
Euclidean Geometry



Pythagoras (570 - 495 BC) and Euclid (323 - 283 BC)

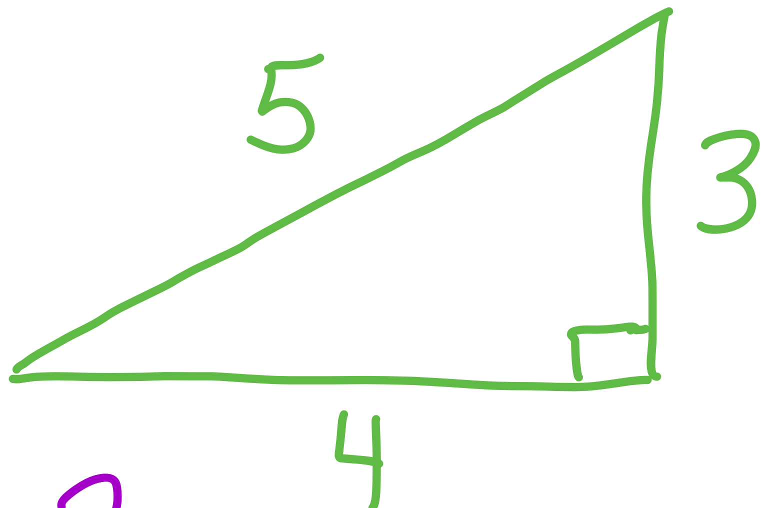
The Rule of Pythagoras

$$a^2 + b^2 = c^2$$



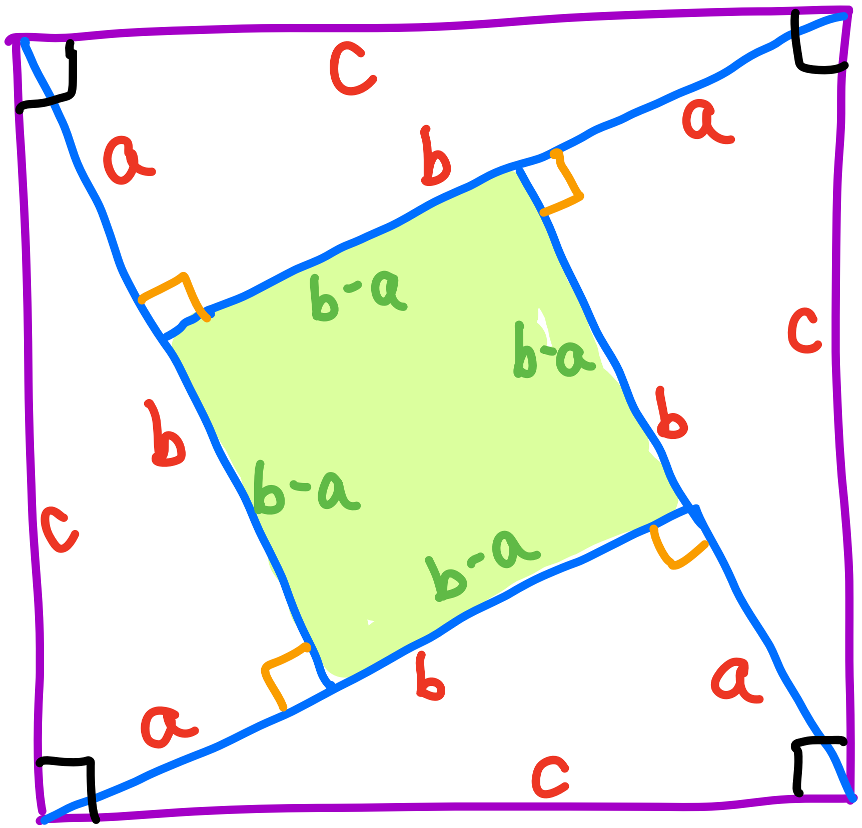
For example,

$$3^2 + 4^2 = 5^2$$



How do we prove this?

Proof #1



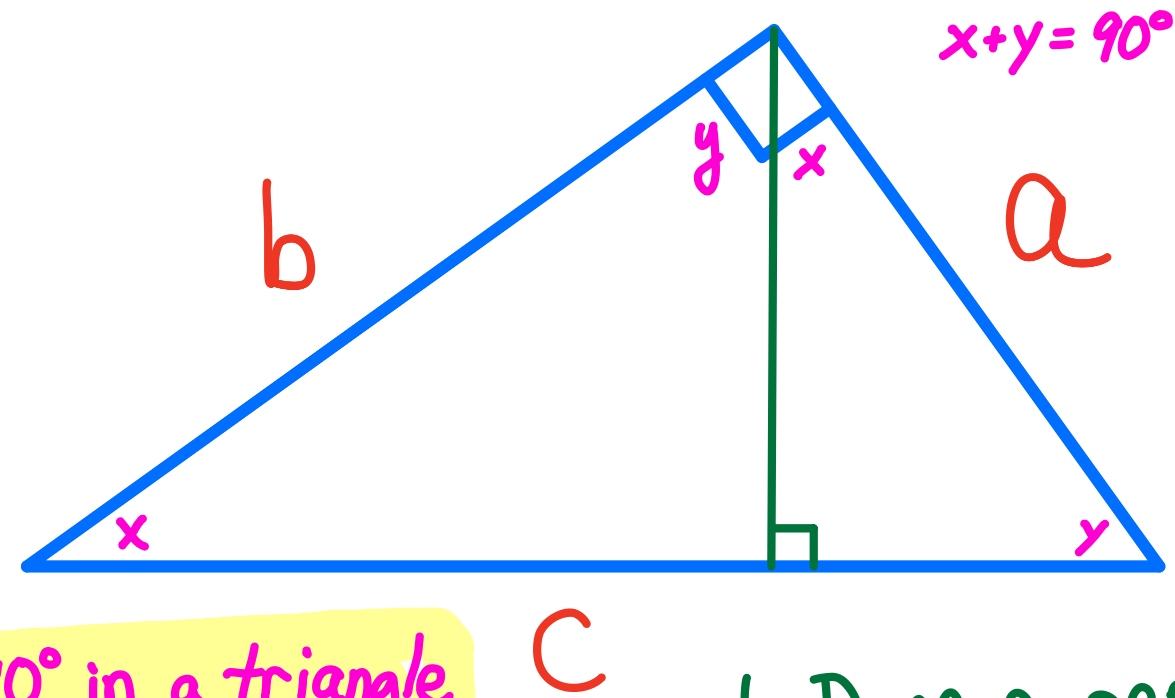
area of 4 right triangles: $4 \cdot \frac{1}{2}ab$
area of inner square: $(b-a)^2$
area of everything: c^2

$$4 \cdot \frac{1}{2}ab + (b-a)^2 = c^2$$

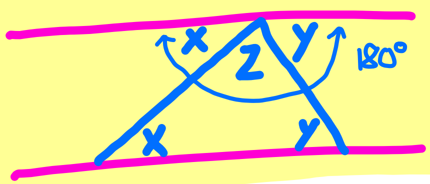
$$2ab + (b^2 - 2ab + a^2) = c^2$$

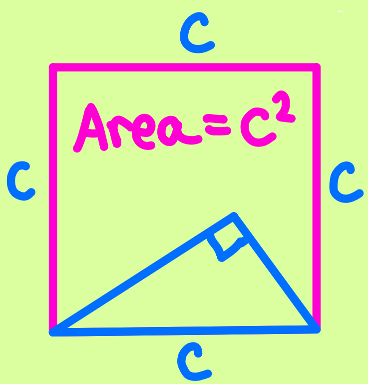
$$a^2 + b^2 = c^2$$

Proof #2



There are 180° in a triangle since

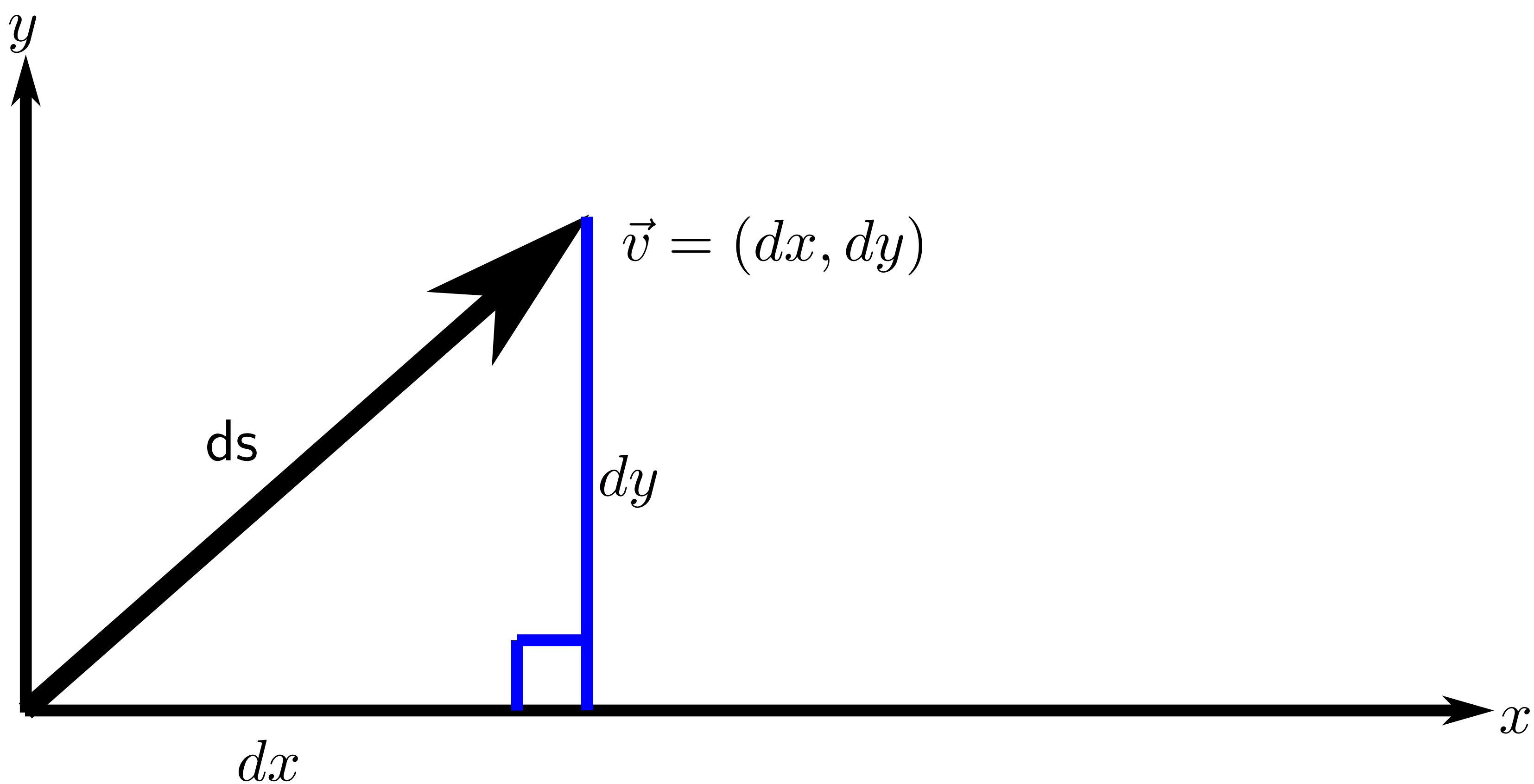




The areas of similar triangles are proportional to their hypotenuses squared.

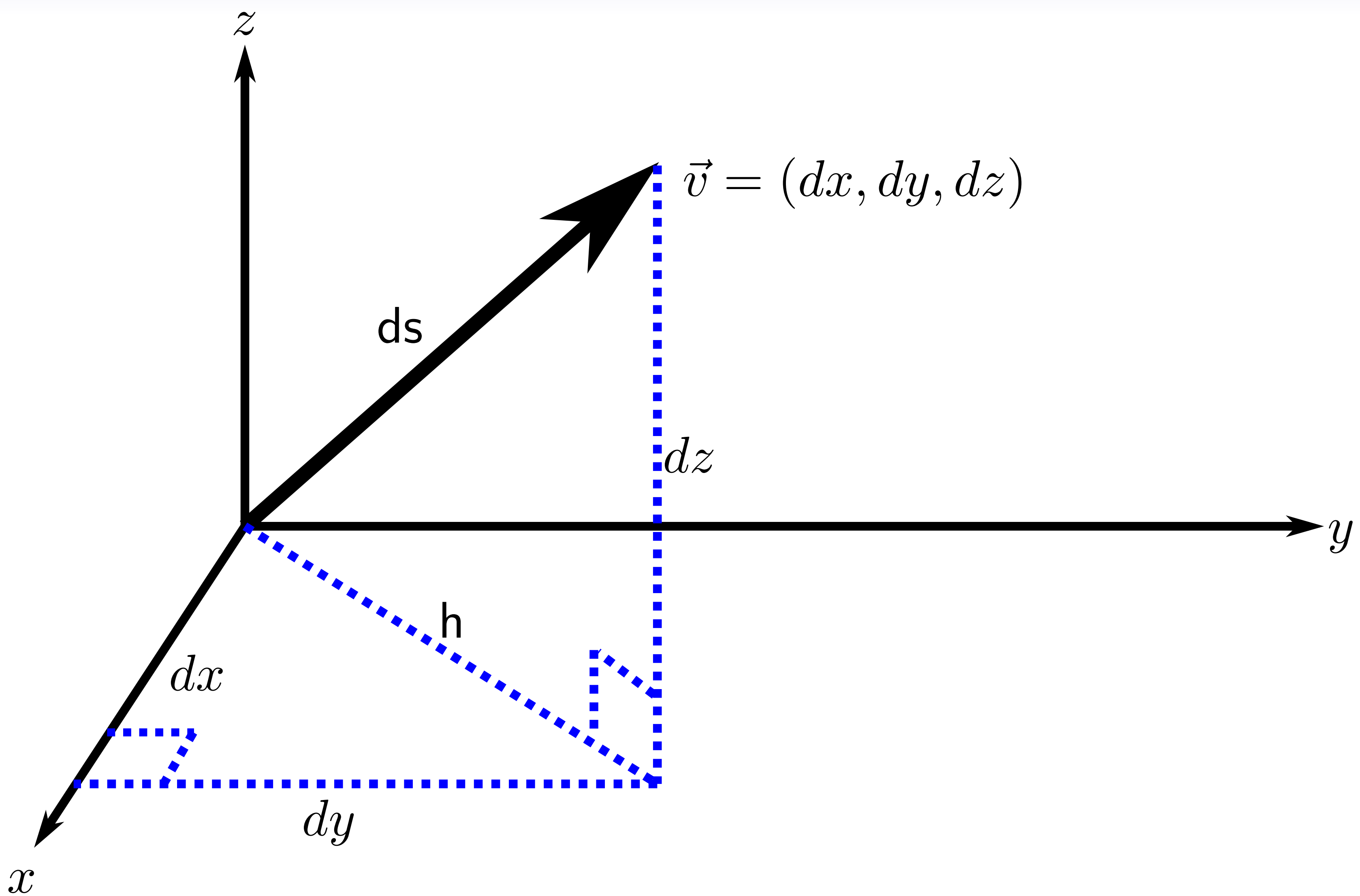
1. Drop a perpendicular
2. Get 3 similar triangles (all same angles)
3. Area of 2 smallest add up to largest.
4. Thus, $a^2 + b^2 = c^2$

The Rule of Pythagoras



Rule of Pythagoras: $ds^2 = dx^2 + dy^2$.

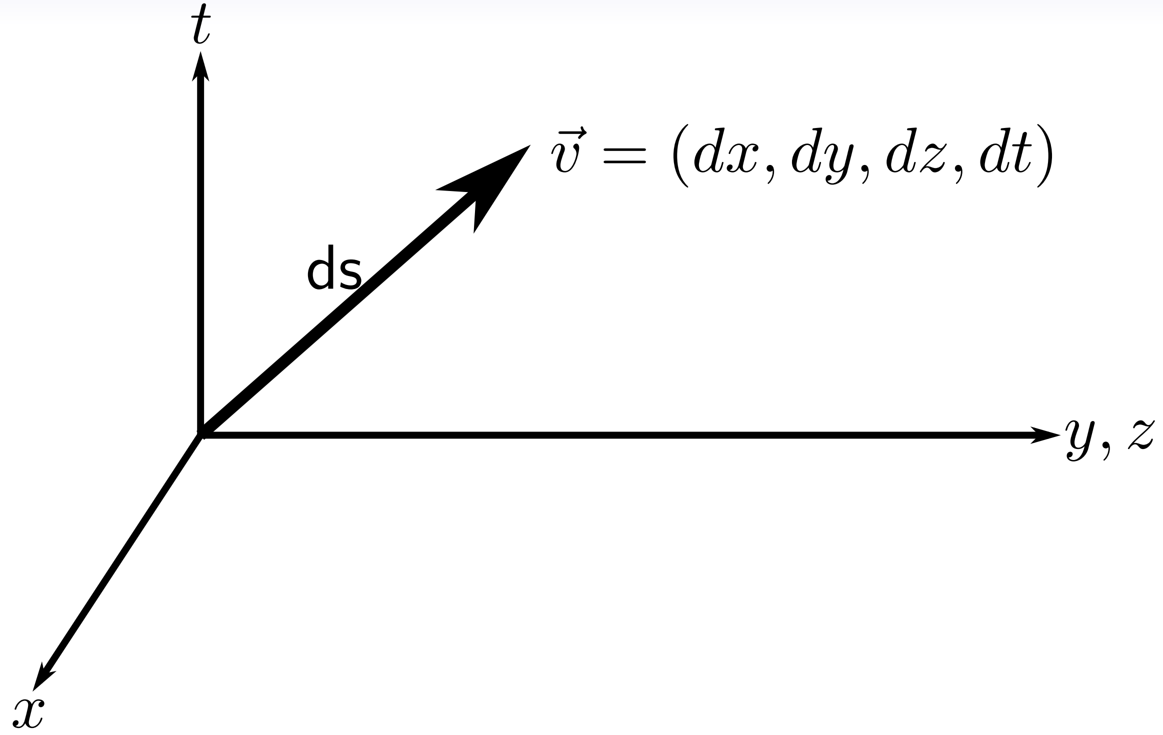
The Rule of Pythagoras in 3 Dimensions



Using the rule of Pythagoras twice, we get

$$ds^2 = h^2 + dz^2 = dx^2 + dy^2 + dz^2.$$

The Rule of Pythagoras in 4 Dimensions

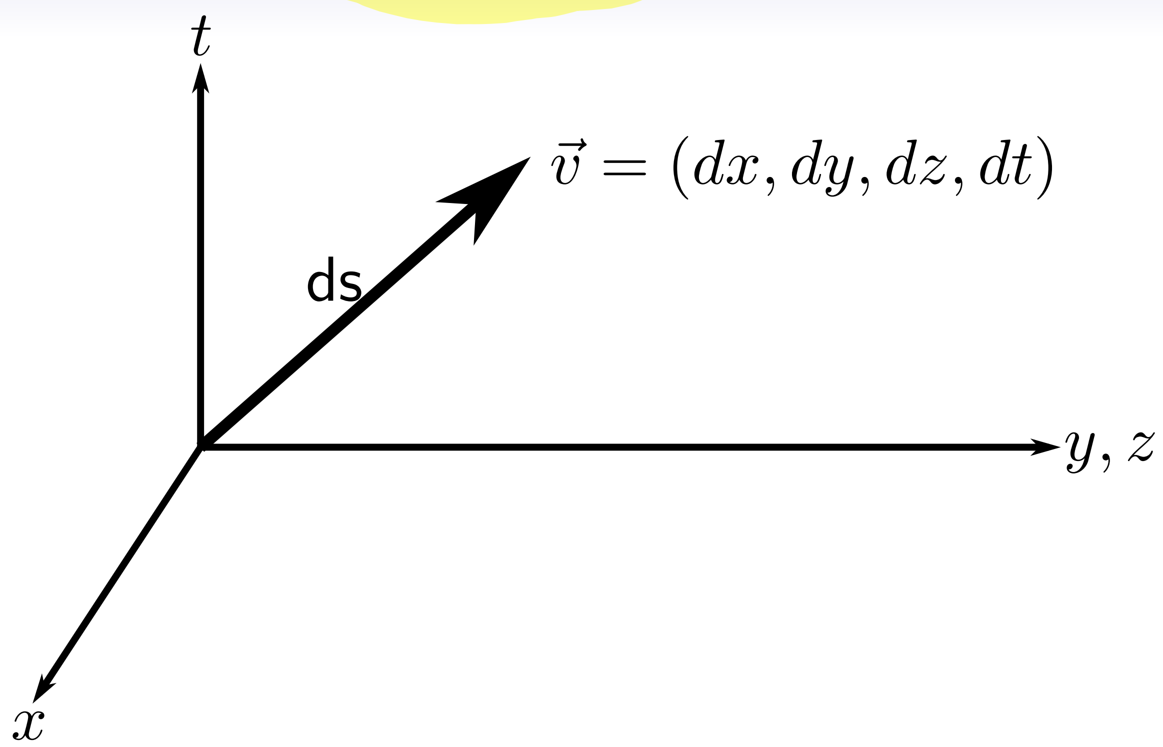


Similarly, following the pattern, in 4 dimensions we get

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2.$$

There is nothing different about any of these 4 dimensions, unlike space and time which are clearly different. How can we modify the geometry to make 1 dimension different from the other 3?

Special Relativity is a Minus Sign in the Rule of Pythagoras

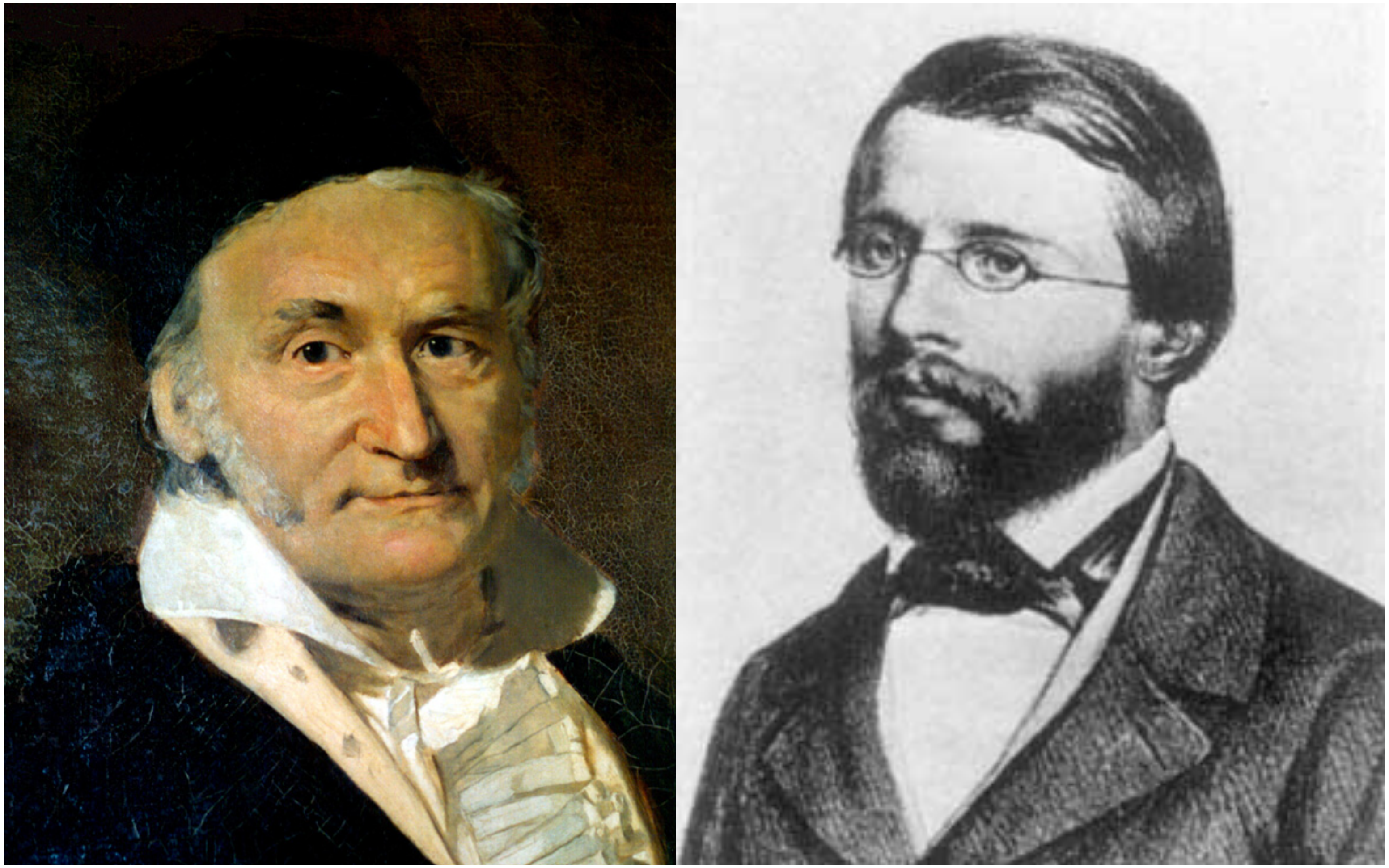


Special Relativity results from studying the geometry of the Minkowski spacetime, where the lengths of vectors are defined by

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2.$$

Notice that there is clearly something different about one of the dimensions now. But are we allowed to do this? Sure, why not!

Riemannian Geometry

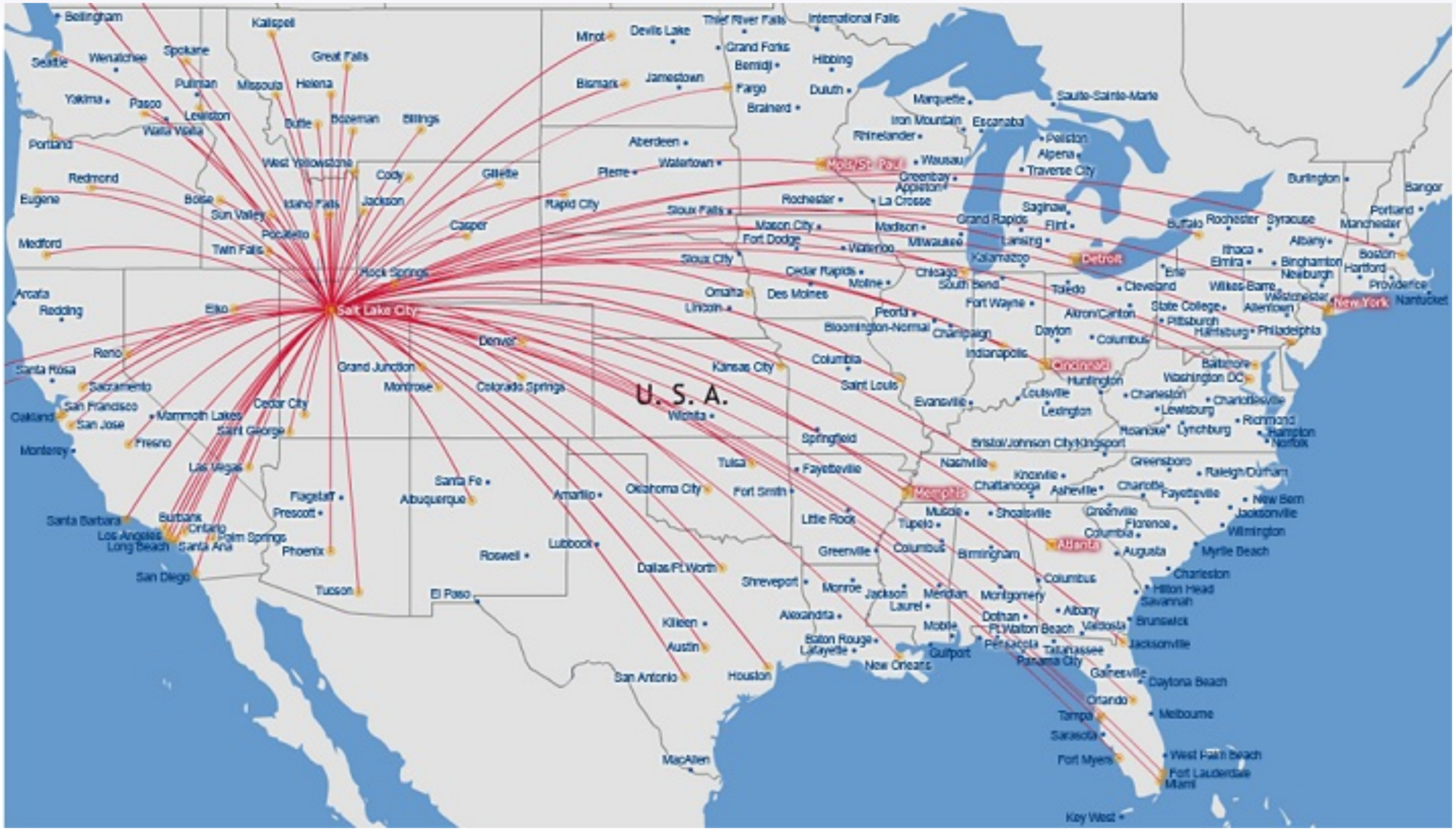


Carl Friedrich Gauss (1777 - 1855) and
Bernhard Riemann (1826 - 1866)

Spheres are Intrinsically 2 Dimensional

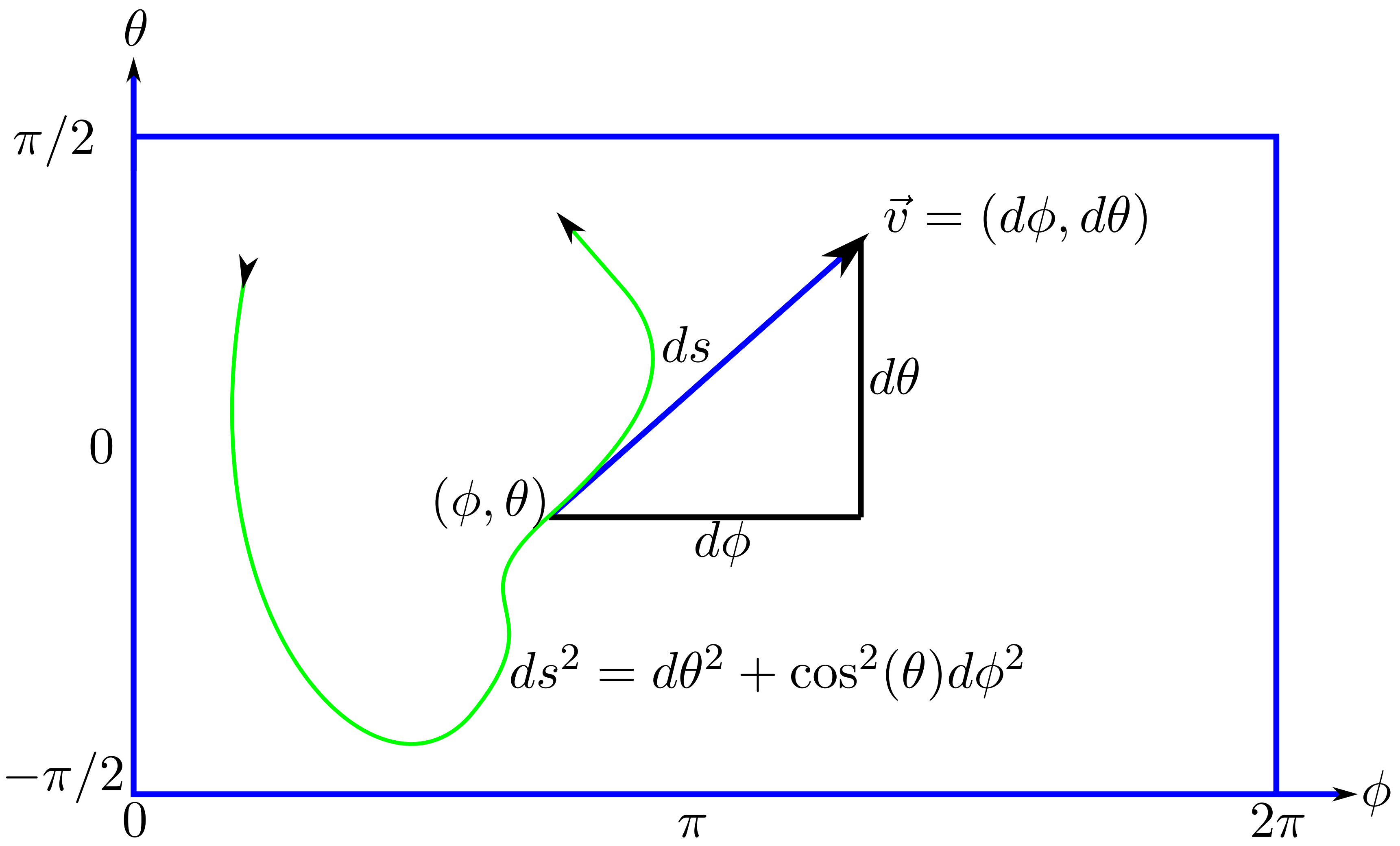
Gauss and Riemann realized that the geometry of a sphere or any other surface may be described by looking at a collection of flat maps of the surface, called an atlas. For example, if you want to drive to the Grand Canyon, you might use an atlas of the US to plan your trip. However, maps usually distort distances somewhat: Greenland and Antarctica are not as big as they appear in Mercator projections of the Earth, as seen on the right.

The Geometry of a Surface



Since maps typically distort distances, the shortest distance between two points is not necessarily a straight line on the map. The true length of a curve may be computed by integrating the lengths of the velocity vectors to the curve. Gauss's key insight was realizing that *all* of the geometry intrinsic to the surface was determined by knowing the length of every vector.

The Geometry of the Unit Sphere



This modified "Rule of Pythagoras" captures the geometry of the unit sphere S^2. What other geometries might there be?

Other Geometries

Well-known examples of other geometries include:

$ds^2 = dx^2 + dy^2$	(Flat 2D Euclidean space)
$ds^2 = dx^2 + \sin^2(x)dy^2$	(The sphere of radius 1)
$ds^2 = R^2dx^2 + R^2\sin^2(x)dy^2$	(The sphere of radius R)
$ds^2 = dx^2 + \sinh^2(x)dy^2$	(Hyperbolic space)
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$$ds^2 = 4dx^2 + 9dy^2 + 16dz^2 - 25dt^2$$

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$$\begin{aligned} ds^2 &= 4dx^2 + 9dy^2 + 16dz^2 - 25dt^2 \\ &= (2dx)^2 + (3dy)^2 + (4dz)^2 - (5dt)^2 \end{aligned}$$

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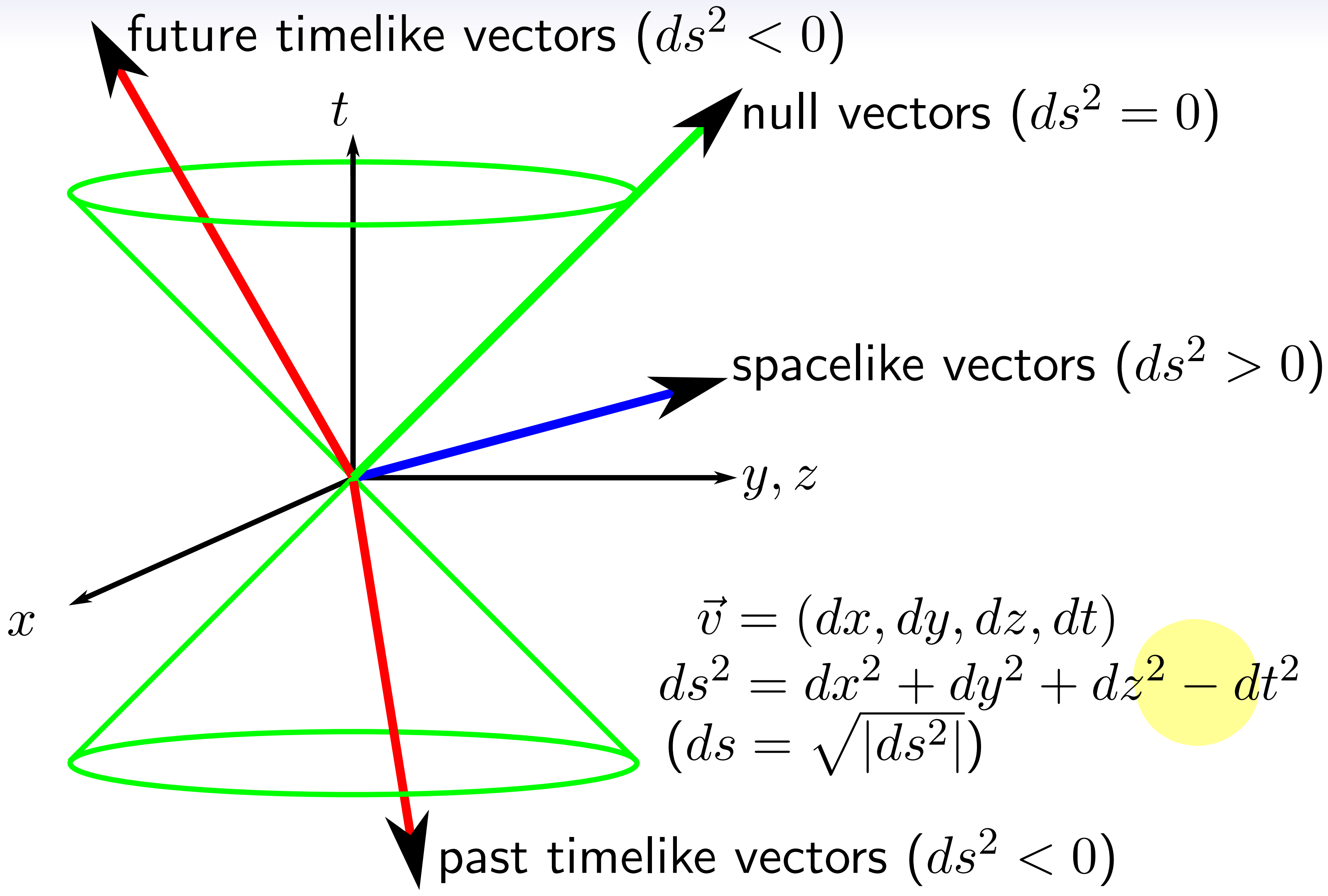
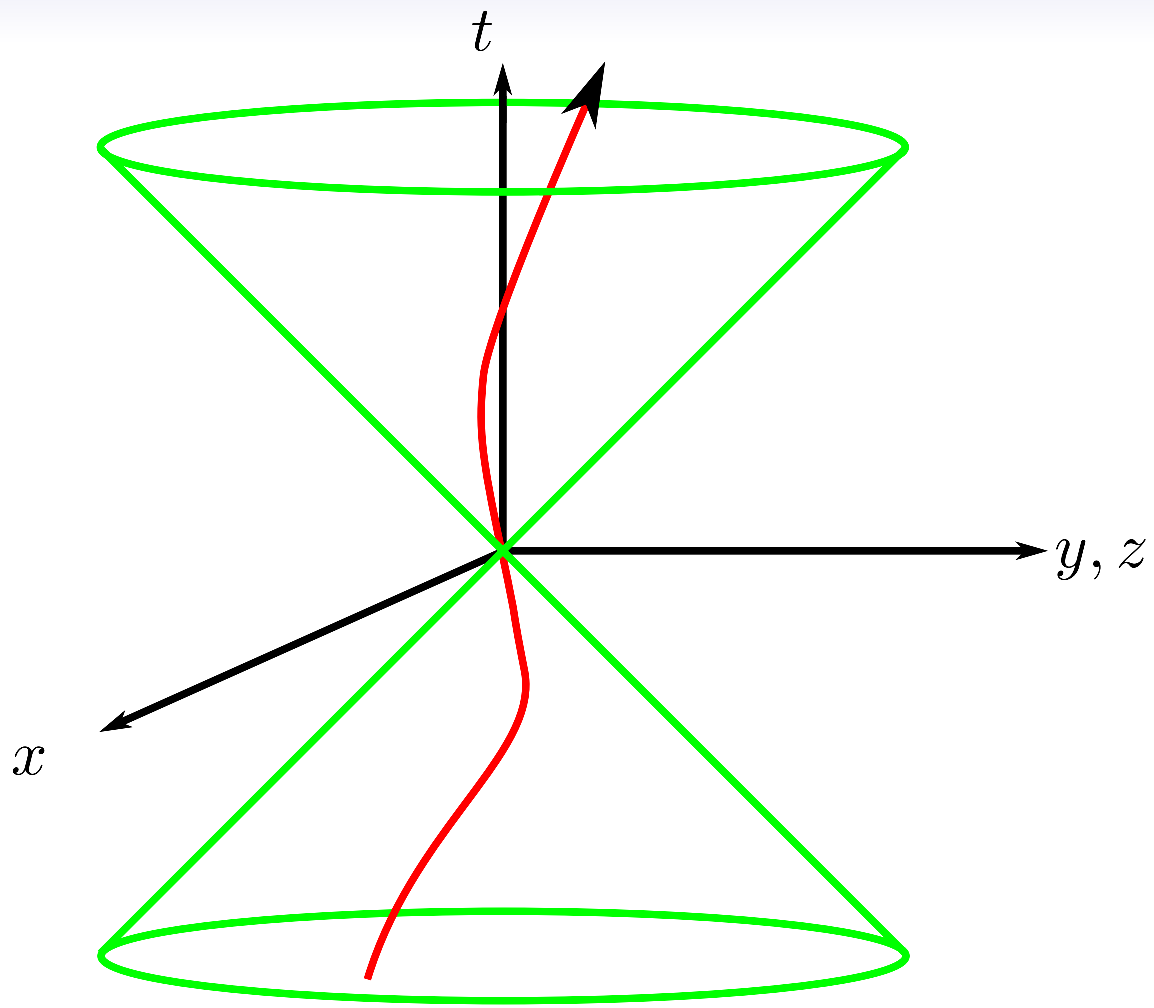
The Geometry of the Minkowski Spacetime



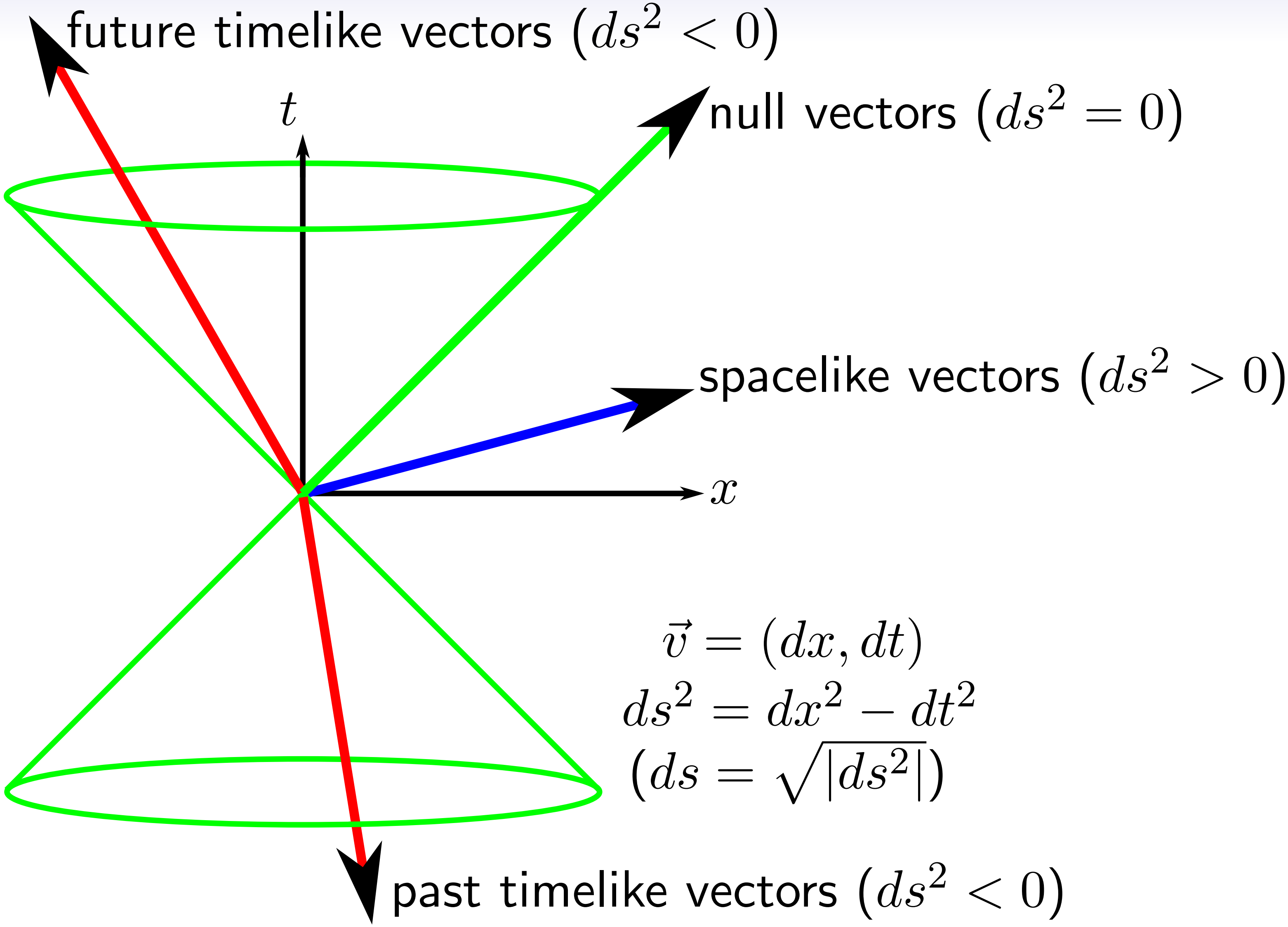
The Minkowski spacetime has 3 types of vectors: spacelike, timelike, and null. The geometry of the null cone naturally divides timelike vectors into future and past components.

The Geometry of the Minkowski Spacetime



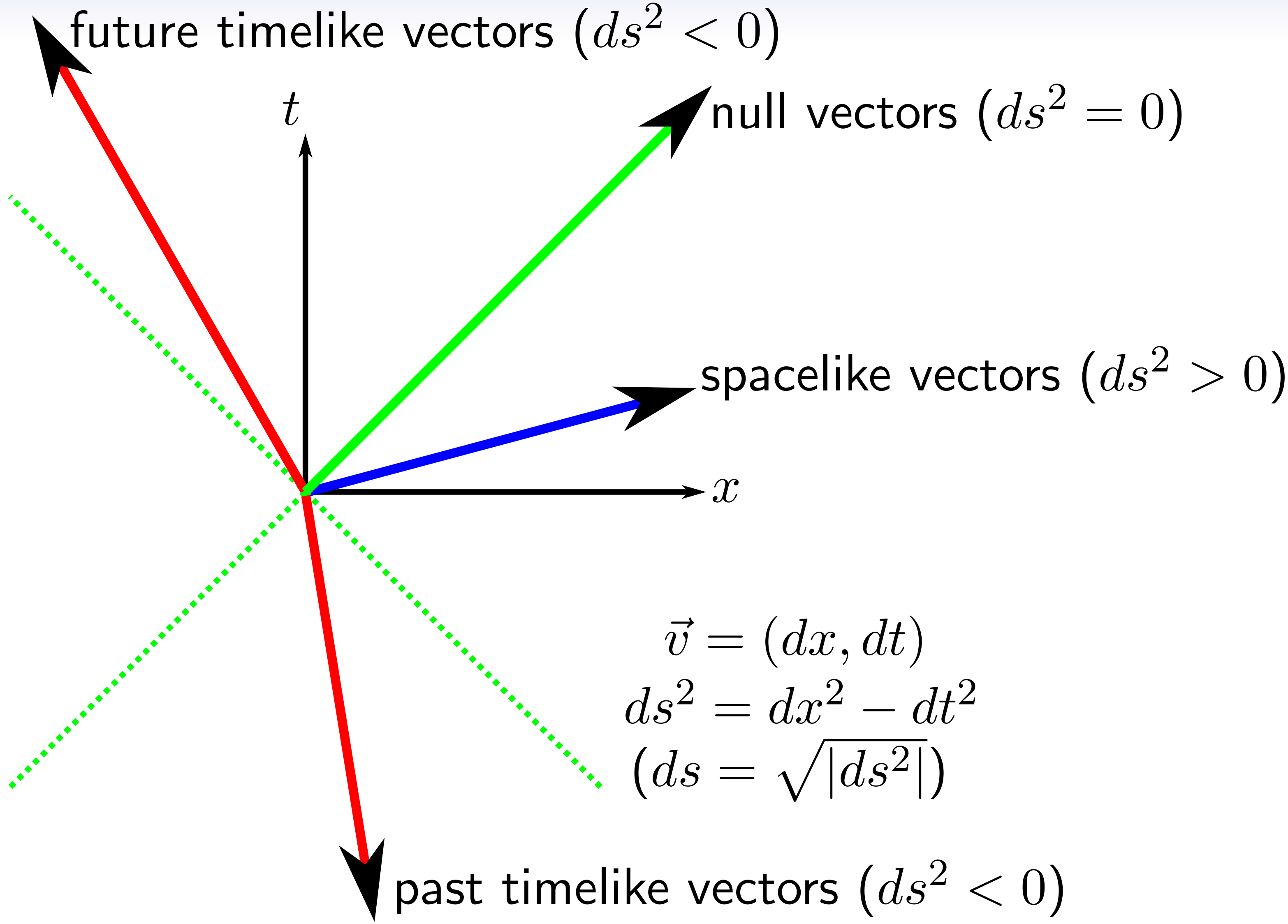
Principle: Geometric quantities correspond to physical observables. For example, the length of your *world line*, which always goes in future timelike directions, equals the time you experience.

The 1+1 Dimensional Minkowski Spacetime



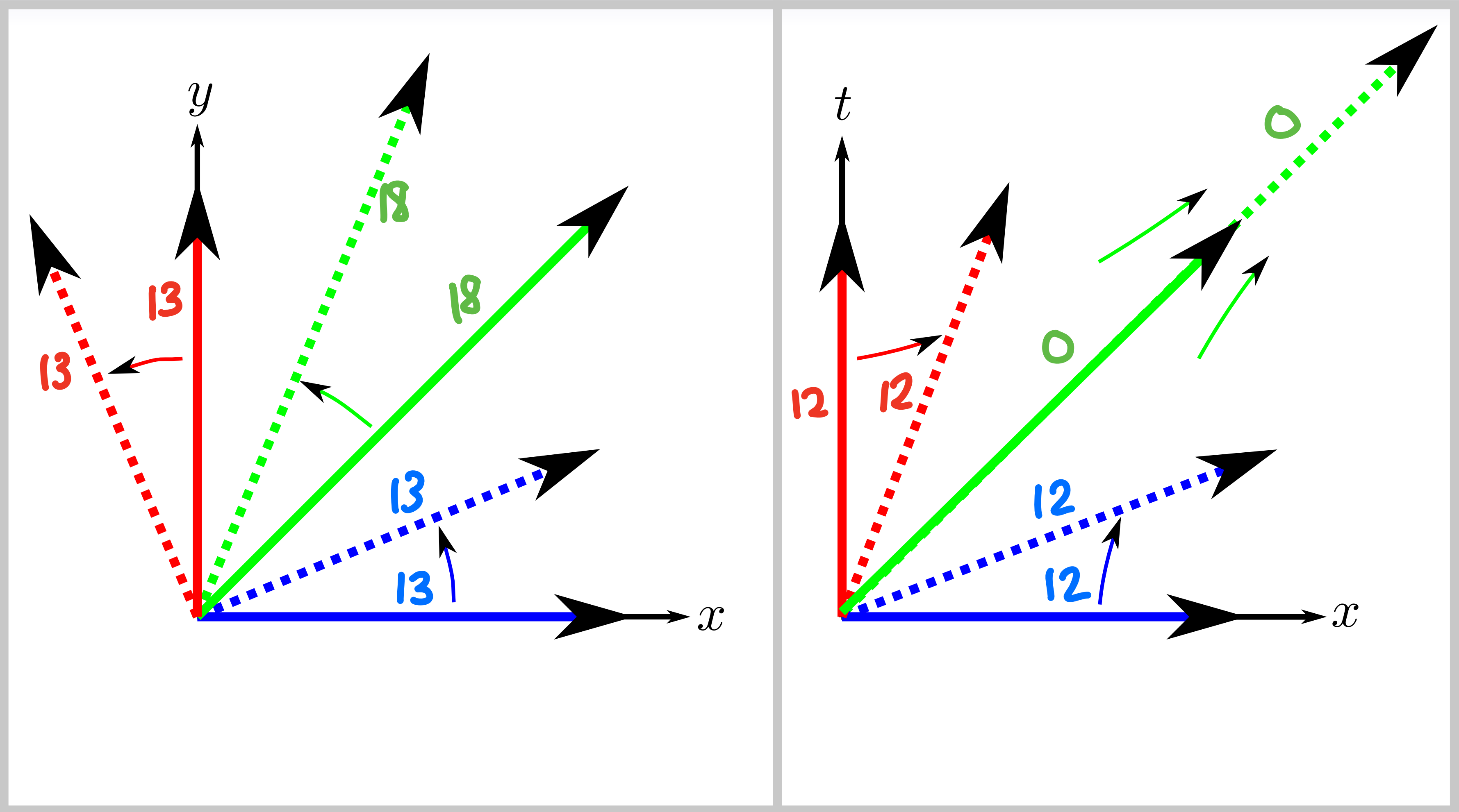
Let's remove the y and z directions for now. The null cone is now really a null "X."

The 1+1 Dimensional Minkowski Spacetime



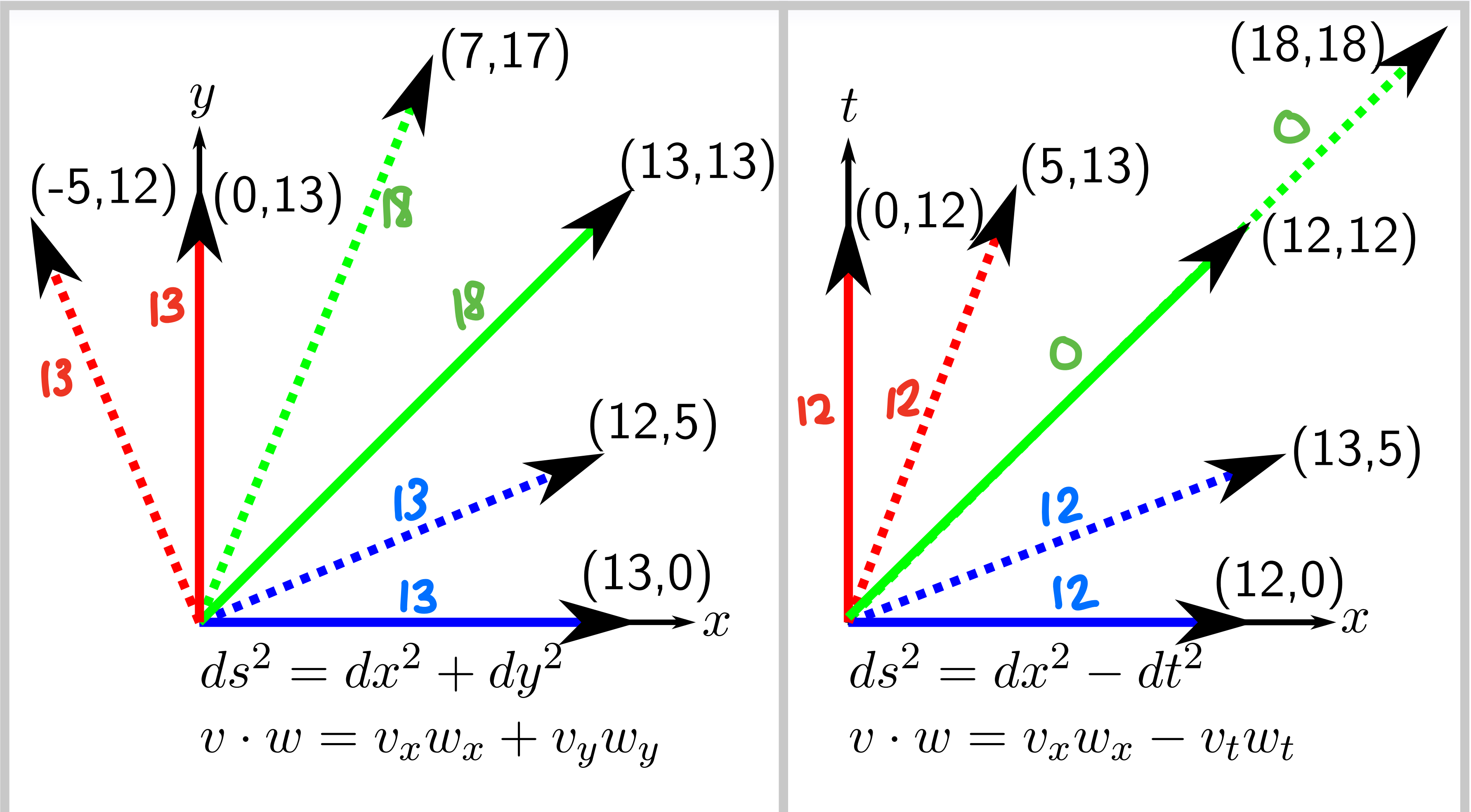
What do “rotations” (which fix the origin and ds^2 for all vectors) look like in this new geometry?

Rotations (left diagram) and Boosts (right diagram)



Analogous to rotations in Euclidean space, *boosts* are linear transformations of the Minkowski spacetime which fix the origin and preserve the dot product between every pair of vectors.

Rotations (left diagram) and Boosts (right diagram)



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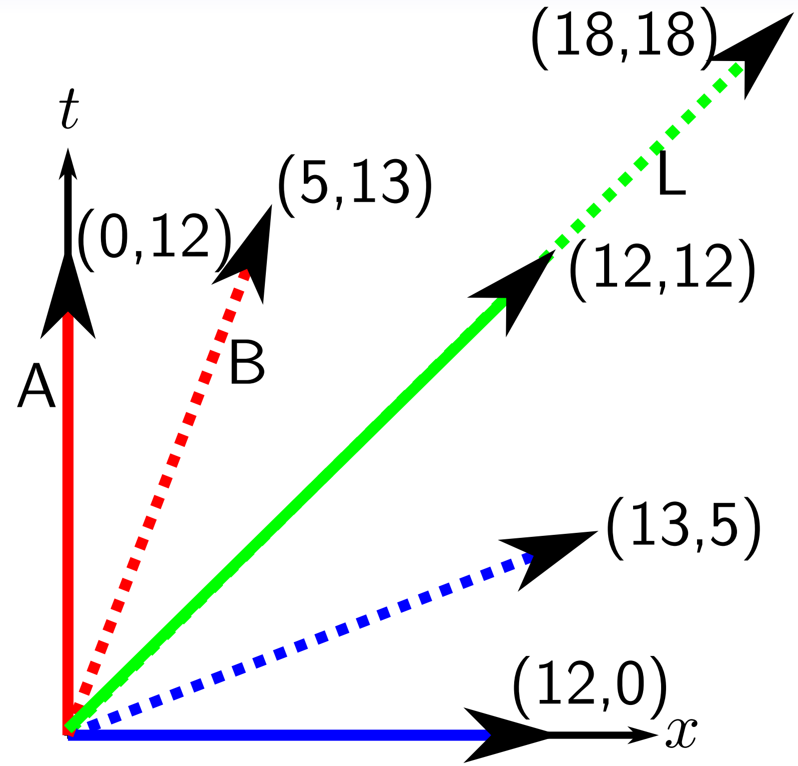
Relative Velocity

A: B has velocity 5/13.
A: L has velocity 1.

B: A has velocity -5/13 (guess).
B: L has velocity ... hmmm.



Suppose A, B, and L all start at the origin and then travel in straight lines as shown. How fast does L appear to be going according to B?

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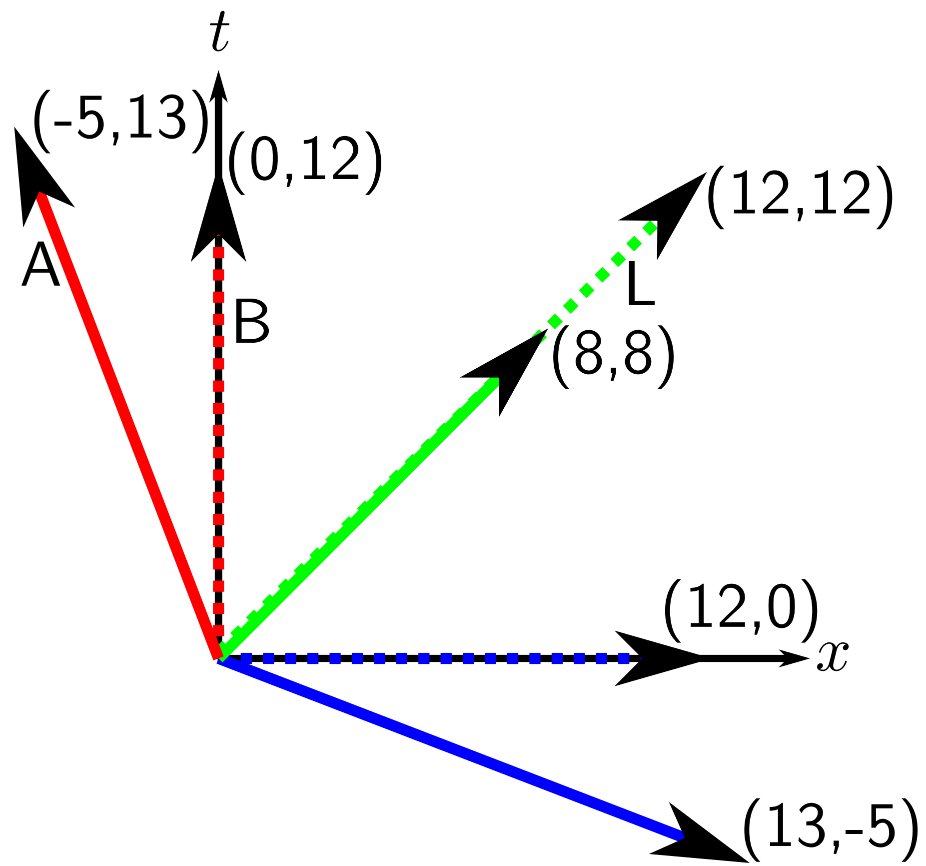

The trick is to rotate, or more precisely boost, the coordinates to make the answer to the problem clear. In this case, boost the coordinate chart so that B is now going in the $(0, 12)$ direction.

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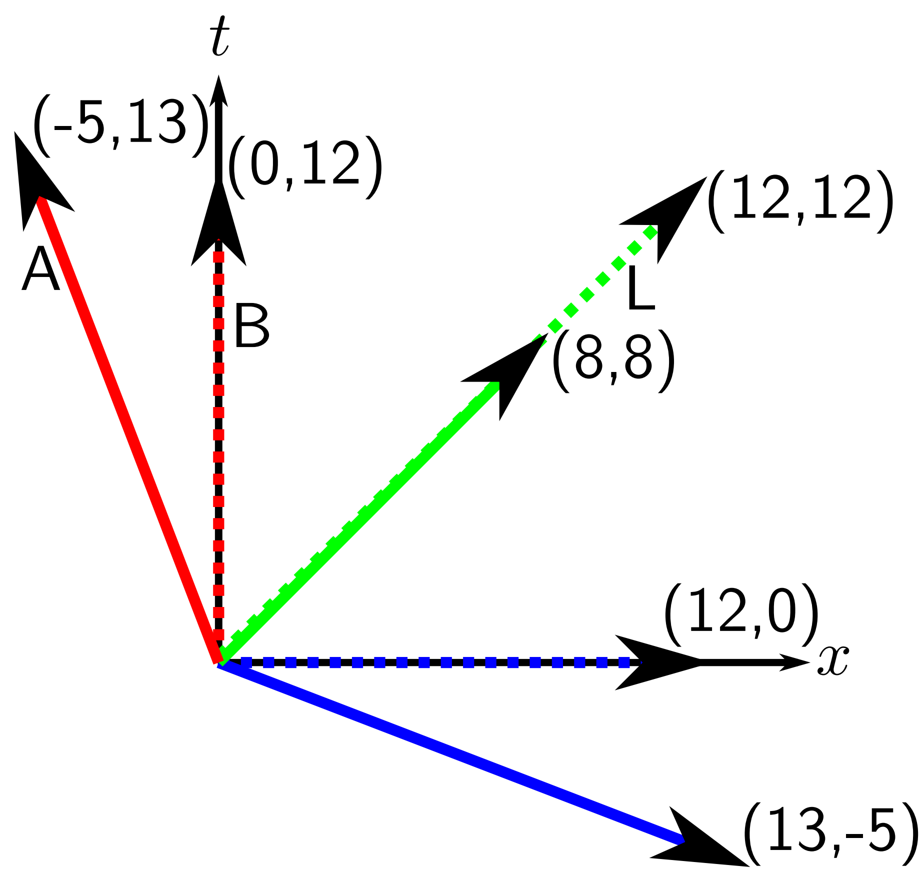

Both A and B agree that L has speed 1, even though A and B are moving with respect to one another. The geometry of the Minkowski spacetime requires that there be a special speed, namely speed 1, that is observed to be the same by all observers.

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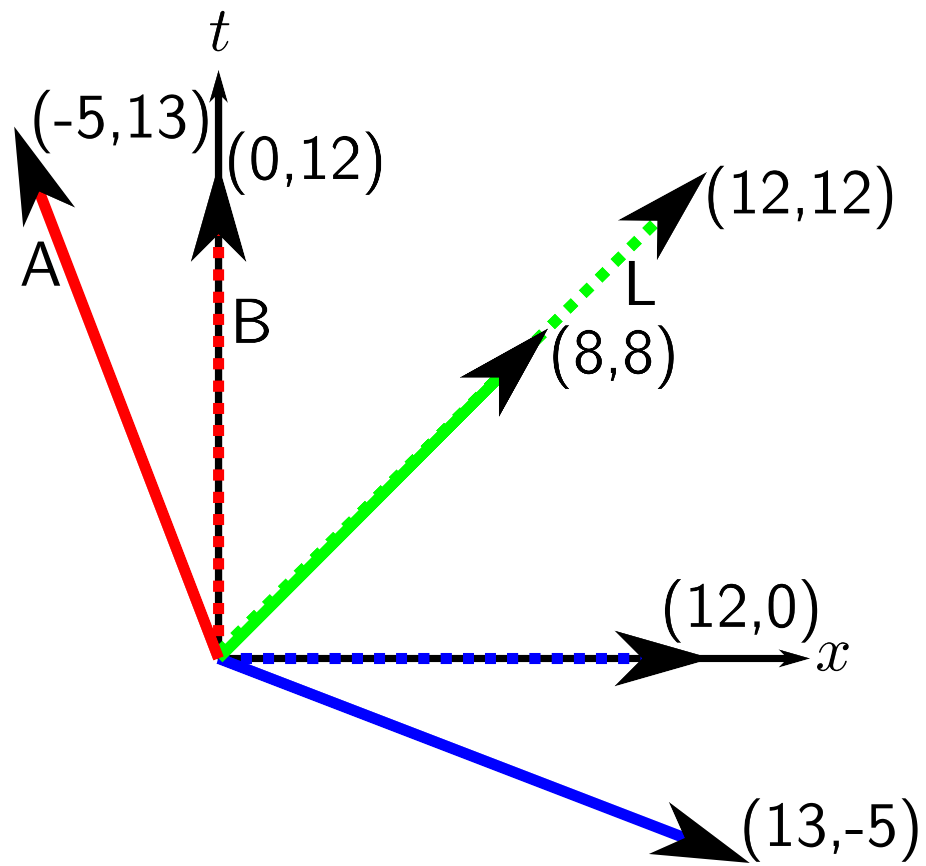
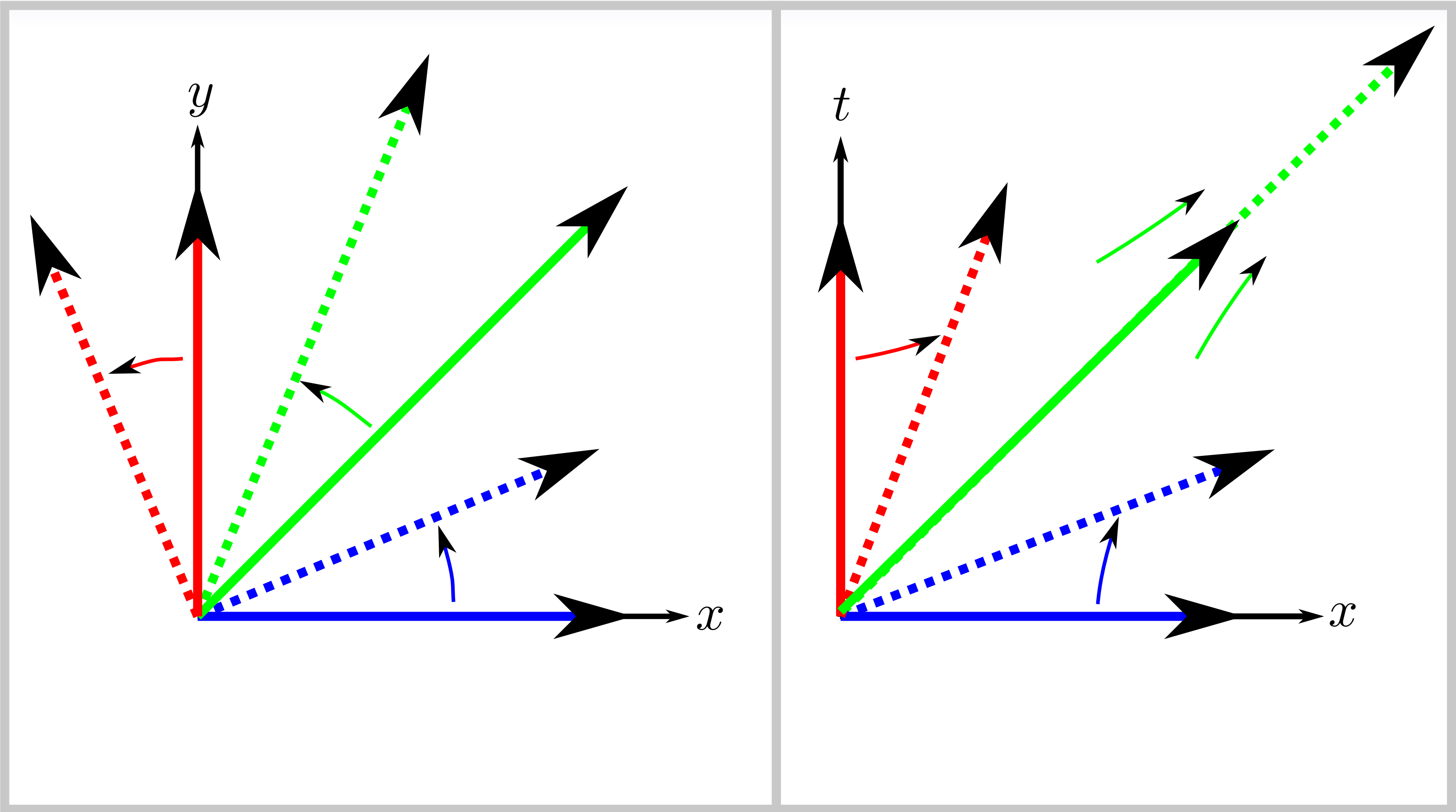


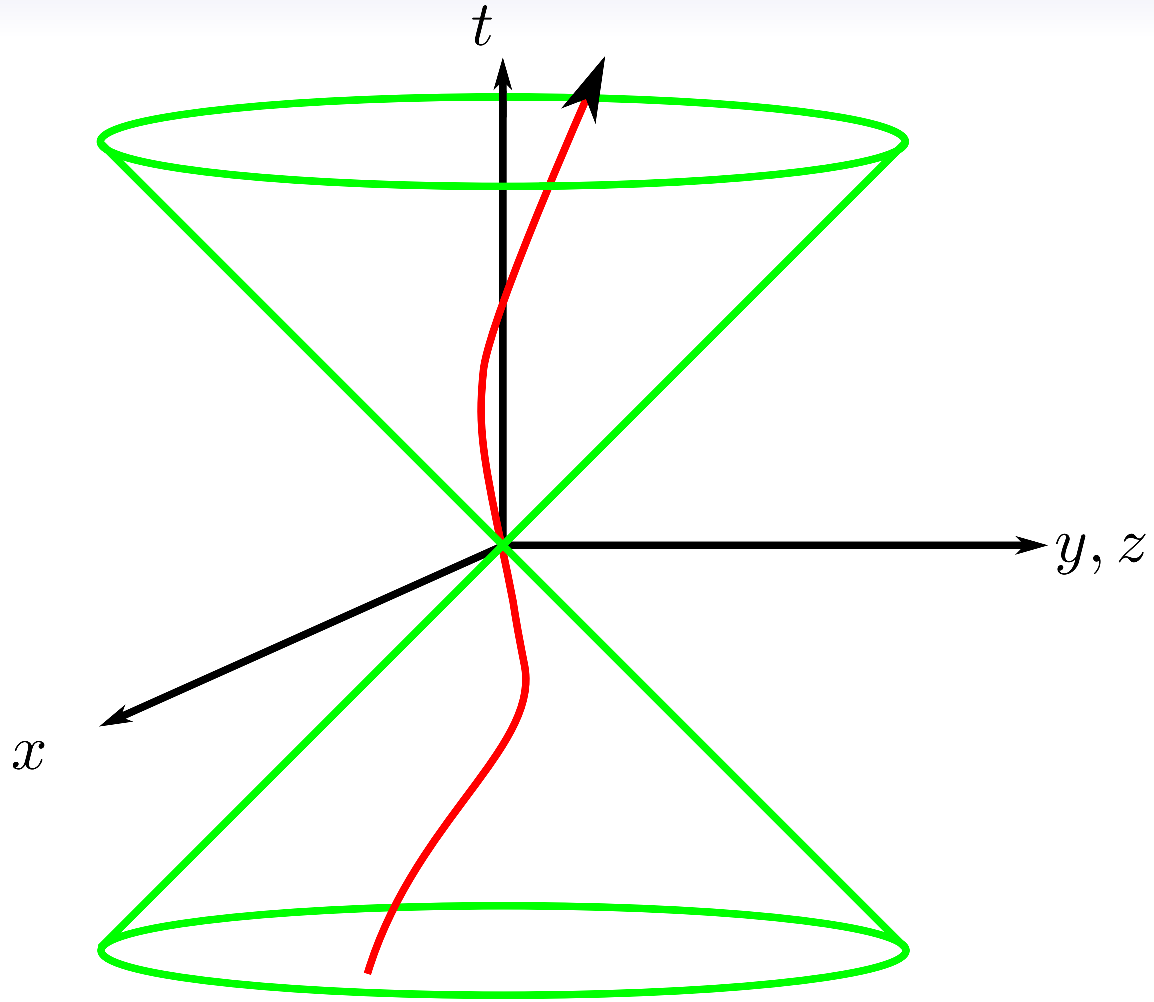
In 1887, the Michelson-Morley experiment determined that the speed of light, c, was the same in every reference frame. This result is consistent with the geometry of the Minkowski spacetime if we simply define $c = 1$. Distance and time then have the same units.

The Constancy of the Speed of Light



Boosts, by definition, preserve ds^2. Hence, null directions must be transformed into other null directions, which all have speed 1. Thus, if we define $c = 1$, the speed of light will be observed to be the same by all observers.

The Twin “Paradox”



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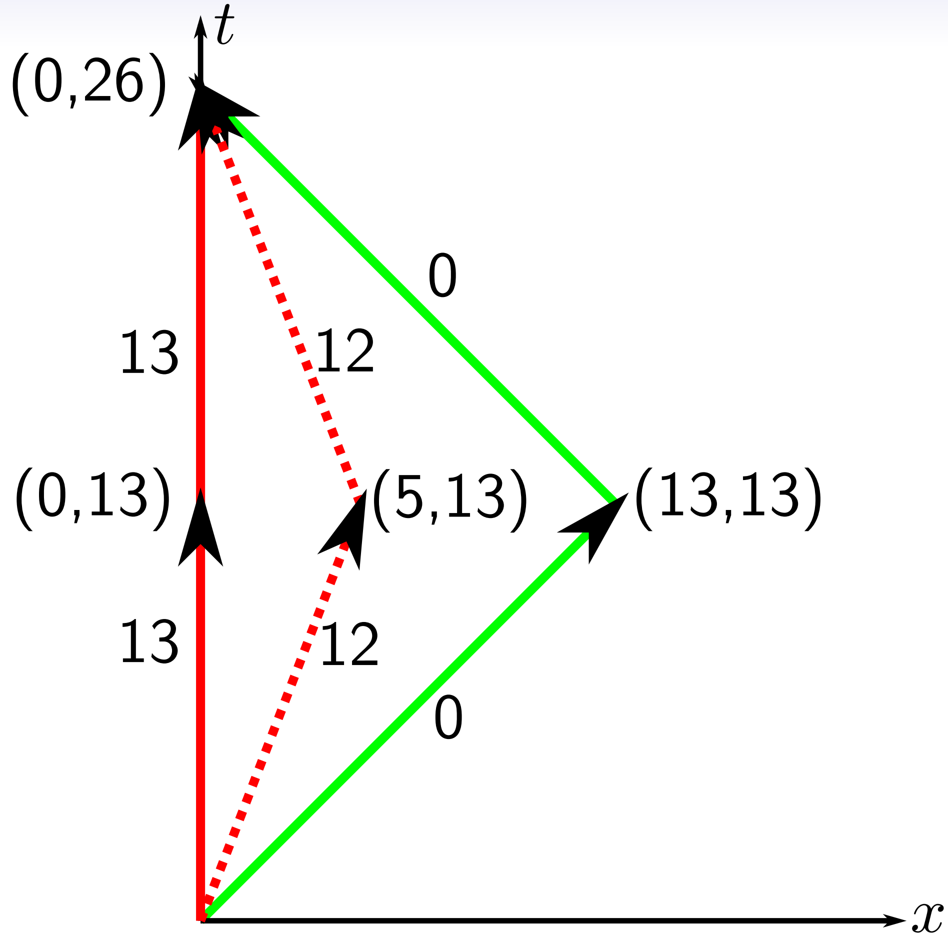
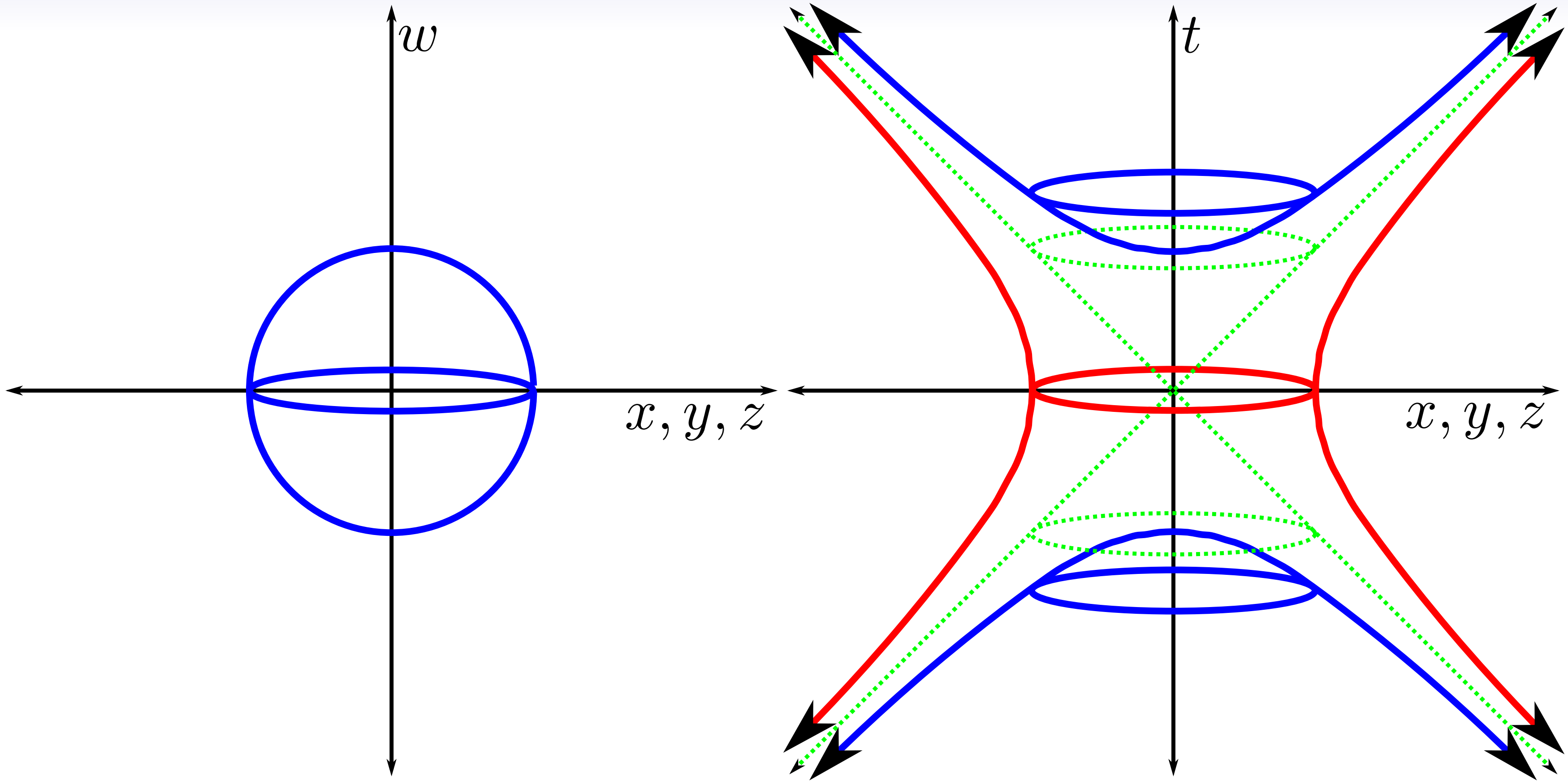
The Twin "Paradox" - just the geometry of spacetime



The *longest* distance between 2 points is a straight line! In this example, the twin who stays on Earth experiences 26 years while the twin who goes at velocity $5/13$, then velocity $-5/13$, experiences only 24 years. An astronaut going close to the speed of light following the green curve would experience almost no time.

The Unit "Spheres" of the Minkowski Spacetime



$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \qquad ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

The Minkowski spacetime has, in some sense, 3 different kinds of spheres, depending on whether the distance squared from the origin is positive, negative, or zero, as drawn on the right. For comparison, the unit sphere of Euclidean space is drawn on the left.

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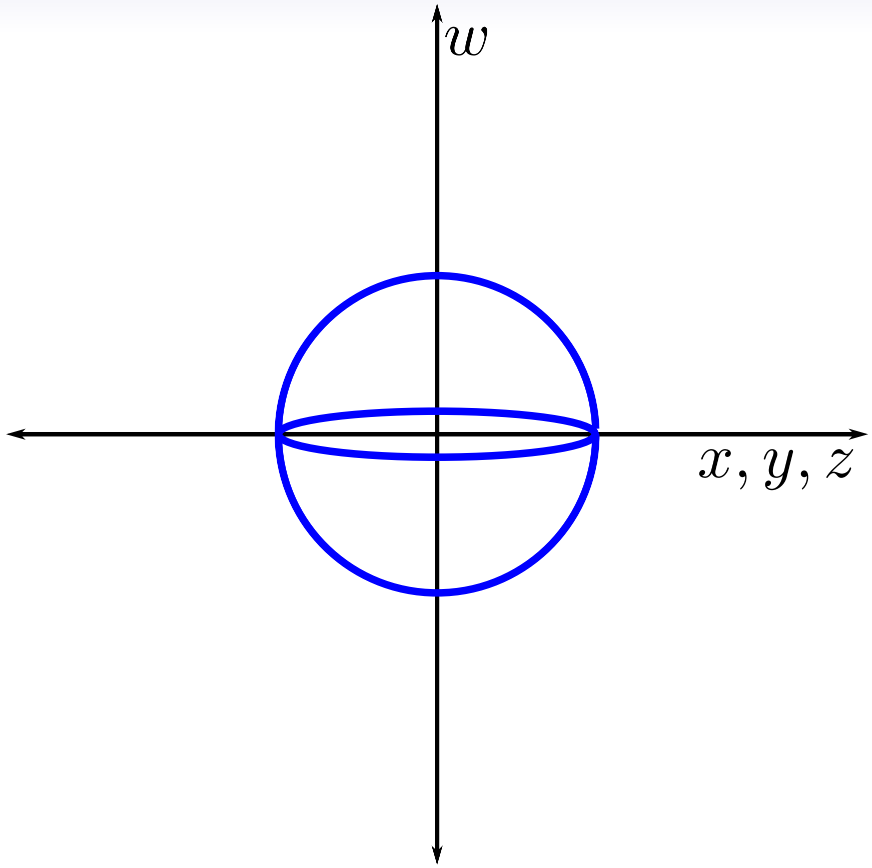

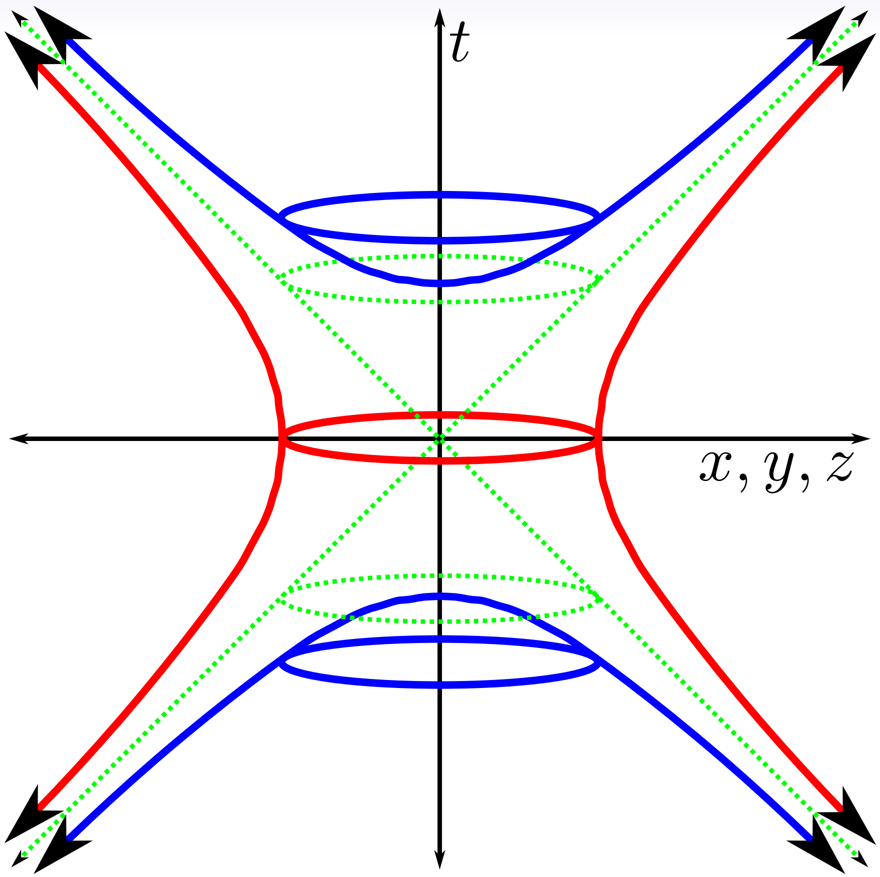

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Just as the unit sphere in Euclidean space has constant curvature, the blue and red hypersurfaces drawn on the right also have constant curvature. Can you guess the geometry of each blue hypersurface?

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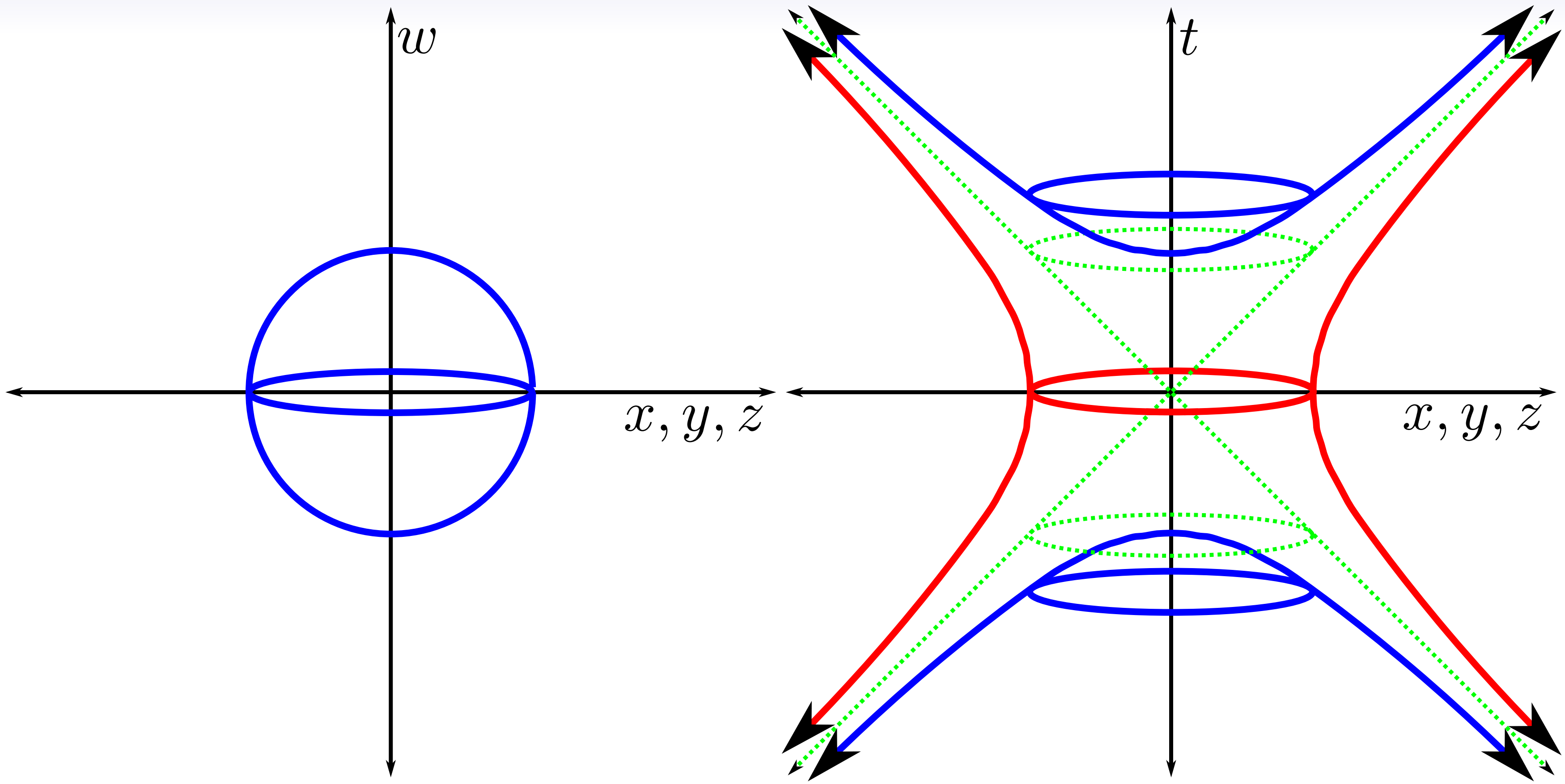

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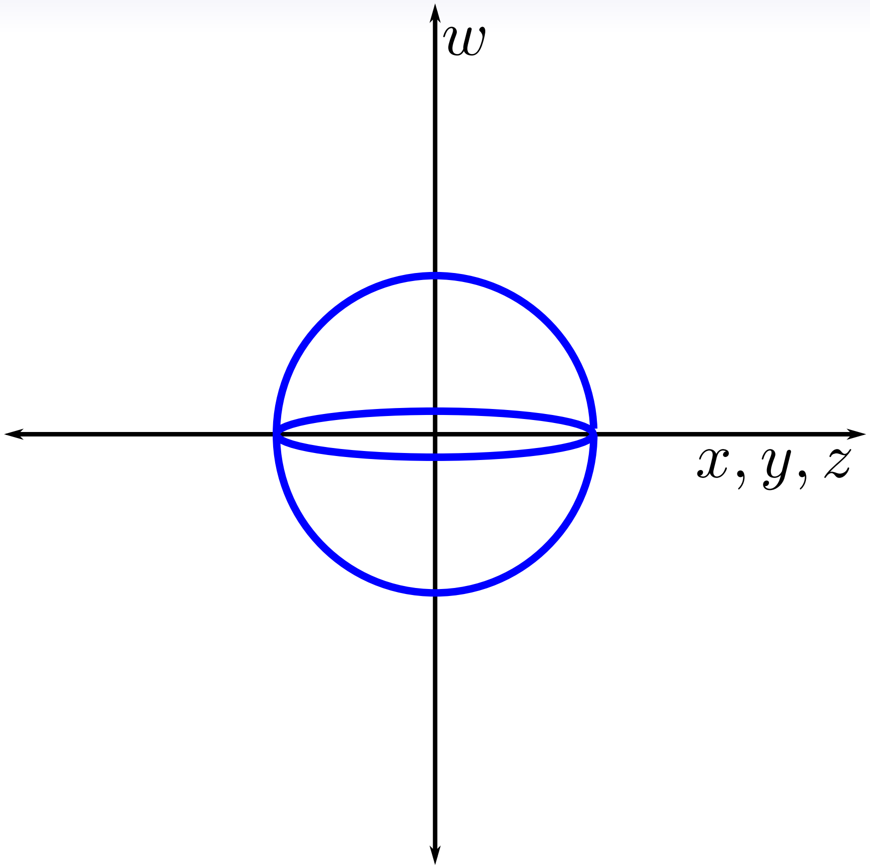
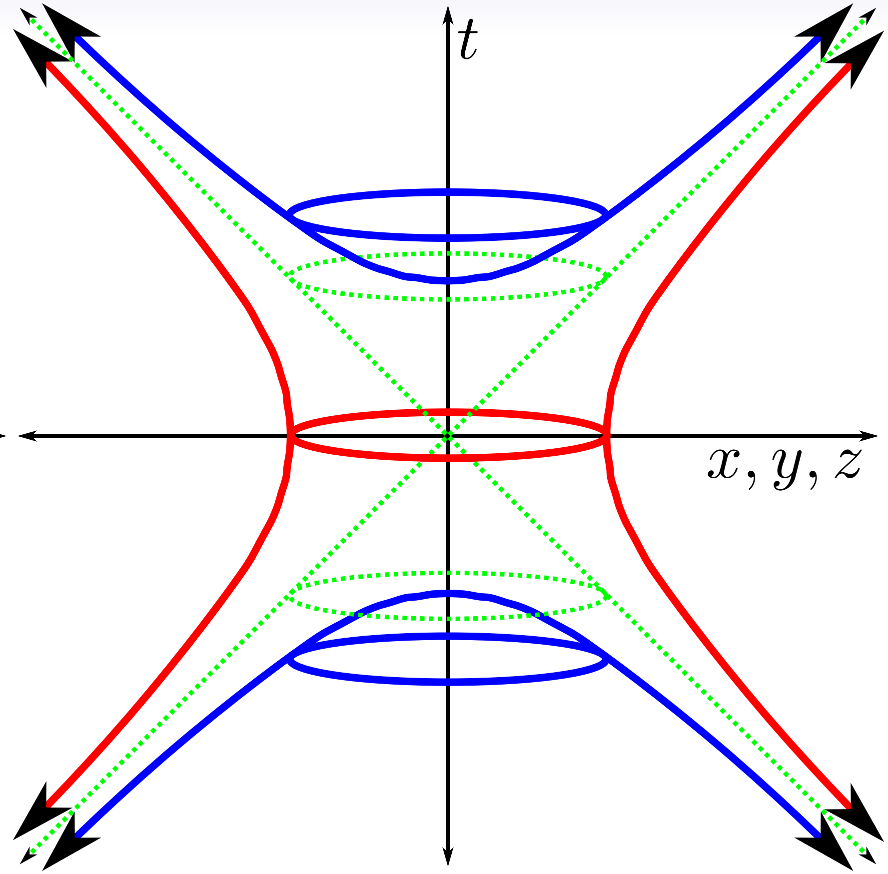
The Unit "Spheres" of the Minkowski Spacetime

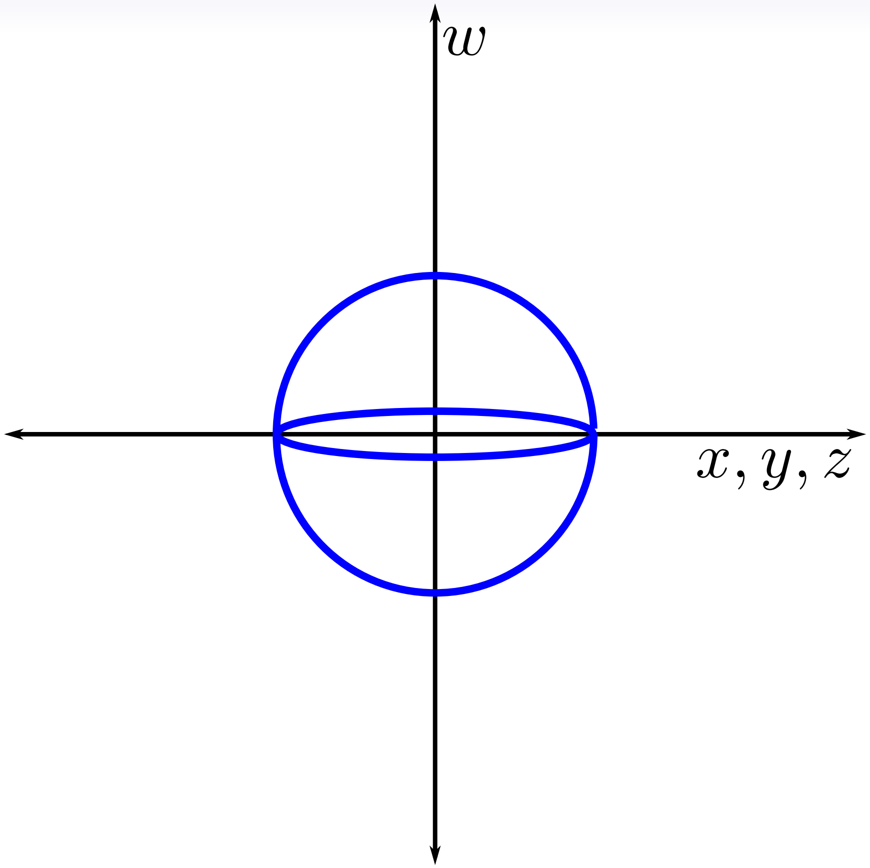


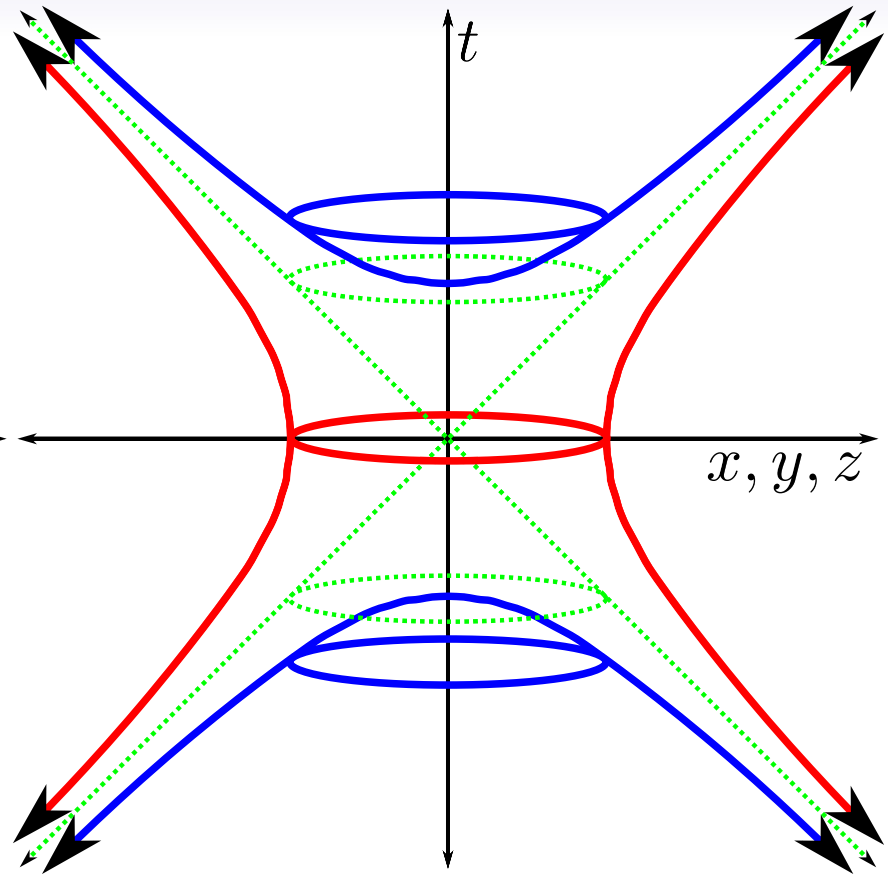


$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \qquad ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

Just as the unit sphere in Euclidean space has constant curvature, the blue and red hypersurfaces drawn on the right also have constant curvature. Can you guess the geometry of each blue hypersurface? ... drum roll please ... Hyperbolic space!

Hyperbolic Space





$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \qquad ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

Hyperbolic space is the spacelike unit sphere of the Minkowski spacetime. Because of the symmetries (both rotations and boosts) of the Minkowski spacetime, every point and every direction of hyperbolic space is the same. Hence, it has constant curvature.

Constant Acceleration

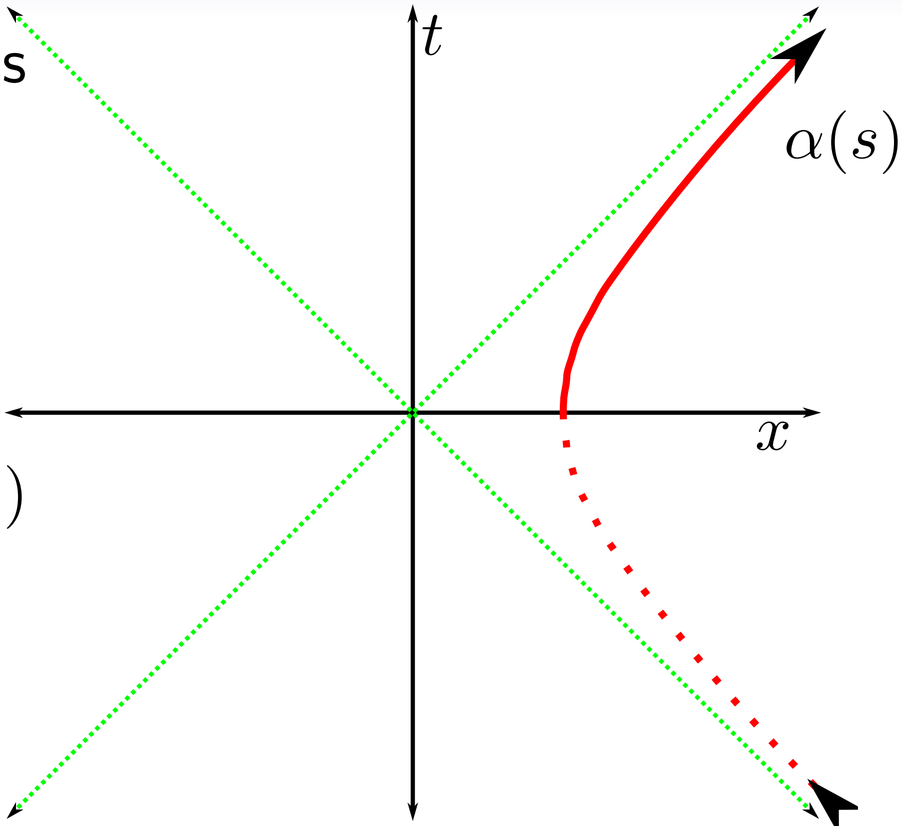
By symmetry, the red curve has constant curvature, just like a circle does in Euclidean space. Physically, this represents *constant acceleration*.

$\alpha(s) = (\frac{1}{a}\cosh(as), \frac{1}{a}\sinh(as))$

$\alpha'(s) = (\cosh(as), \sinh(as))$

$|\alpha'(s)| = 1$

Thus, s is the time experienced by the astronaut.



$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$

For small s, we recover the Newtonian analogue:

$$d = \frac{1}{a}(\cosh(as) - 1) \approx \frac{1}{a}\frac{1}{2}(as)^2 = \frac{1}{2}as^2.$$

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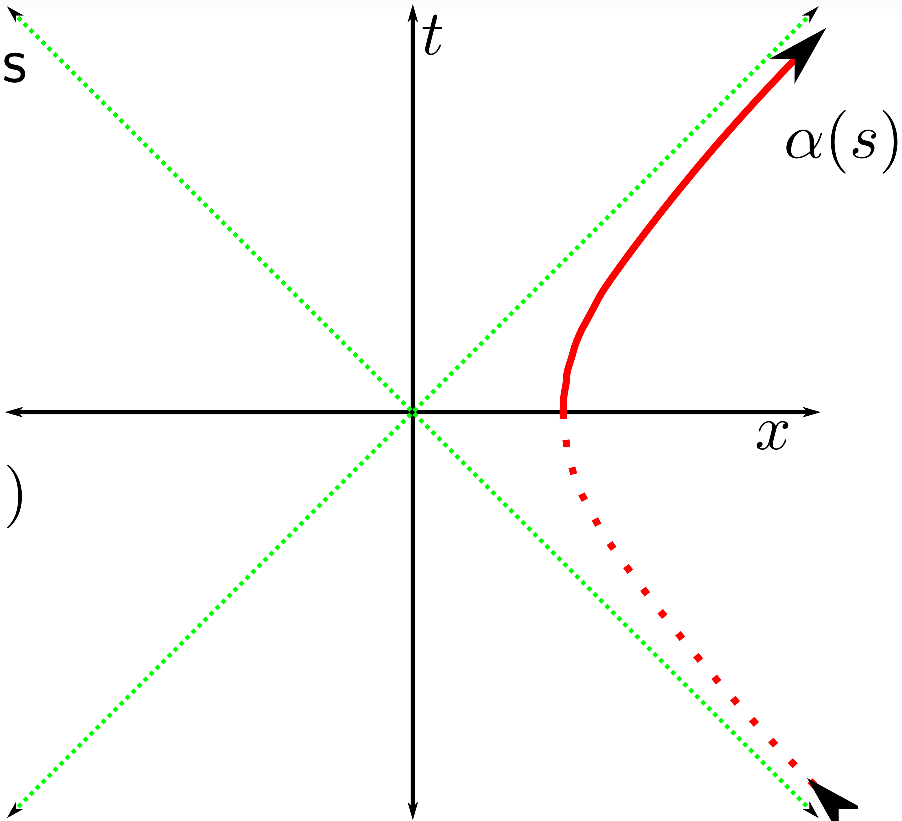


$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2$$

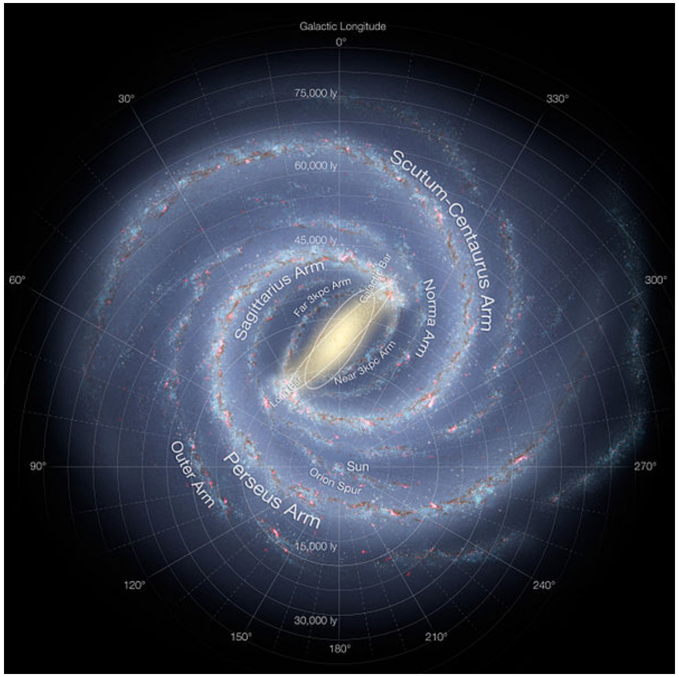
For large s, though, the distance traveled grows *exponentially!*

$$d = \frac{1}{a}(\cosh(as) - 1) \approx \frac{1}{2a}e^{as}$$

Alien Abduction

So, if you were abducted by aliens after this lecture, how far away from Earth could they take you in your lifetime?

Alien Abduction



- The star Proxima Centauri is about 4 light years away.
- The Milky Way is about 100,000 light years in diameter.
- The Andromeda galaxy is about 2,500,000 light years away.
- The edge of the observable universe is about 45,000,000,000 light years away.

Alien Abduction

Assuming Special Relativity, nothing can accelerate past the speed of light. So even if the aliens had a space craft that could go close to the speed of light, they could not take you farther than 100 light years from Earth ...

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If the aliens accelerate their spacecraft much faster than $1g = 9.8m/s^2$, they might kill you. Let's assume the aliens want to keep you alive and accelerate their spacecraft at precisely $1g$, and that they can do this for as long as they like.

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Plugging into our constant acceleration formula, we get ...

Alien Abduction

d - distance traveled; s - time experienced by the travelers

$$\frac{d}{\text{1 light year}} = \cosh\left(\frac{s}{\text{1 year}}\right) - 1$$

Some sample approximate values:

s (in years)	d (in light years)
0	0
1	0.5
2	3
3	9
4	25
5	75
10	10,000
15	1,500,000
20	250,000,000
25	35,000,000,000
30	5,000,000,000,000

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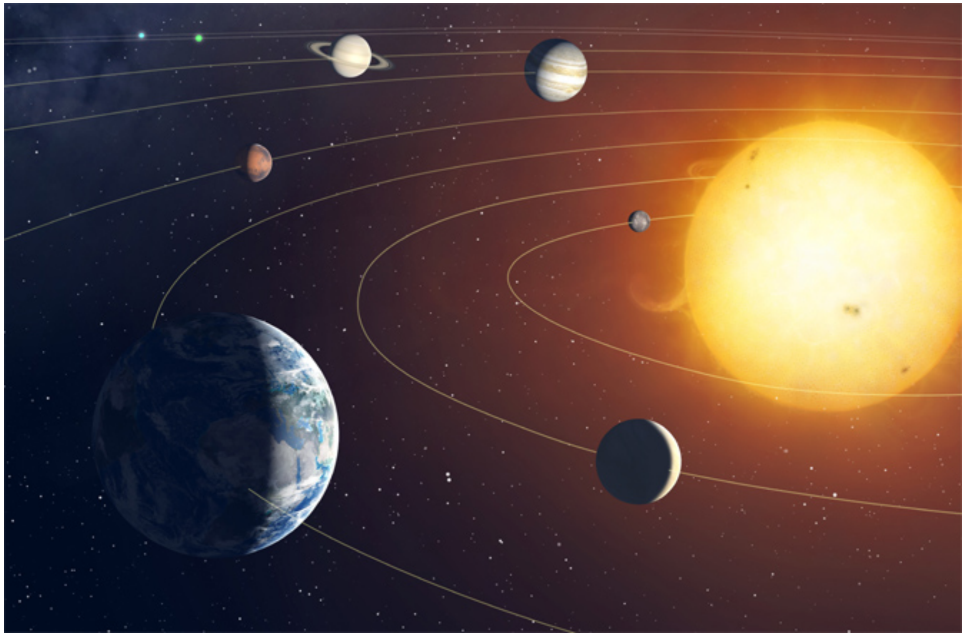
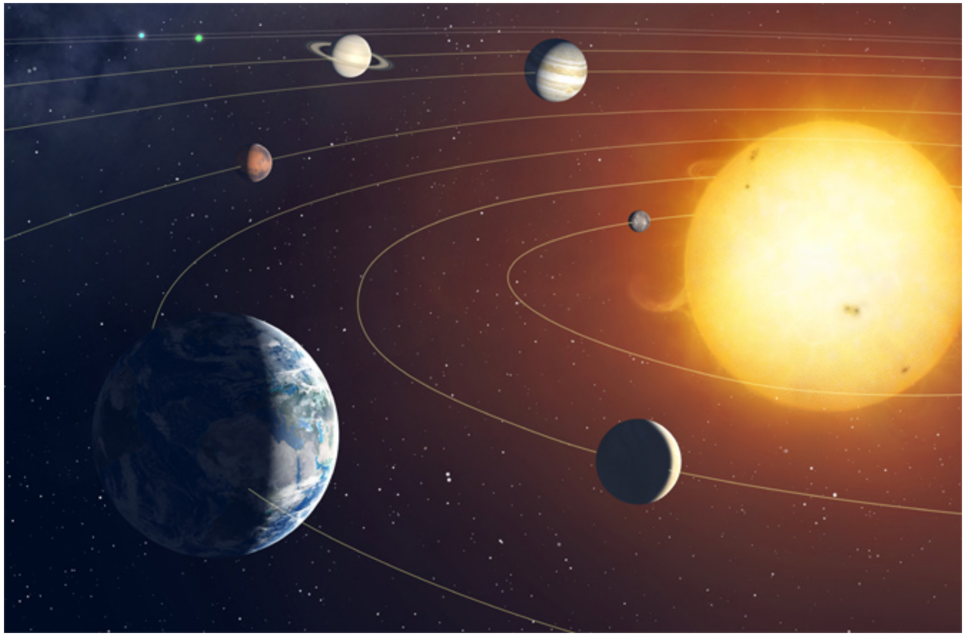
25 years to accelerate, 25 years to decelerate, and then you would be at rest, 70 billion light years from Earth!

Space Exploration and Time Travel



Meanwhile, 70 billion years have passed on the Earth.

Space Exploration and Time Travel



Meanwhile, 70 billion years have passed on the Earth.

In effect, a spaceship which can accelerate at 1g for as long as you like is both a tool for space exploration and a time machine. If such spaceships exist someday, space explorers might head out in various directions and agree to meet back at Earth, 1 million or 1 billion years in the future.

General Relativity

Everything we've discussed so far is Special Relativity, based on the flat geometry of the Minkowski spacetime

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2,$$

which came from changing a sign in the rule of Pythagoras.

What about other spacetime metrics, like

$$ds^2 = f(x, y, z, t)(dx^2 + dy^2 + dz^2) - h(x, y, z, t)dt^2,$$

for some functions f and h?

What happens when we *remove the assumption* that the spacetime metric is flat?

General Relativity

Principle: Geometric quantities correspond to physical observables. What does the curvature of a spacetime correspond to? Einstein's great idea, which is a consequence of his happiest thought, can be summed up in 3 words:

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$$G = 8\pi T$$

That is, the curvature of spacetime (geometry) corresponds to matter density (physics)!

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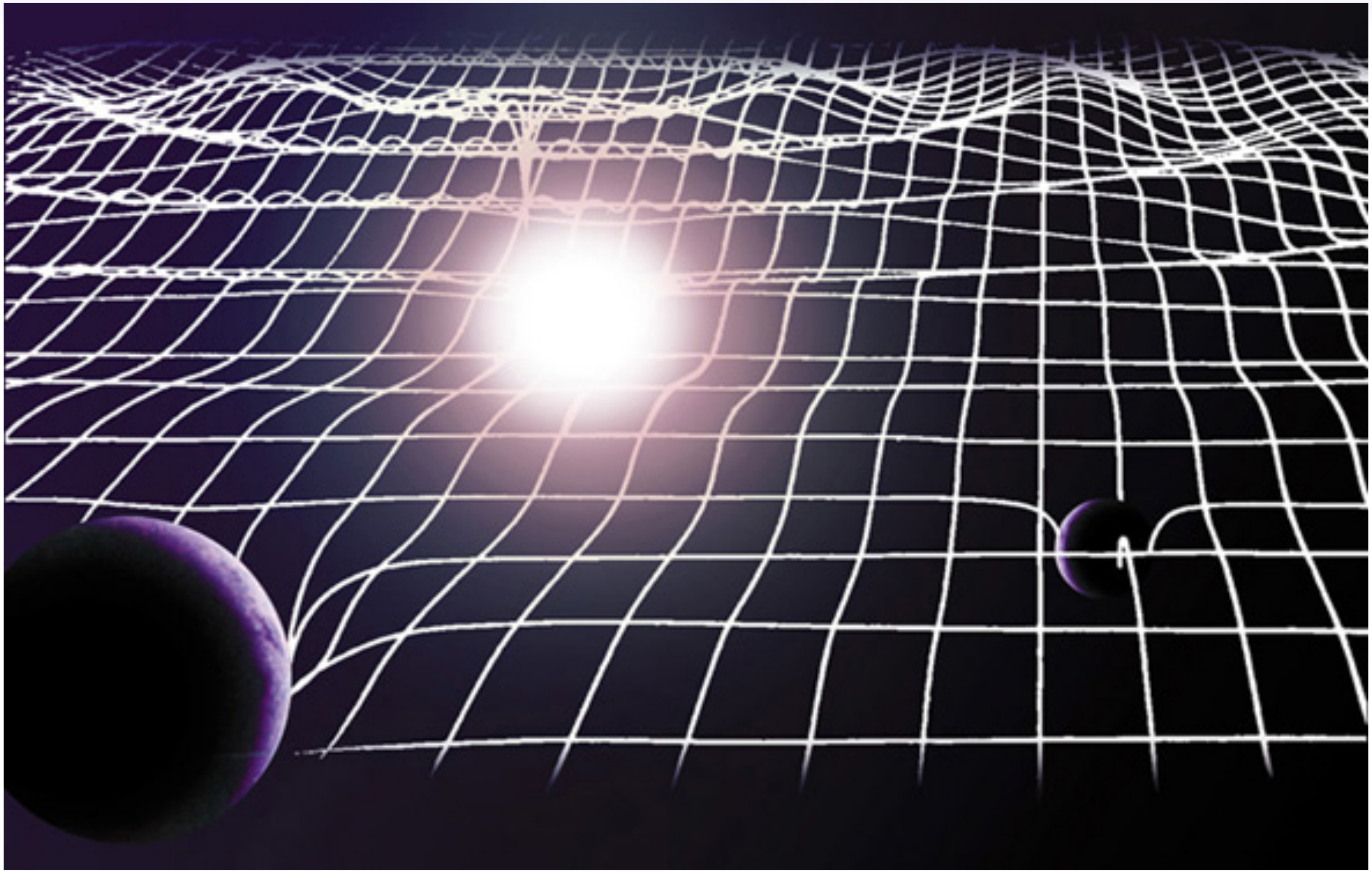
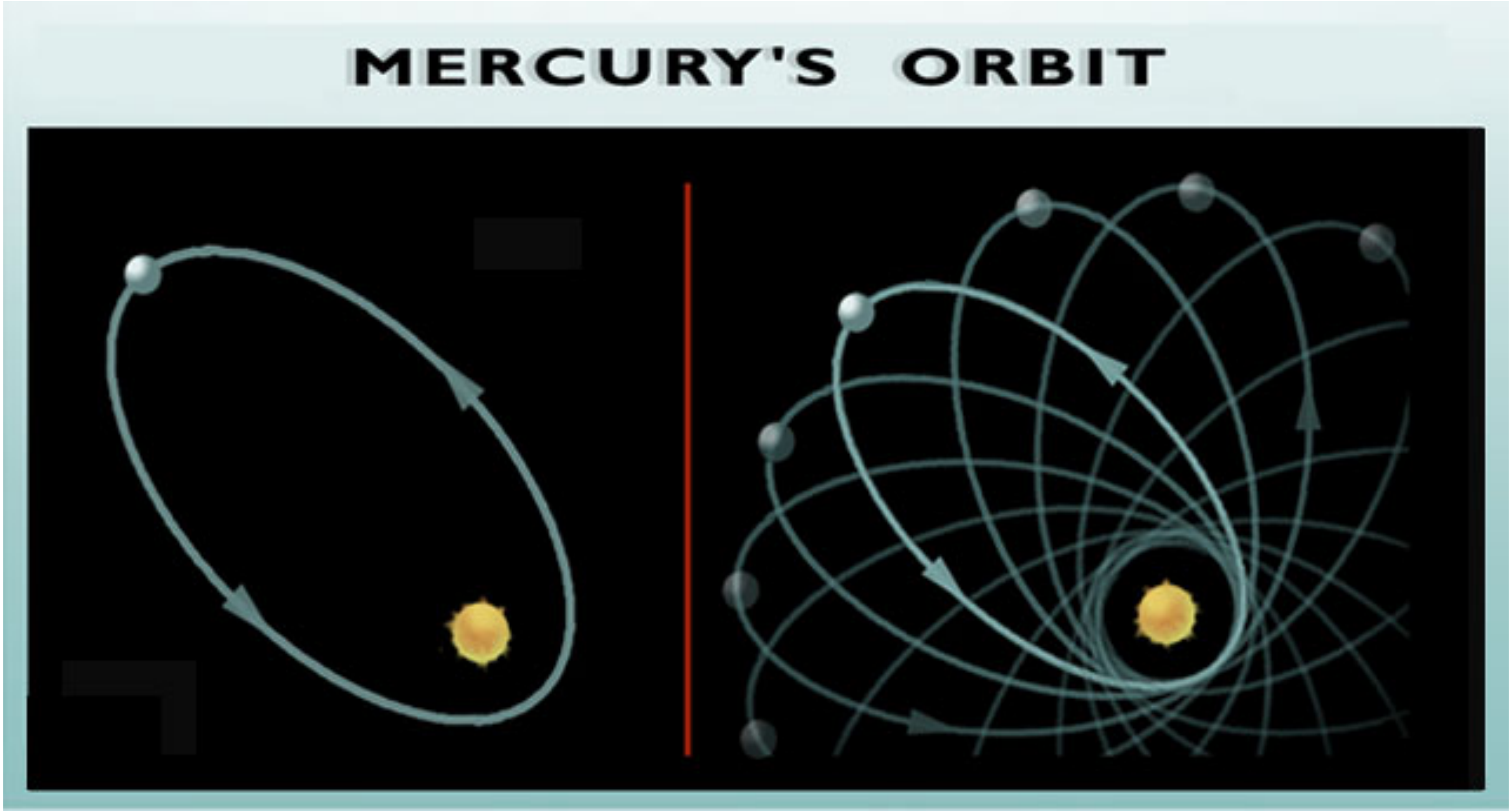
Let's review the successes of general relativity.

Successes of General Relativity: Gravity



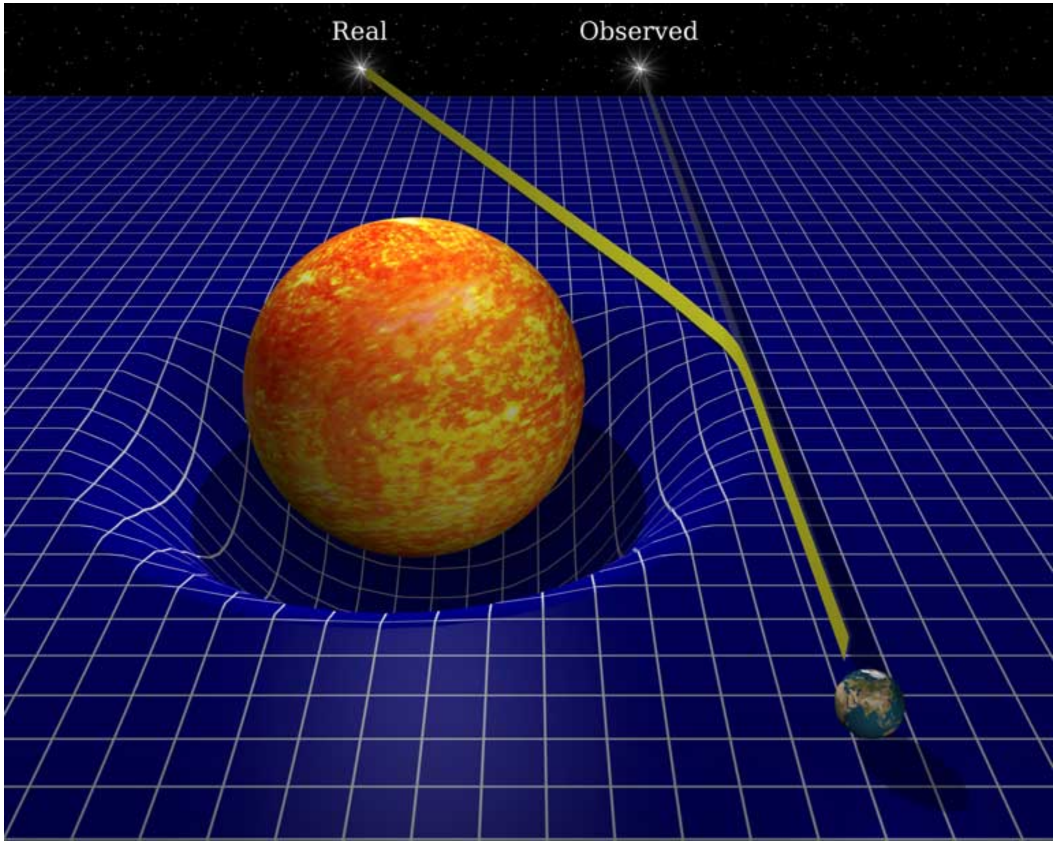
The Earth goes around the Sun because the mass of the Sun curves spacetime, not because of some mysterious $1/r^2$ force law assumed as an axiom without any explanation as to what the mechanism for gravity might be.

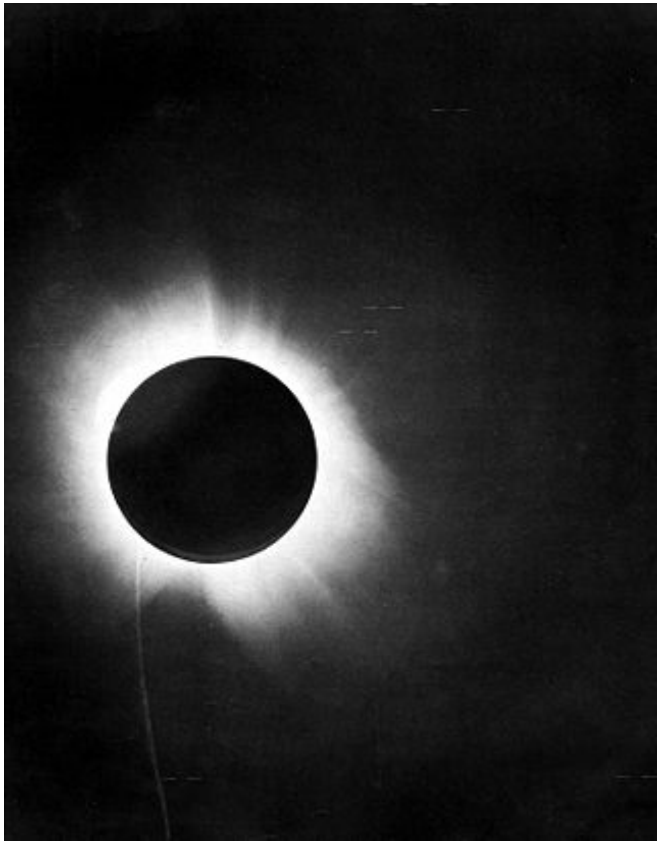
Successes of General Relativity: The Orbit of Mercury



Newtonian physics predicts a precession of 1.5436° per century, not 1.5556° per century, observed since Verrier in 1859. In 1915, Einstein showed that General Relativity gets it right.

Successes of General Relativity: Gravitational Lensing



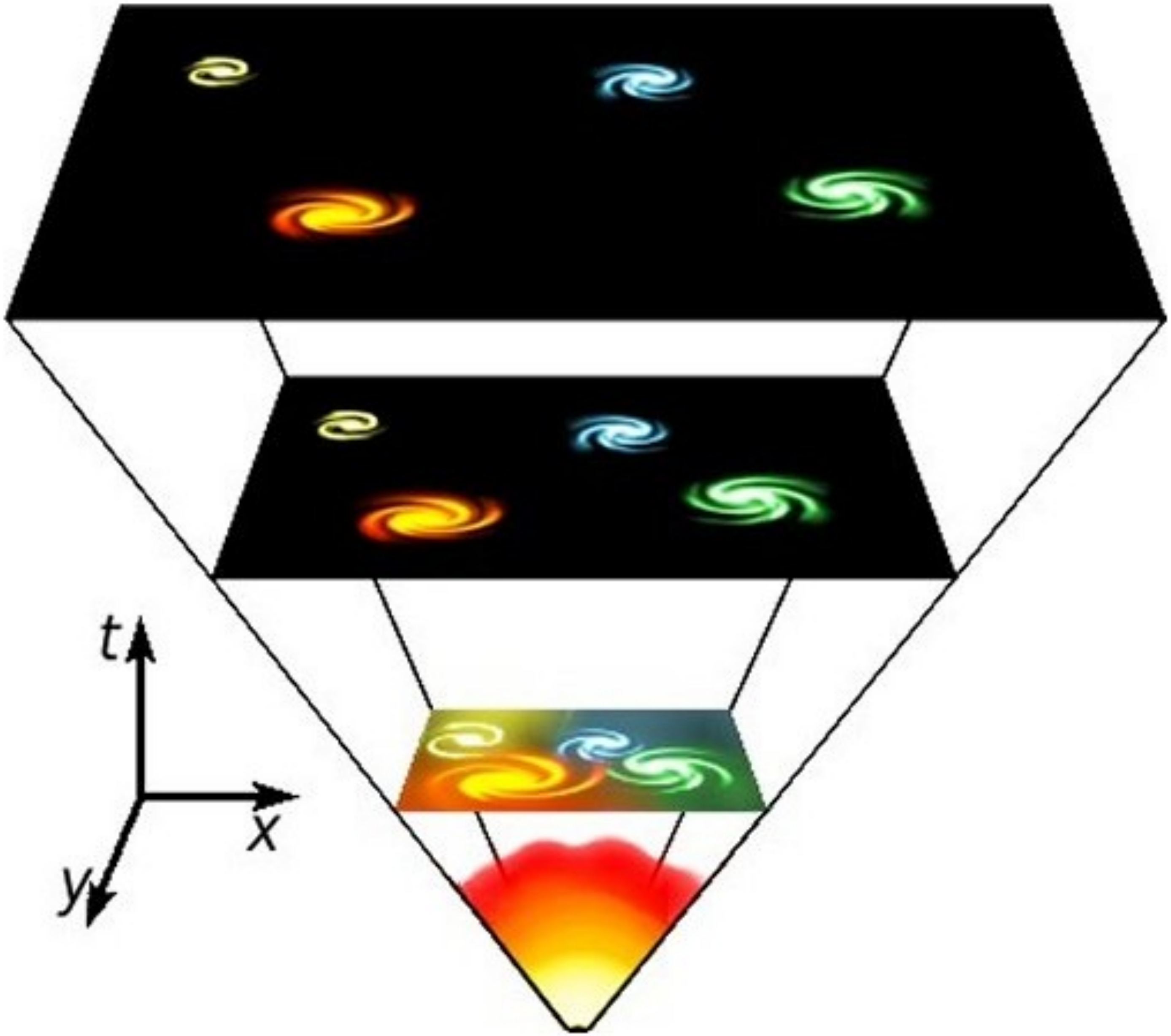


General Relativity predicts *twice* the bending angle for light that Newtonian physics predicts and agrees with observations, as observed by Eddington in 1919, on an island off the west coast of Africa during a solar eclipse.

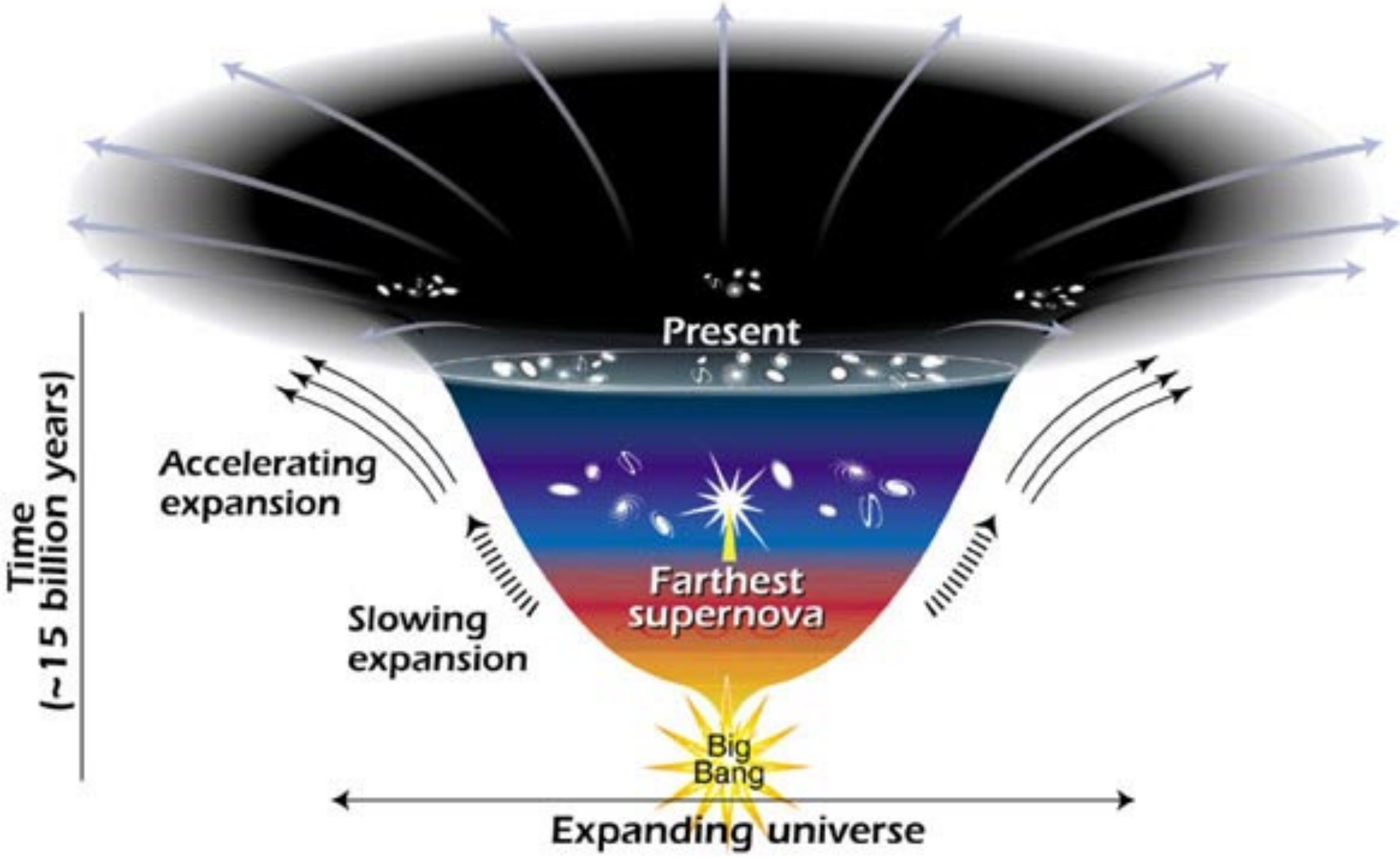
Successes of General Relativity: Gravitational Lensing


After Eddington, Einstein becomes a celebrity, still the only scientist to receive a ticker tape parade in NYC, as he did in 1921. Still, it's not like he won the Super Bowl or anything ...

Successes of General Relativity: The Big Bang



Successes of General Relativity: The *Accelerating* Expansion of the Universe



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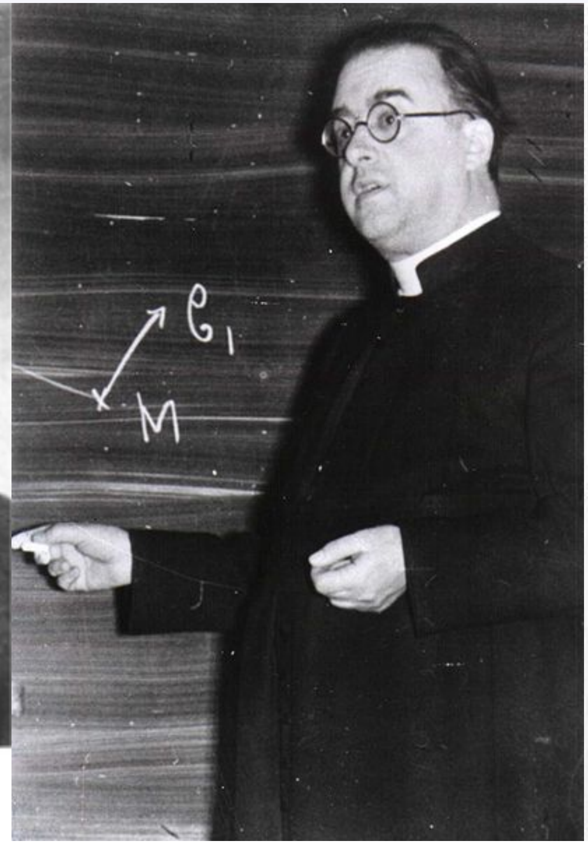


Big Bang cosmologies predicted by General Relativity were discovered in 1922 by Alexander Friedmann (left, who died in 1925) and independently in 1927 by George Lemaitre (middle), years before Edwin Hubble's (right) landmark discovery of the expanding universe in 1929.

Successes of General Relativity: The *Accelerating* Expansion of the Universe

Photo: Lawrence Berkeley National Lab

Saul Perlmutter

Photo: Belinda Pratten, Australian National University

Brian P. Schmidt

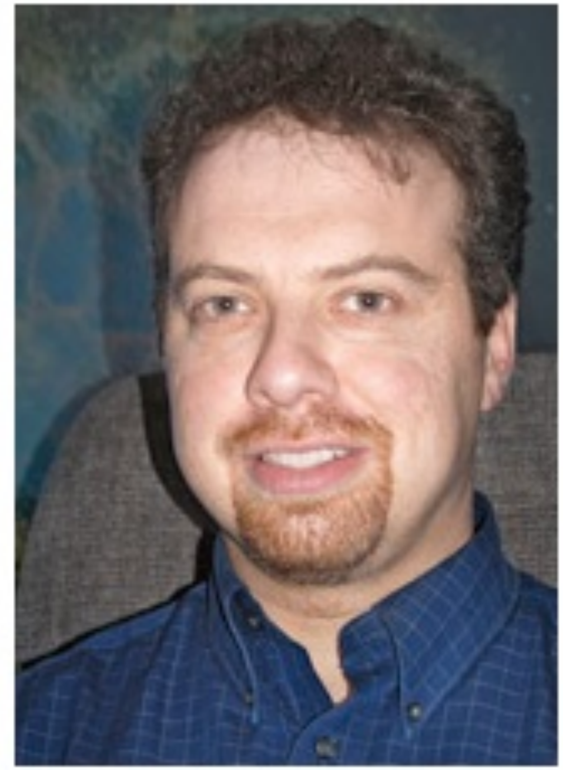
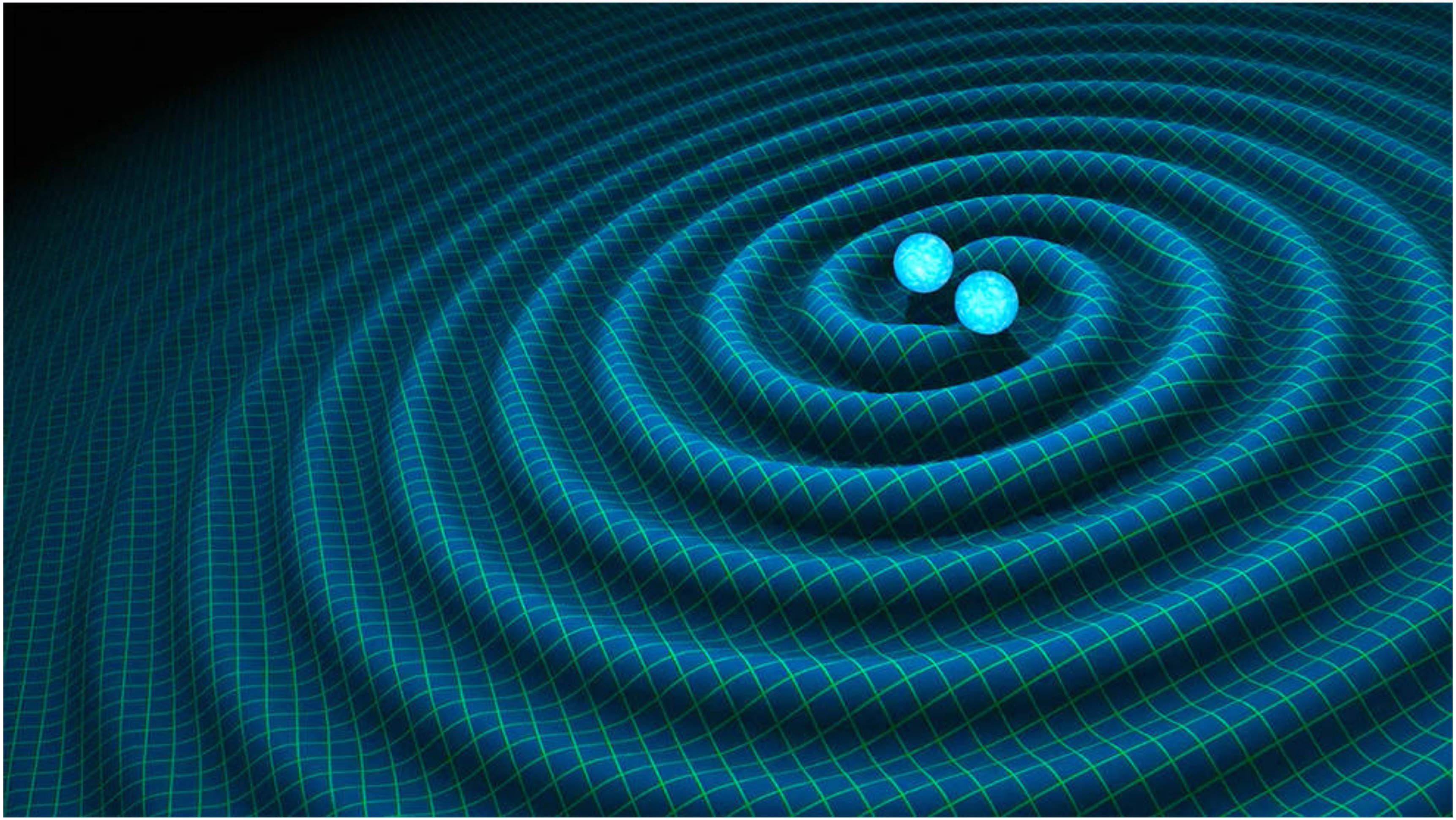


Photo: Scanpix/AFP

Adam G. Riess

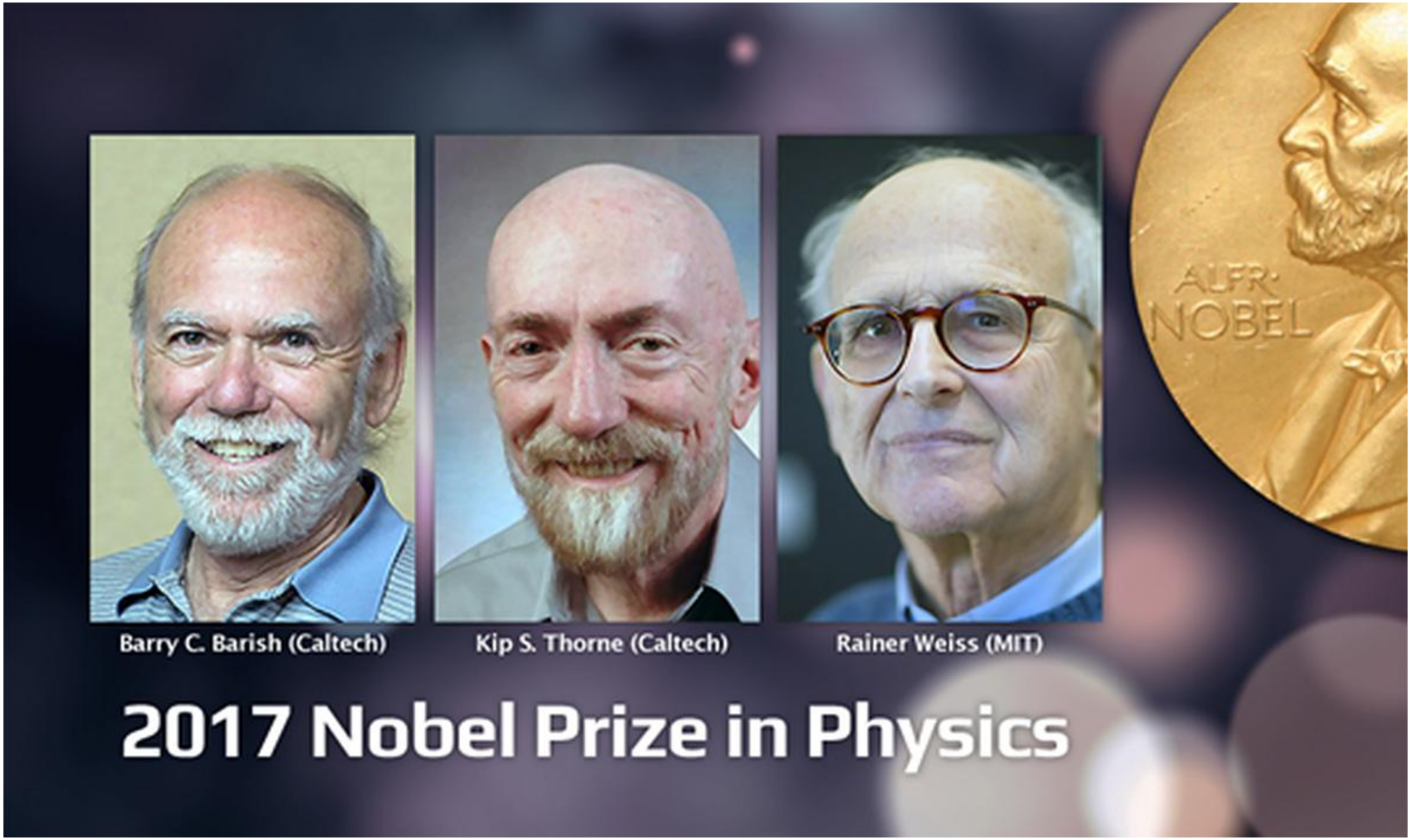
The Nobel Prize in Physics 2011 was awarded *"for the discovery of the accelerating expansion of the Universe through observations of distant supernovae"* with one half to Saul Perlmutter and the other half jointly to Brian P. Schmidt and Adam G. Riess.

Successes of General Relativity: Gravitational Waves



At 09:51 GMT on 14 September 2015, the Laser Interferometer Gravitational Wave Observatory (LIGO) detected gravitational waves produced by two merging black holes.

Successes of General Relativity: Gravitational Waves



Successes of General Relativity: Black Holes

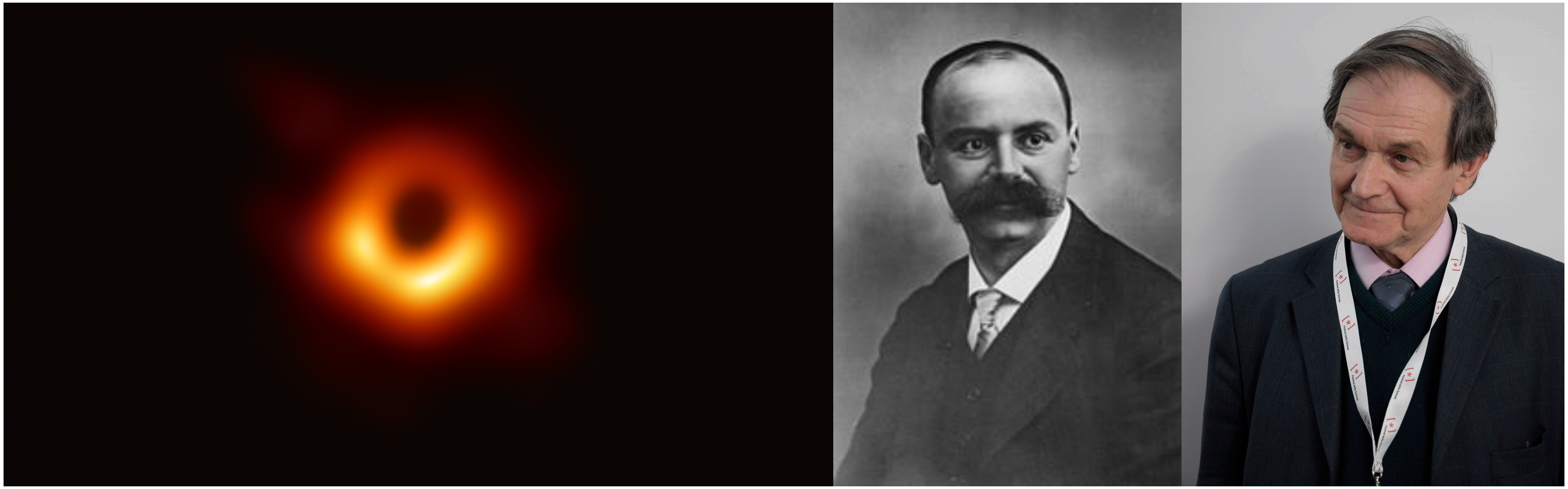
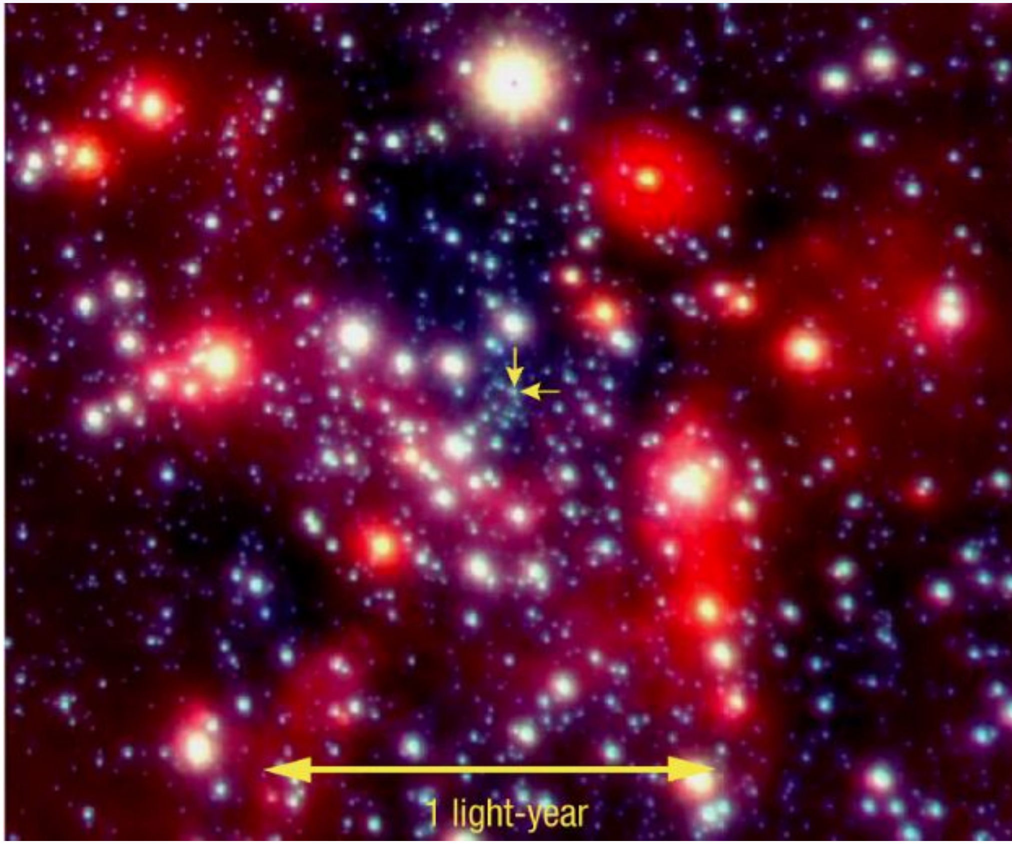
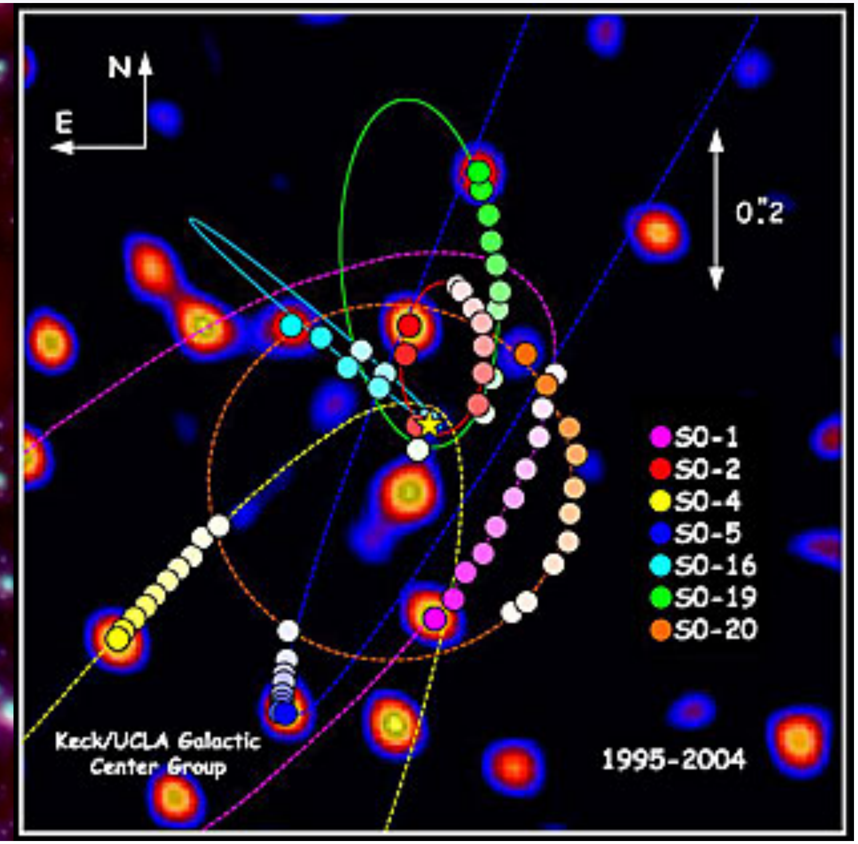


Image of a black hole captured by the international network of radio telescopes called the Event Horizon Telescope. Schwarzschild found the exact black hole solution to the highly nonlinear Einstein vacuum equations in 1915. Sir Roger Penrose won the 2020 Nobel Prize in Physics for his work on black holes.

Successes of General Relativity: Black Holes



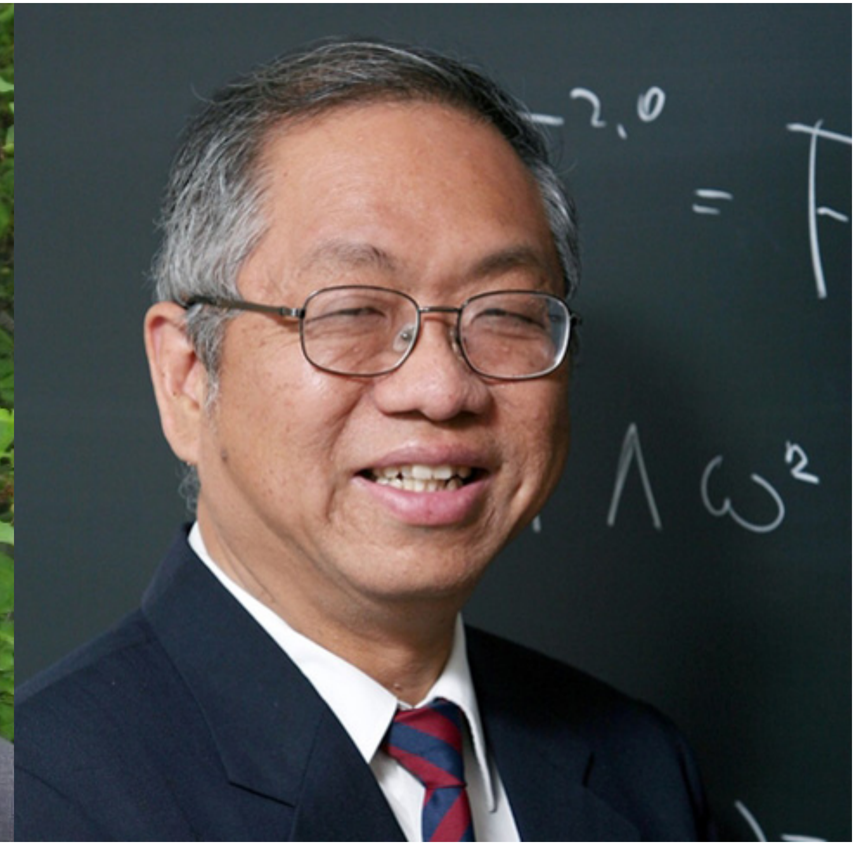


The supermassive black hole Sagittarius A* (4 million solar masses) at the center of the Milky Way Galaxy. The first black hole ever observed, Cygnus X - 1, was discovered in 1970. Today it is believed that most large galaxies have supermassive black holes at their centers.

Successes of GR: The Positive Mass Theorem

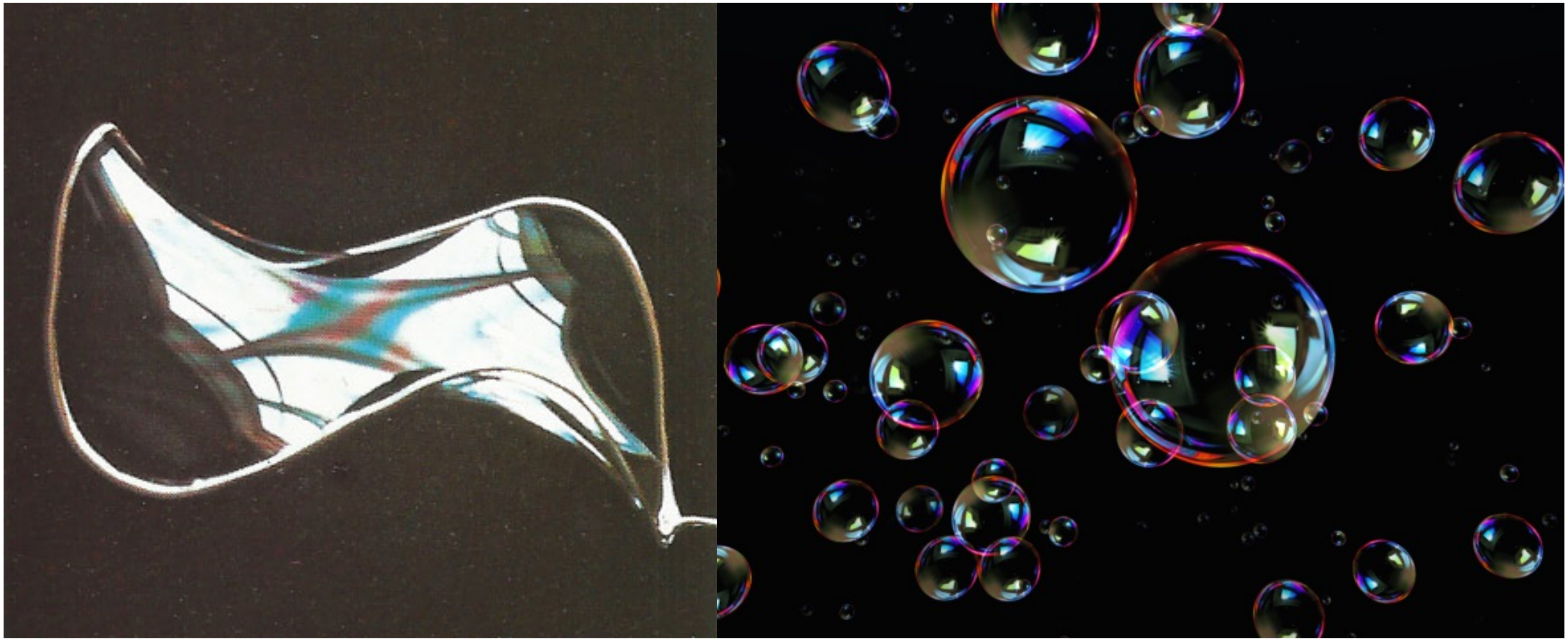
The Positive Mass Theorem is the precise mathematical theorem that proves gravity is attractive as opposed to repulsive. It is the reason when you jump up, you come down, as opposed to accelerating upward into outer space forever!

Successes of GR: The Positive Mass Theorem



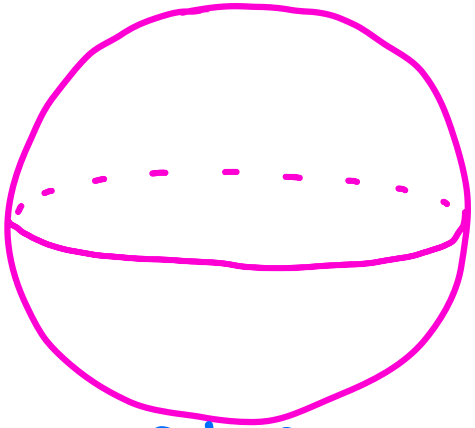
The Positive Mass Theorem was first proved by Schoen and Yau in 1979. Richard Schoen won the 2017 Wolf Prize. Shing-Tung Yau won the 2010 Wolf Prize.

Successes of GR: The Positive Mass Theorem



Surpringly, Schoen and Yau used minimal area surfaces, like soap films, to prove that gravity is attractive. This insight has led to many more discoveries of how geometry and analysis can be used to understand physics, thereby enriching mathematics in surprising ways as well.

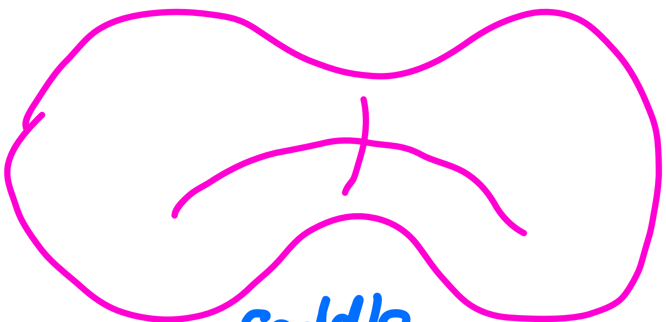
Curvature


sphere

Positive

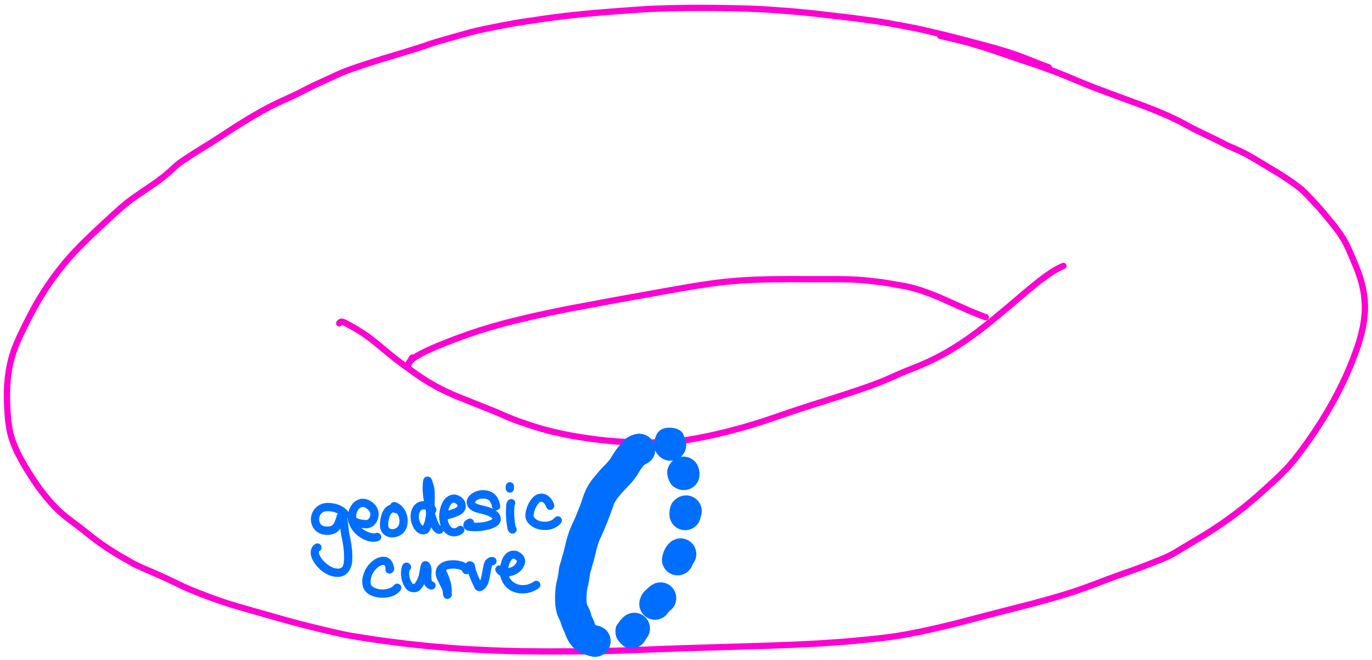

flat piece of paper

Zero


saddle

Negative

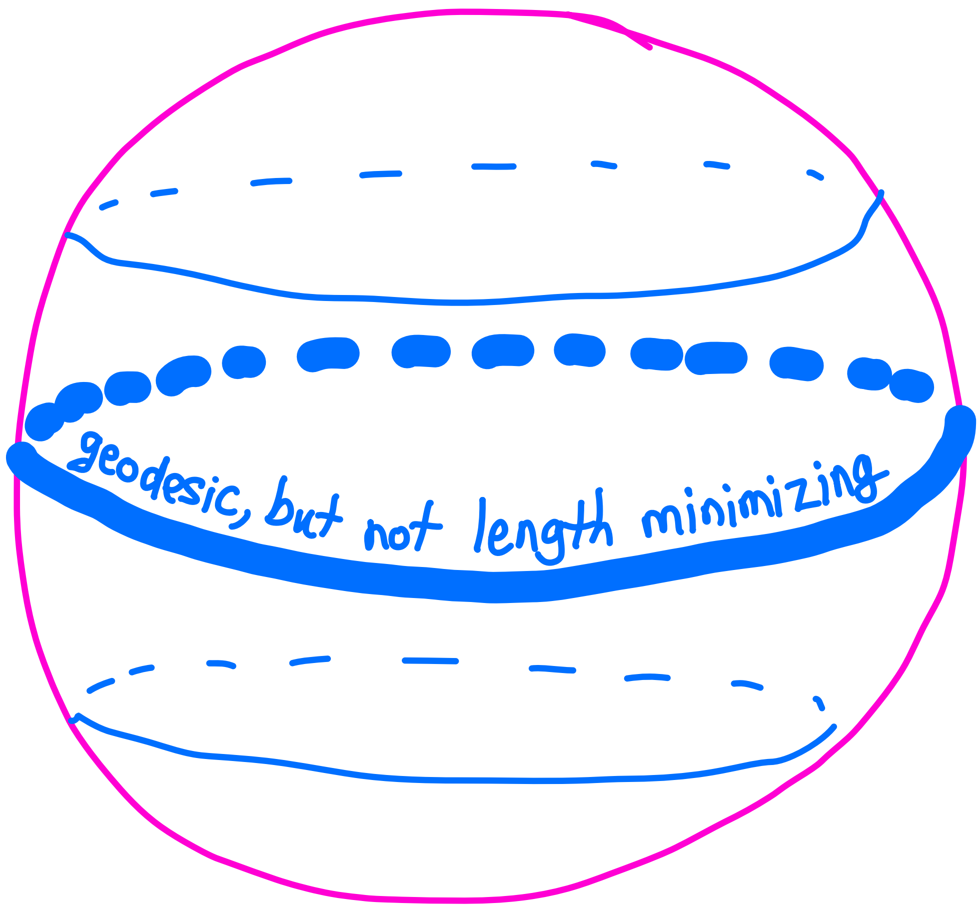
Length Minimizing Curves

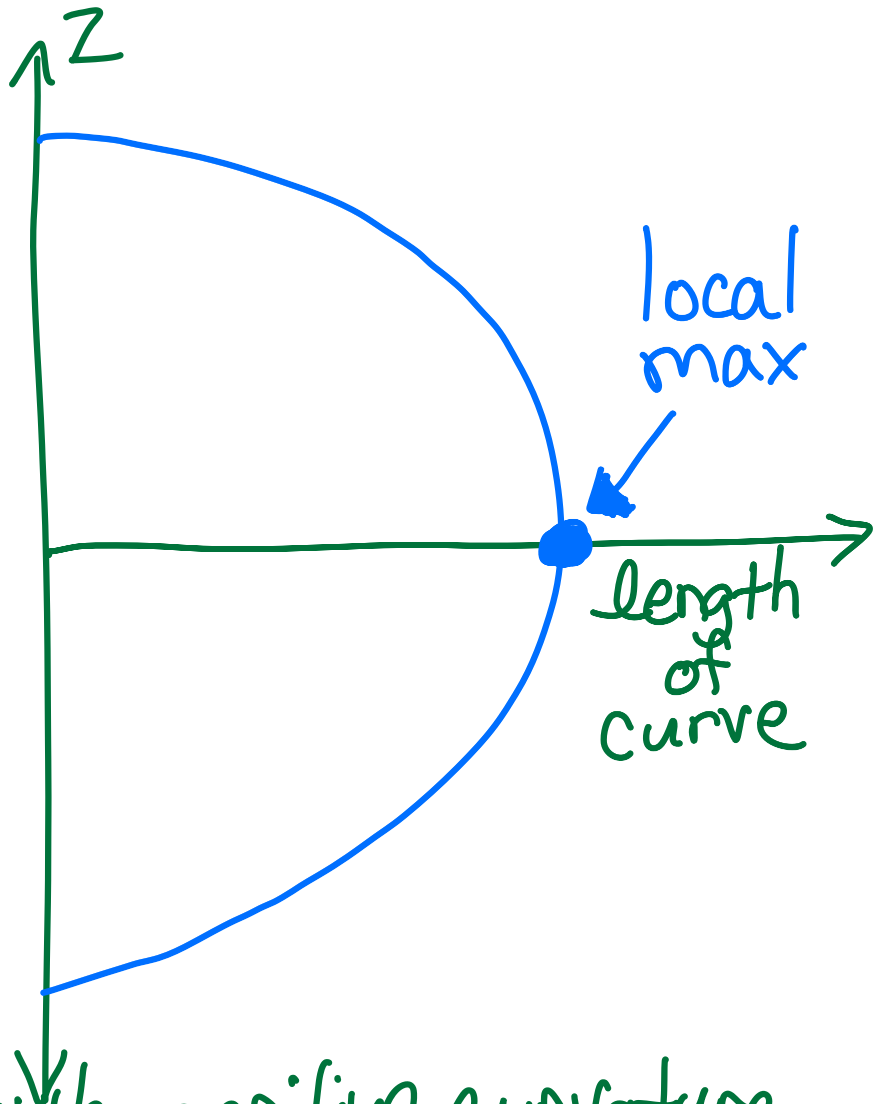


Shortest curve among nearby curves.

Same as a slippery rubber band.

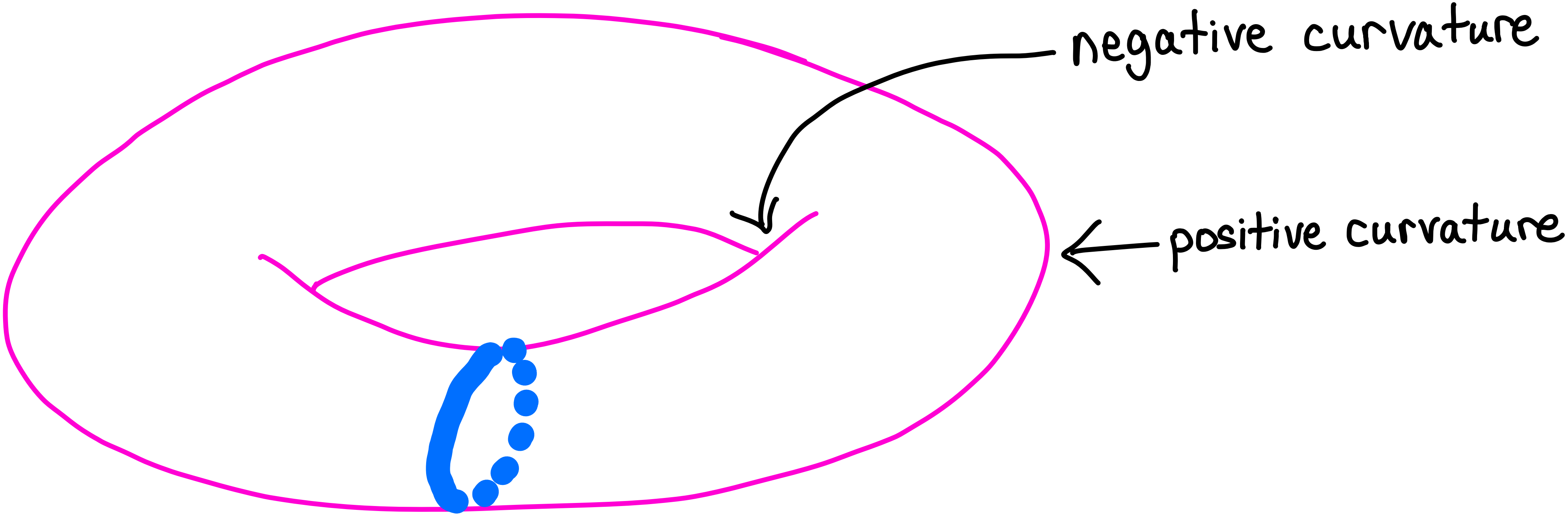
Positive Curvature & Instability





<u>Theorem</u>: An oriented surface with positive curvature does not have any length minimizing curves.

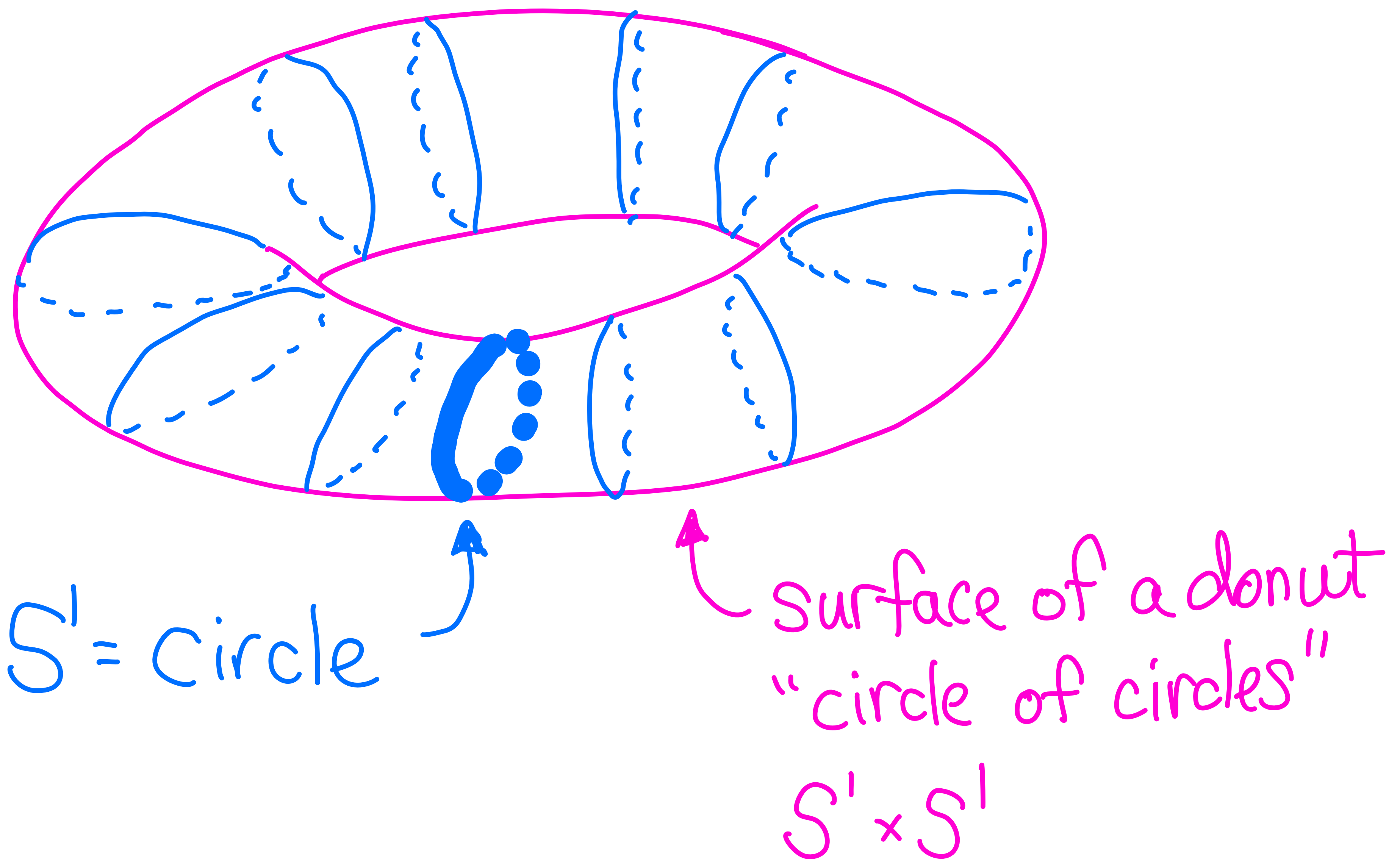
A Torus can not have Positive Curvature everywhere



1. Topologically, there must be a length minimizing curve.
2. But positive curvature prohibits a length minimizing curve.

Q.E.D.

A Torus can not have Positive Curvature everywhere



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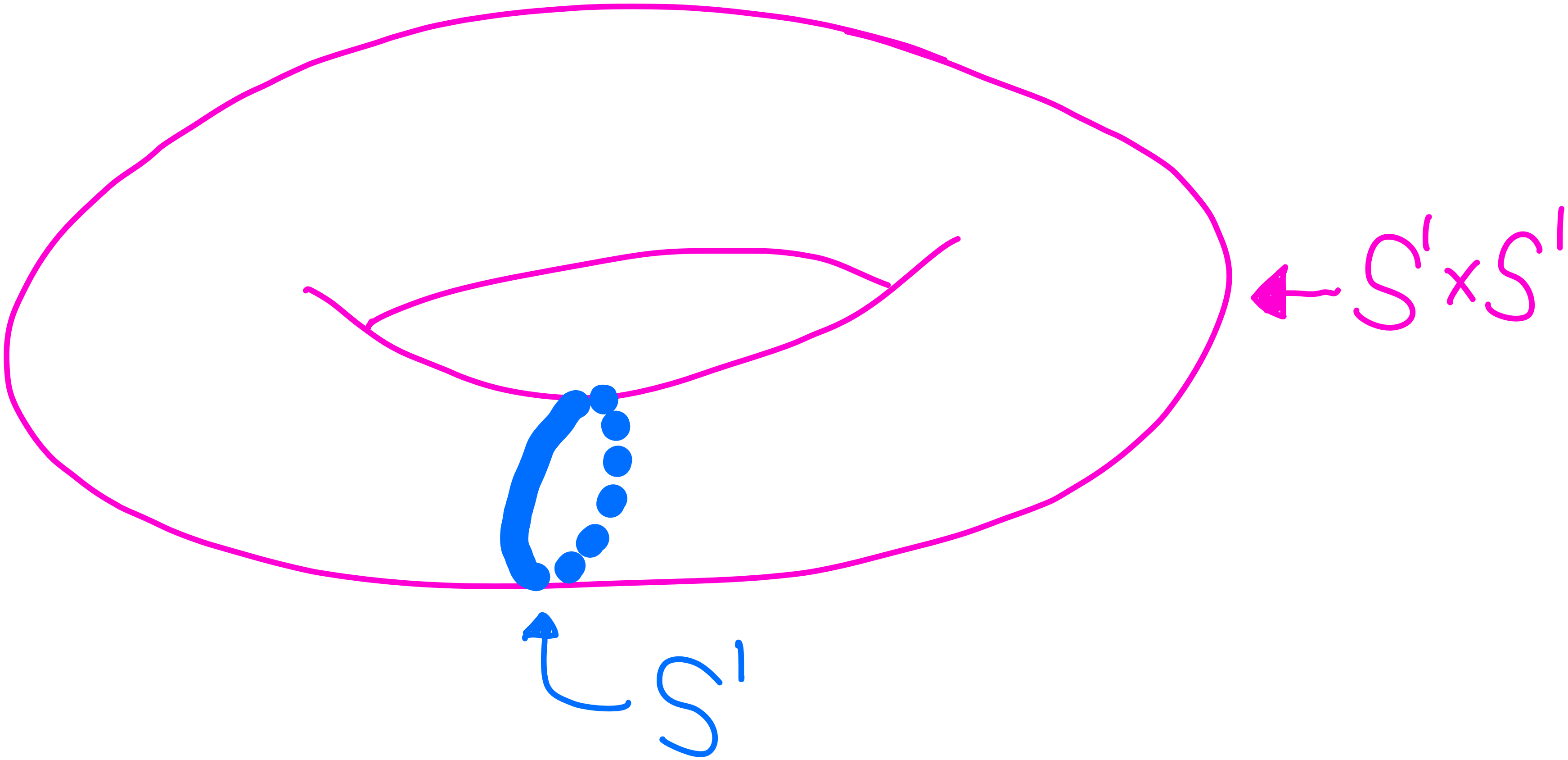
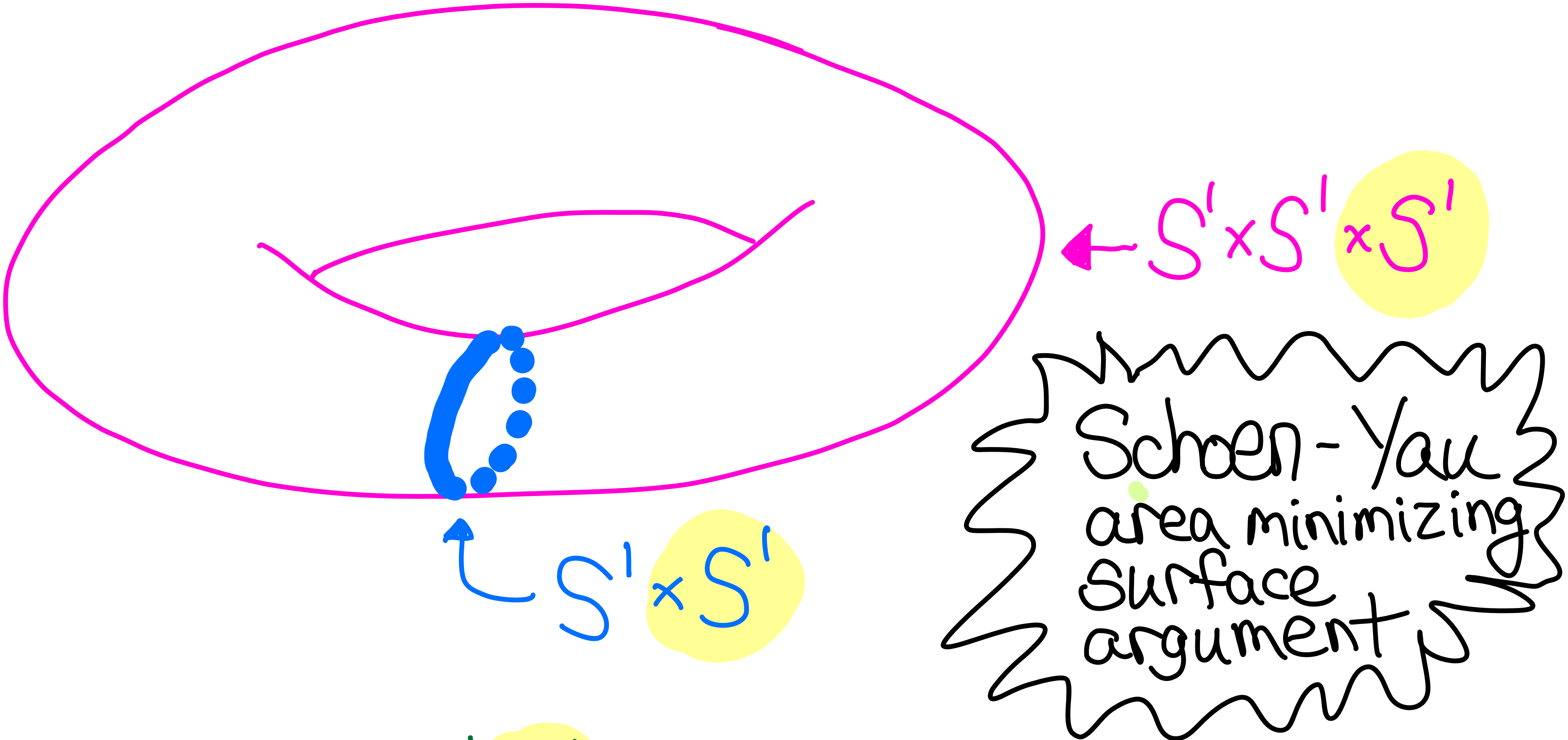


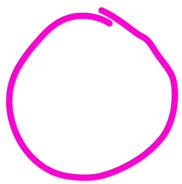
Theorem: $S^1 \times S^1$ does not admit a metric of positive curvature.

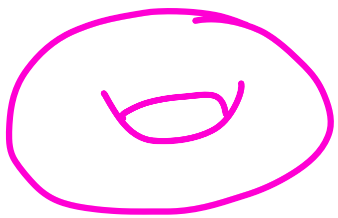
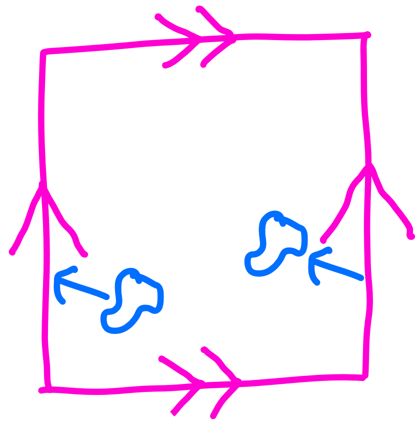
A Torus can not have Positive Curvature everywhere



<u>Theorem</u>: $S^1 \times S^1 \times S^1$ does not admit a metric of positive scalar curvature.

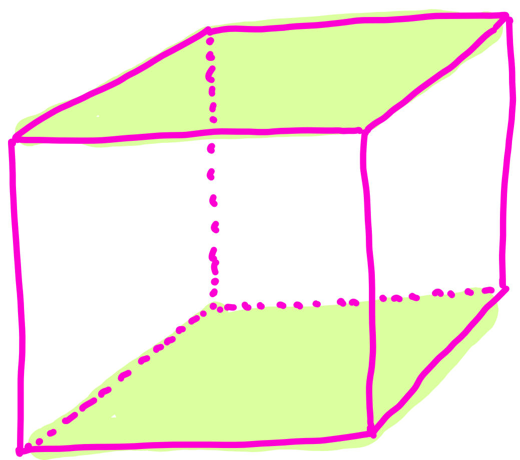
Visualizing Tori

 $=$ S^1 $=$ 

 $=$ $S^1 \times S^1$ $=$ 

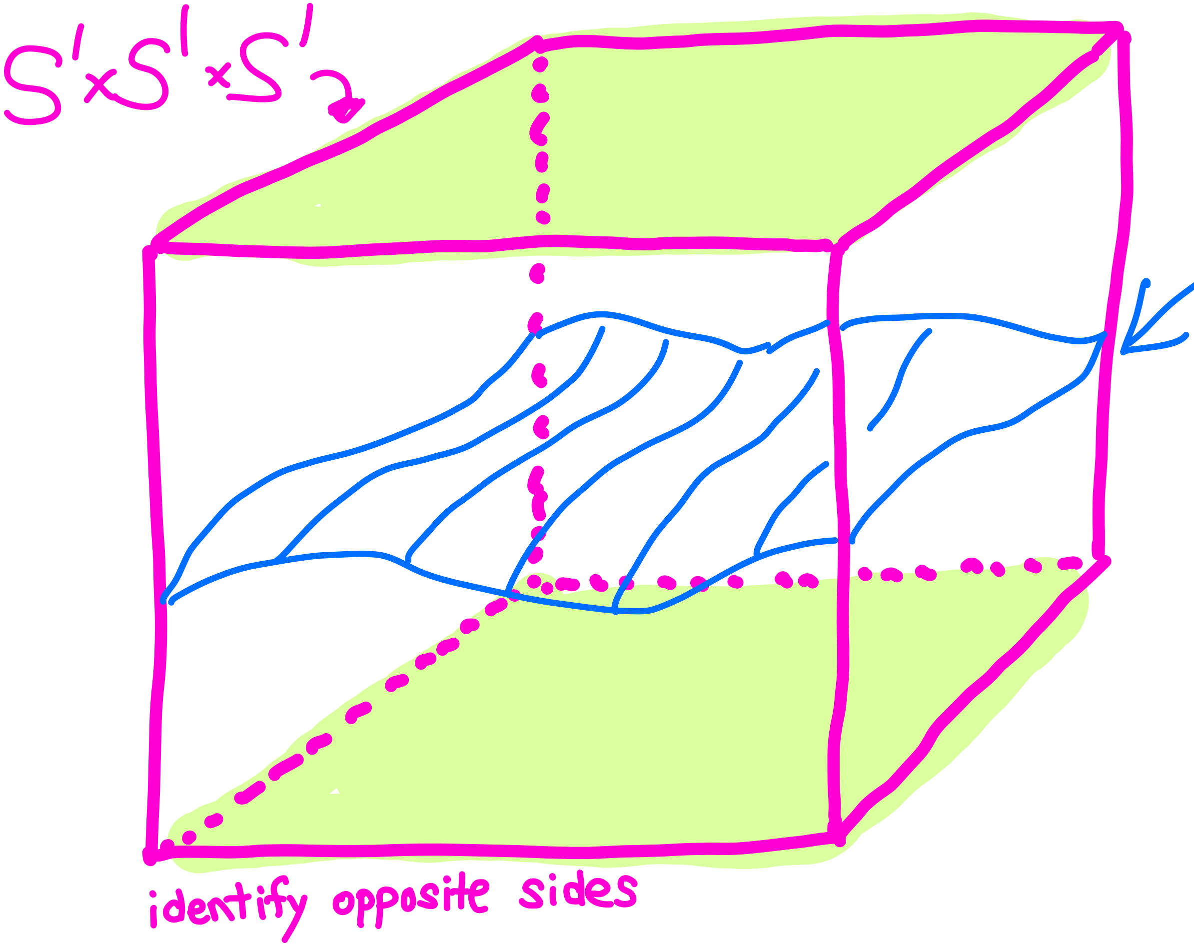
identify opposite edges

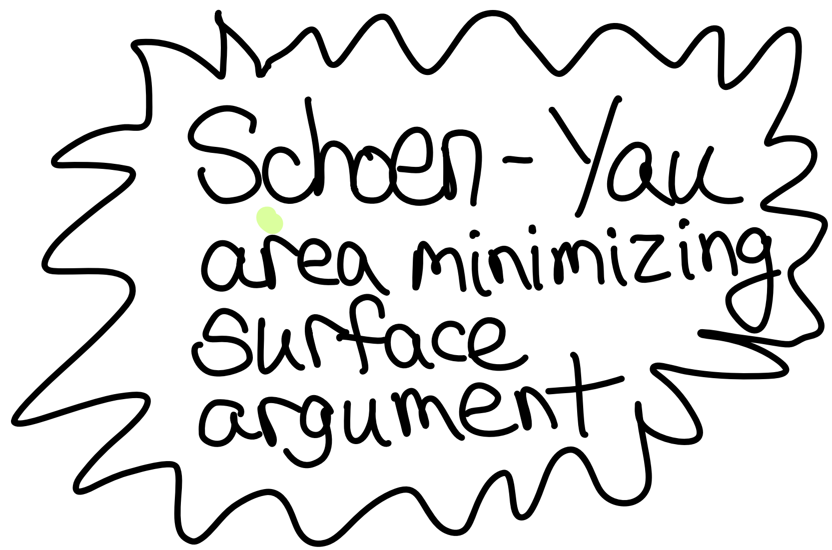
Atari Asteroids game

$S^1 \times S^1 \times S^1 =$ 

identify opposite sides

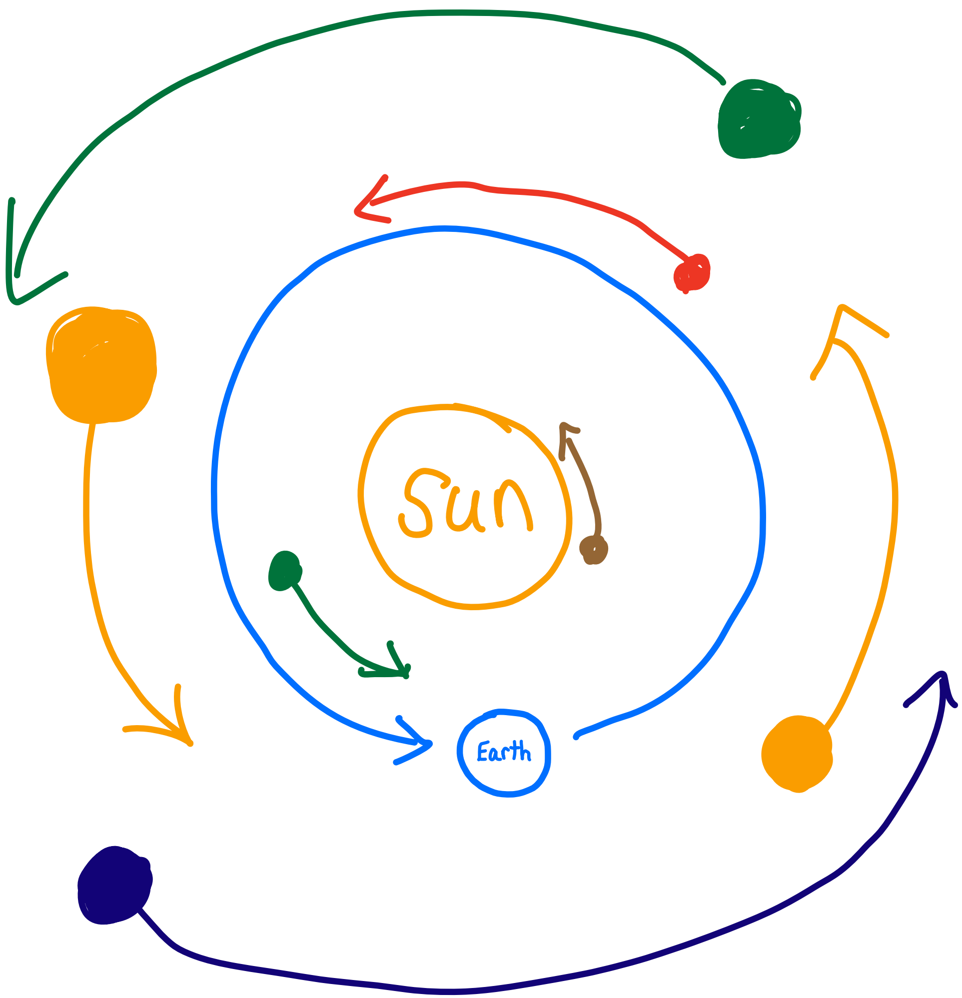
A Torus can not have Positive Curvature everywhere





Theorem: $S^1 \times S^1 \times S^1$ does not admit a metric of positive scalar curvature.

The Solar System



at some instant in time

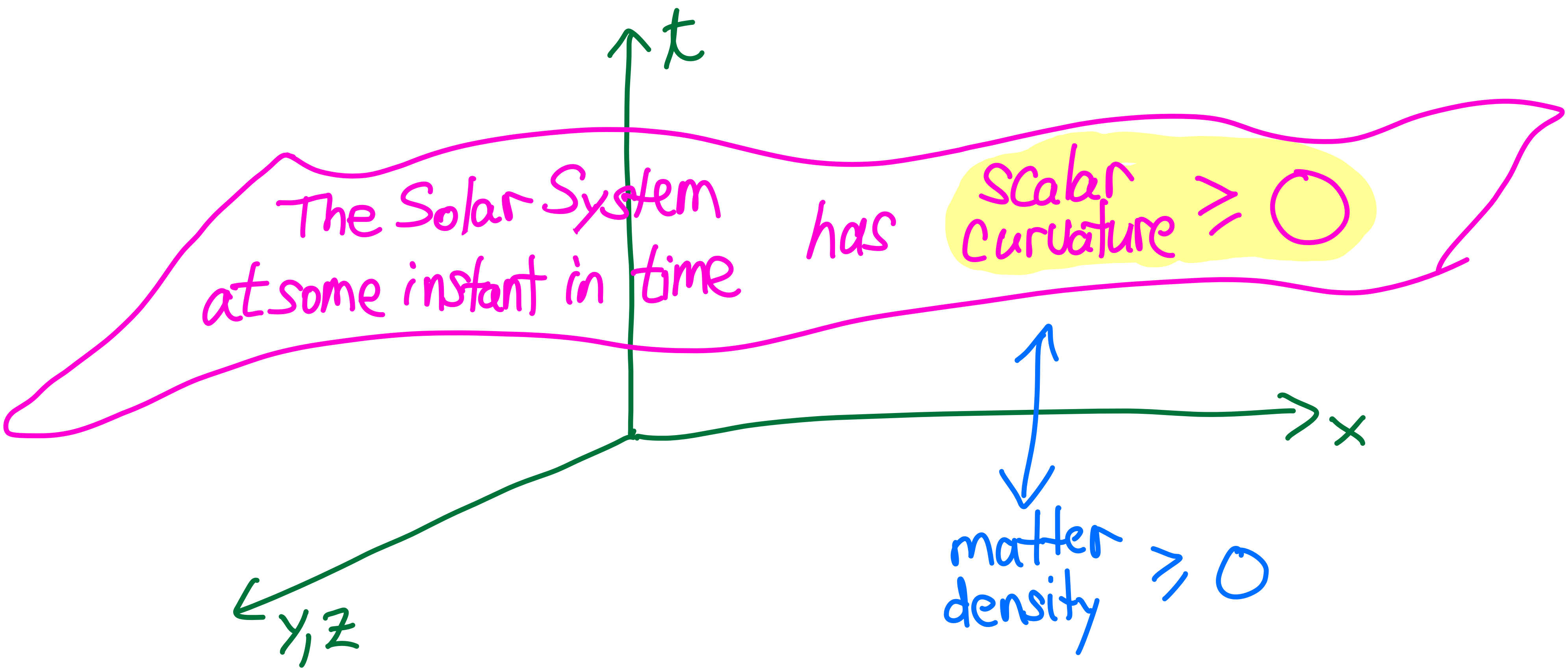
has positive (or zero) matter density everywhere.

The Einstein Equation

$$G = 8\pi T$$

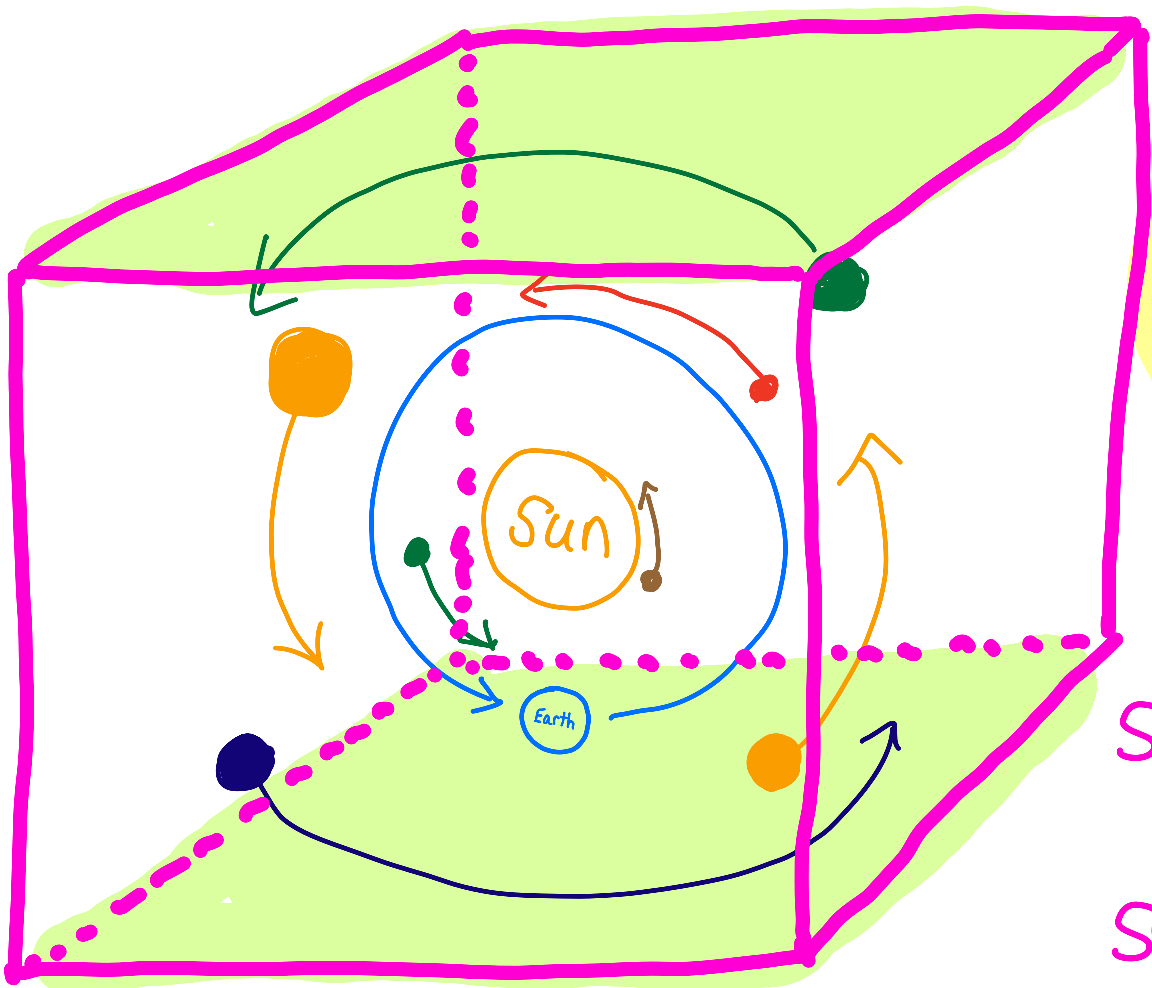
Curvature = Matter Density

at every point in the spacetime of the universe



The Solar System in a Cube

If the **total gravitational mass** ≤ 0, then calculations show that we can identify opposite sides using the Lohkamp trick such that we get a metric on $S^1 \times S^1 \times S^1$ with positive scalar curvature (after spreading the matter around a little bit). Thus,



total gravitational mass > 0.

Schoen - Yau, 1979

scalar curvature ≥ 0 everywhere

scalar curvature > 0 somewhere

Thus, when you jump up, you come down.

Thank you!

The Mass of the Universe

70% **Dark Energy**
(the cosmological constant of **General Relativity** used to explain the observed *accelerating* expansion of the universe)

25% **Dark Matter**

5% **Regular Baryonic Matter**
(Gas, Dust, Planets, Stars, etc., composed of particles described by the **Standard Model of Particle Physics** and **Quantum Field Theory**)

Which theory best describes Dark Matter?

Big Questions

There is roughly five times more dark matter in the universe than regular baryonic matter represented by the periodic table.

Also, most of the mass of galaxies is dark matter.

1. What is the nature of dark matter?

2. Does dark matter have something to do with spiral structure in galaxies?

The Puzzle of the Spirals

"Much as the discovery of these strange forms may be calculated to excite our curiosity, and to awaken an intense desire to learn something of the laws which give order to these wonderful systems, as yet, I think, we have no fair ground even for plausible conjecture."

Lord Rosse (1850)

"A beginning has been made by Jeans and other mathematicians on the dynamical problems involved in the structure of the spirals."

Curtis (1919)

"Incidentally, if you are looking for a good problem..."

Feynman (1963)

The Puzzle of the Spirals

"The old puzzle of the spiral arms of galaxies continues to taunt theorists. The more they manage to unravel it, the more obstinate seems the remaining dynamics. Right now, this sense of frustration seems greatest in just that part of the subject which advanced most impressively during the past decade - the idea of Lindblad and Lin that the grand bisymmetric spiral patterns, as in M51 and M81, are basically compression waves felt most intensely by the gas in the disks of those galaxies. Recent observations leave little doubt that such spiral "density waves" exist and indeed are fairly common, but no one still seems to know why.

To confound matters, not even the N-body experiments conducted on several large computers since the late 1960s have yet yielded any decently long-lived regular spirals."

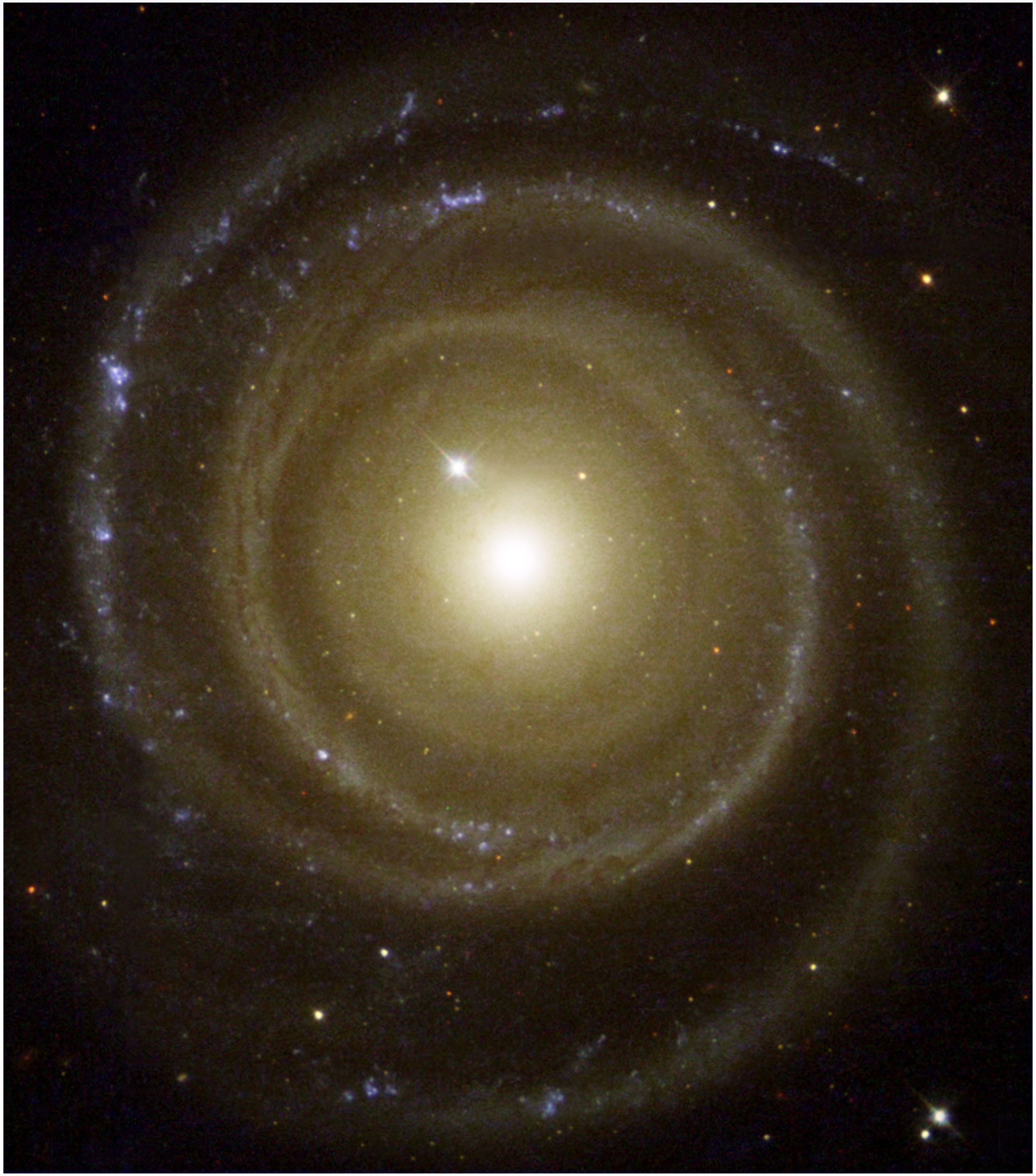
Toomre (1977)

Spiral Galaxy M81

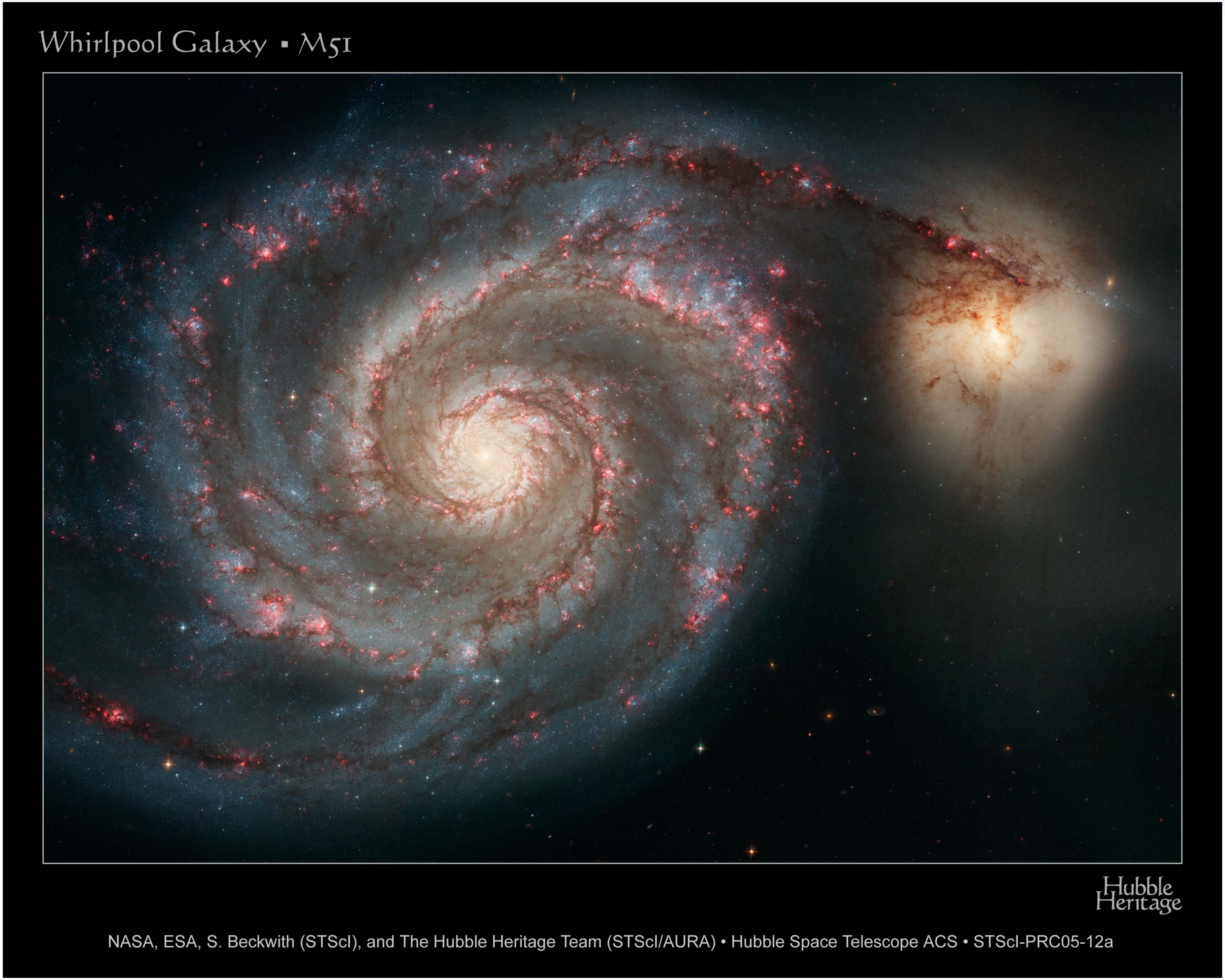
Spiral Galaxy M74

Spiral Galaxy NGC1365

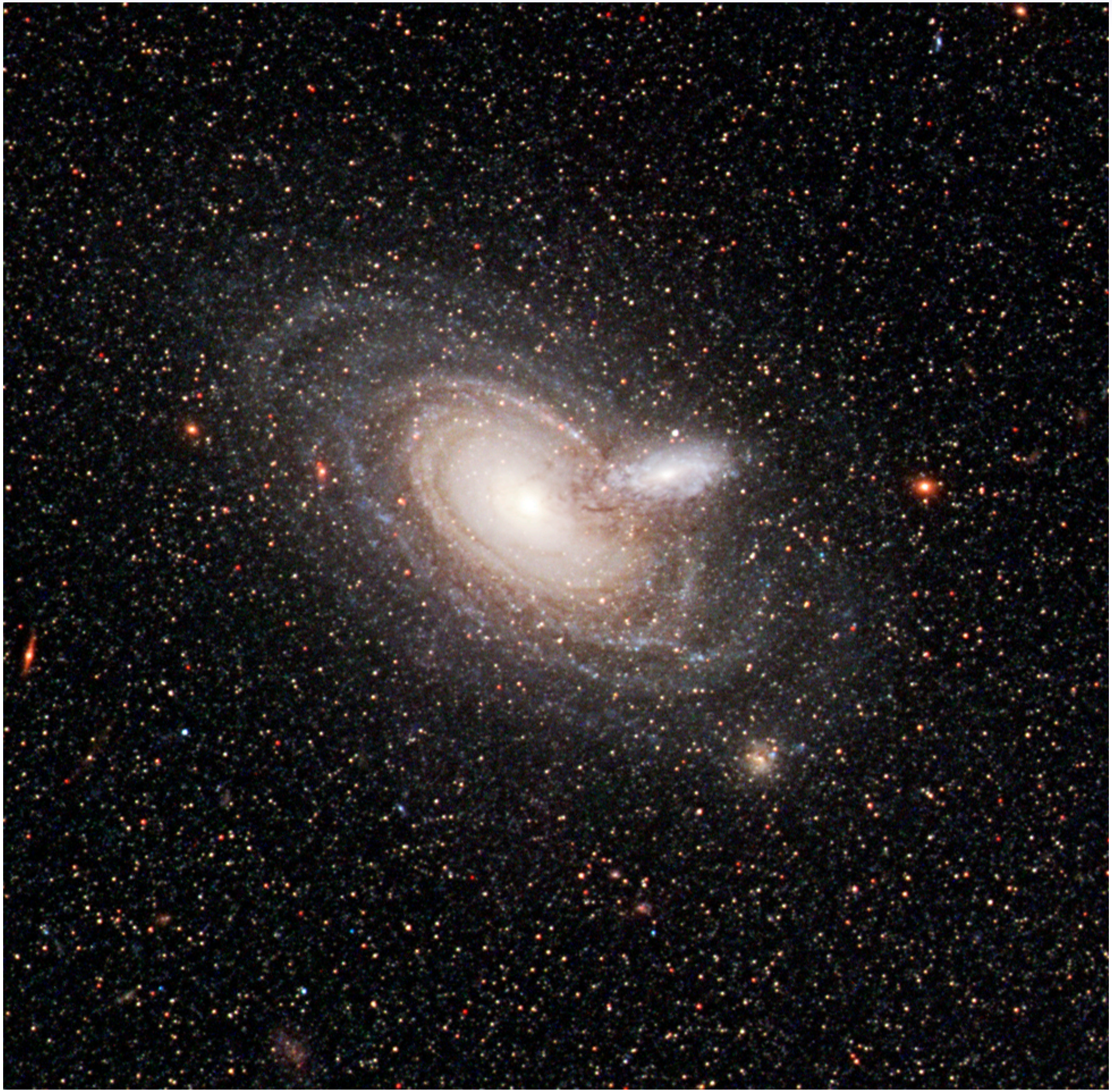
Spiral Galaxy NGC4622



Spiral Galaxy M51, the Whirlpool Galaxy



Spiral Galaxies 2MASX J00482185-2507365



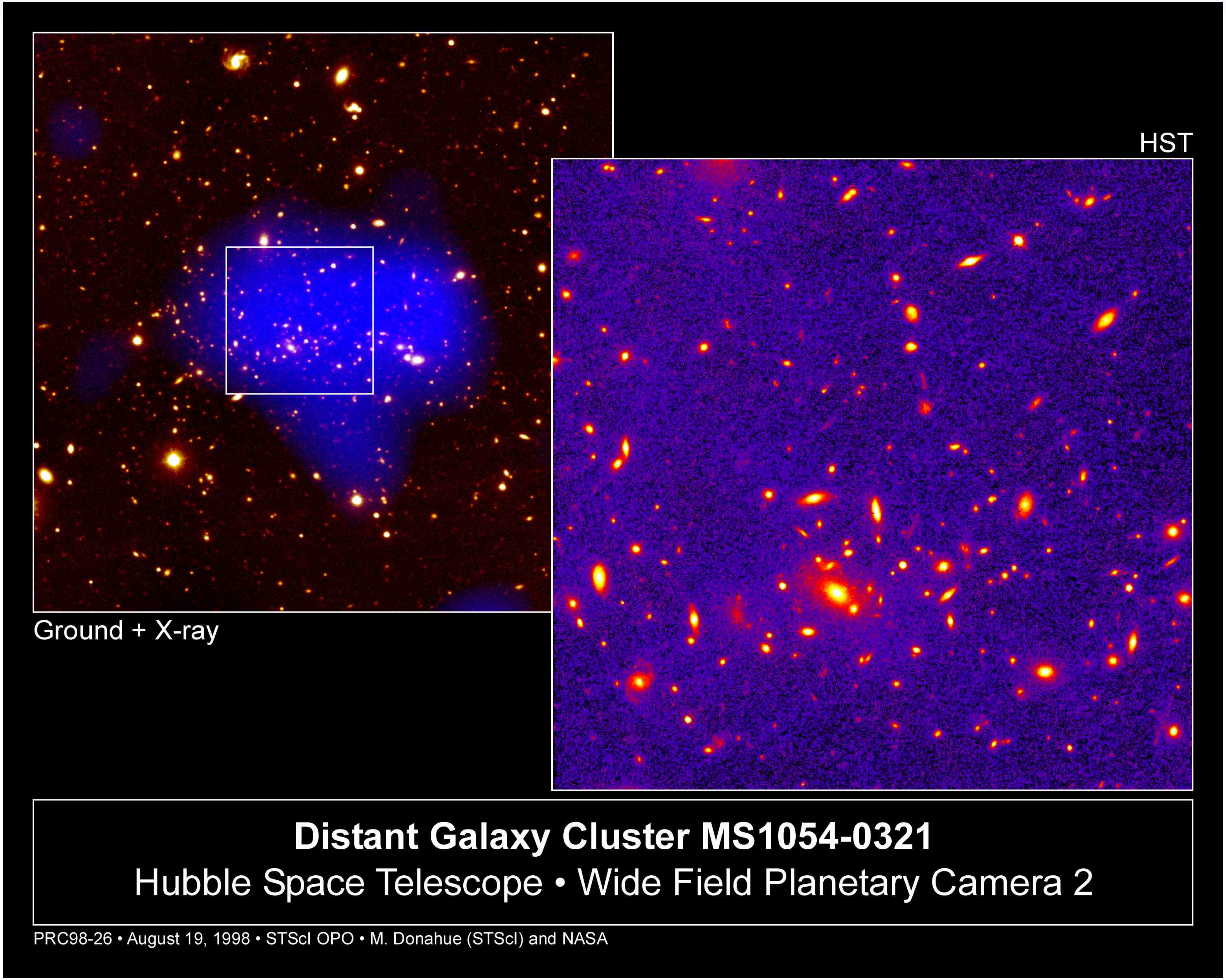
Spiral Galaxy NGC3314

Spiral Galaxies ARP274

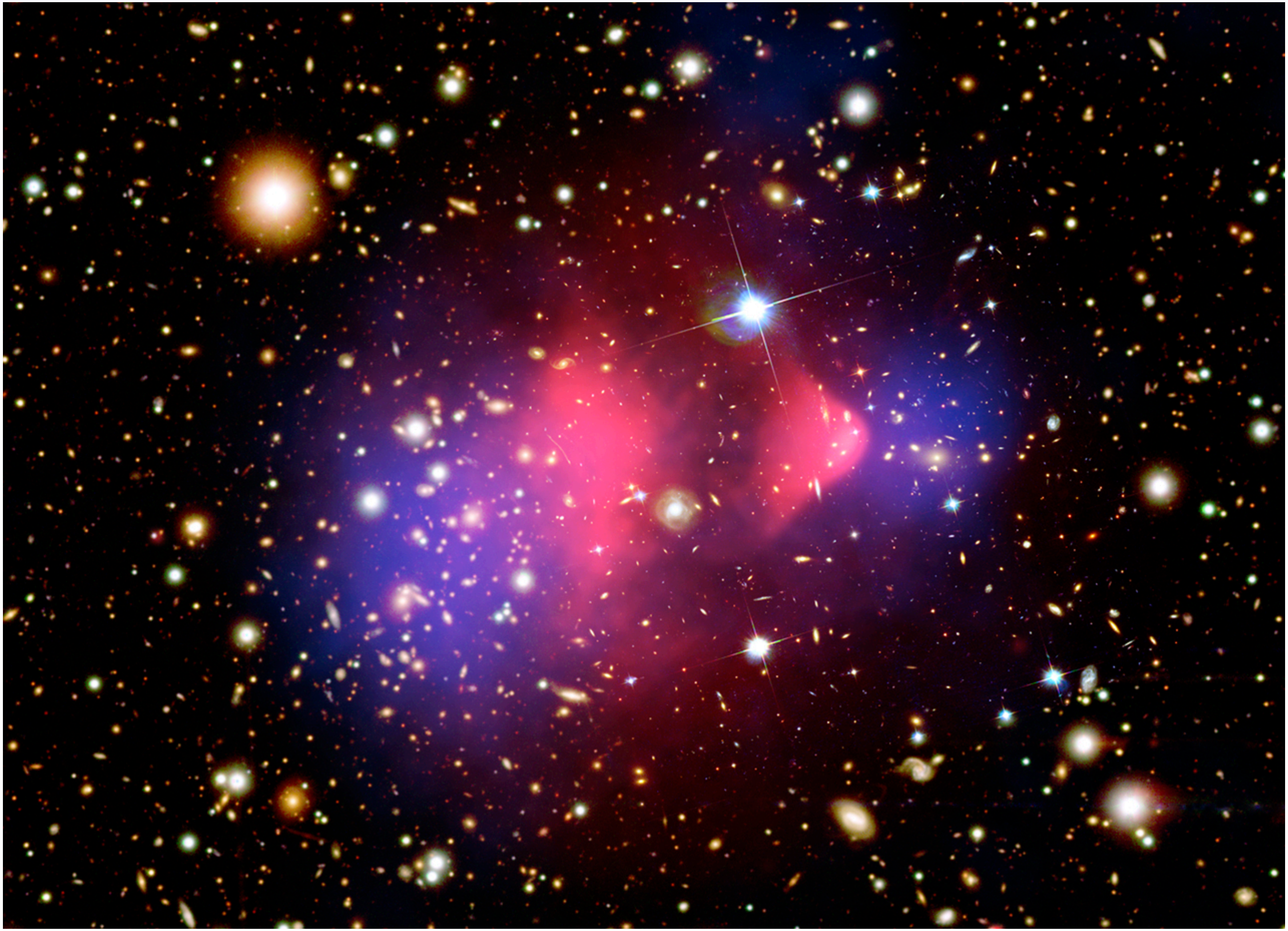
Elliptical Galaxies

Figure: Elliptical galaxies contain ellipsoidal shaped collections of stars in mostly radial orbits. Two examples are M87 (left) and NGC1132 (right).

Galaxy Cluster MS1054-0321



The Bullet Cluster



SPIRALS

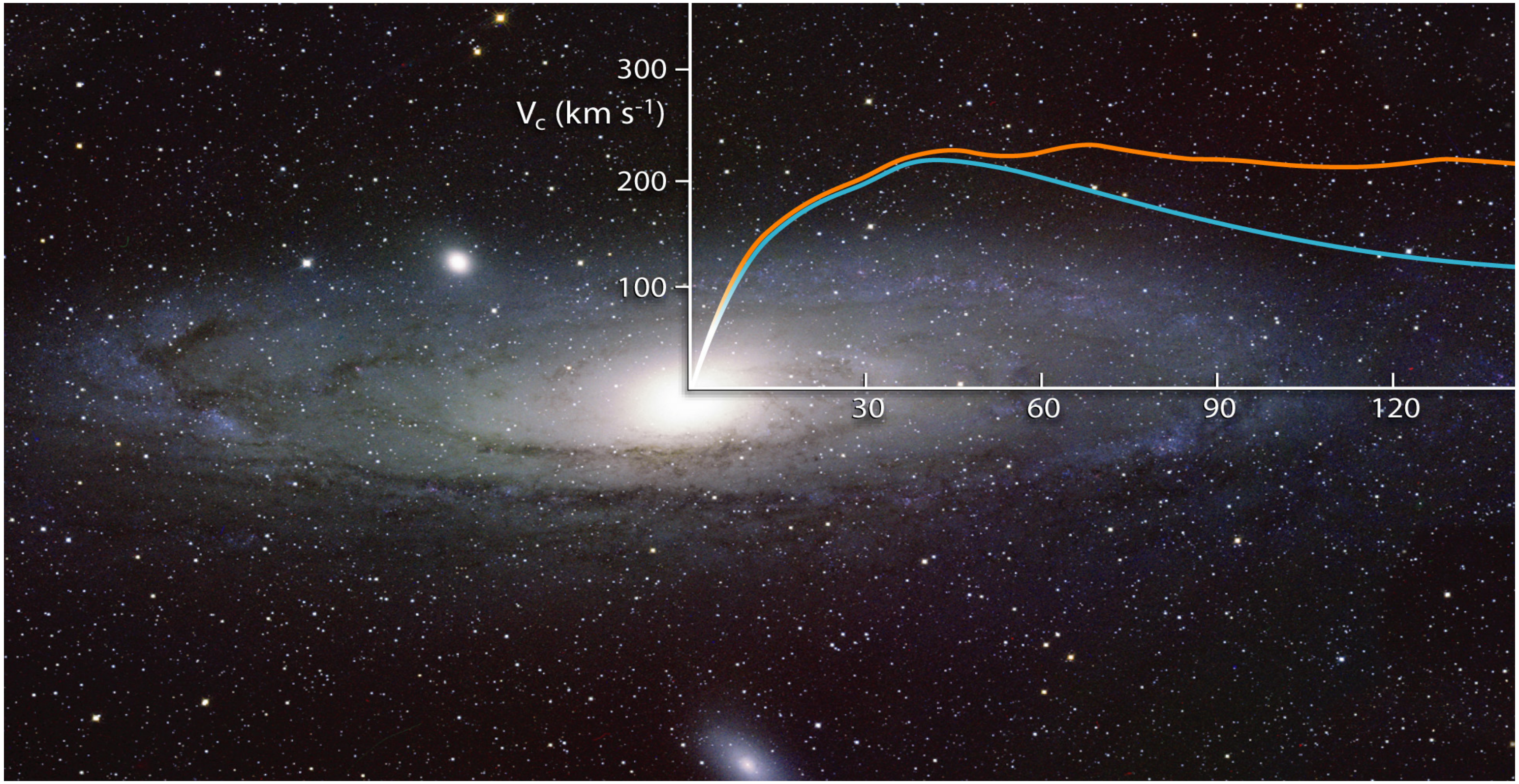


Figure: From the Dark Matter Awareness Week presentation. Presentation review at arXiv:1102.1184v1 by Paolo Salucci, Christiane Frigerio Martins, and Andrea Lapi.

What about Dark Matter and Spiral Galaxies?

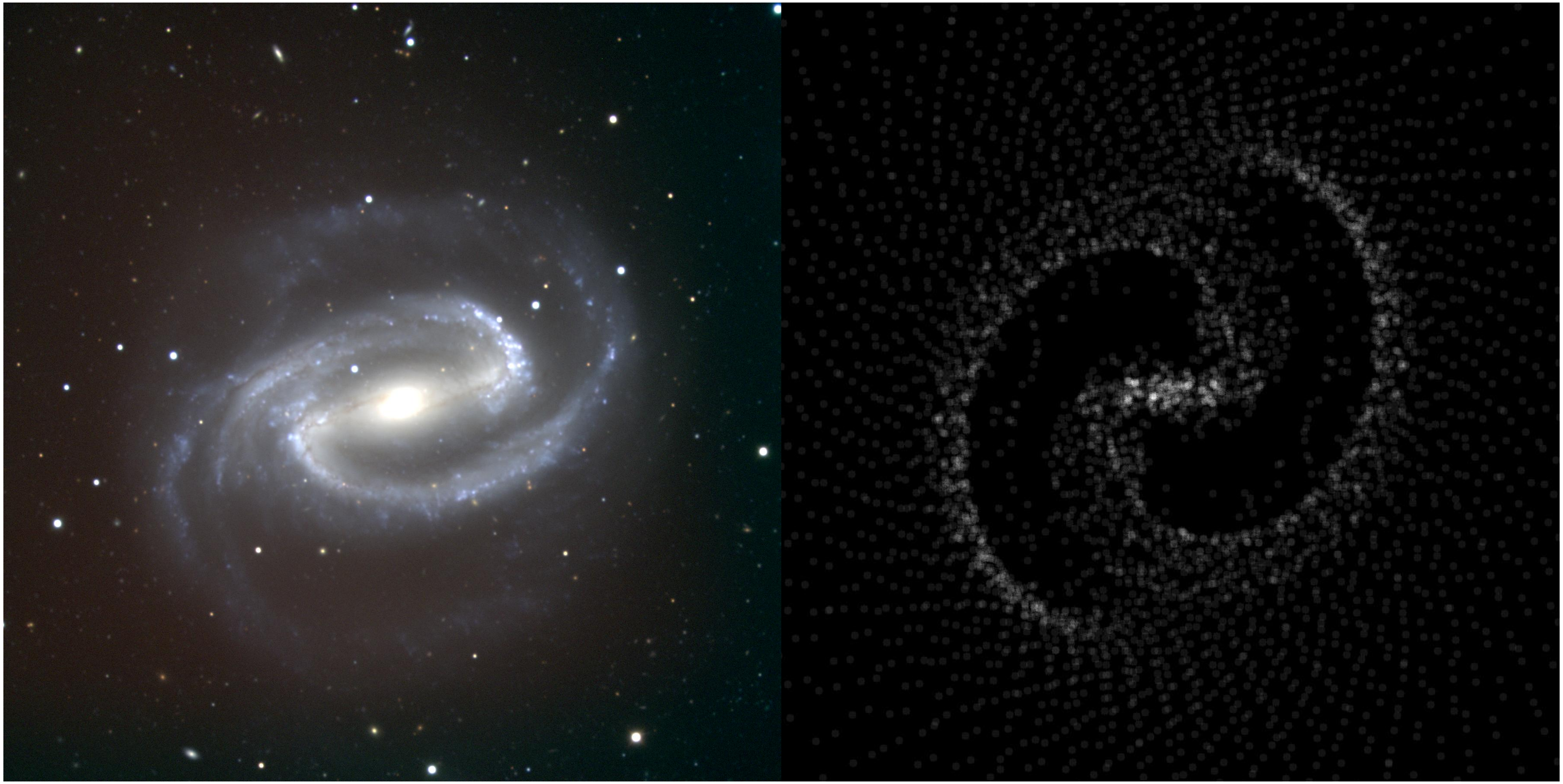
Three main ideas of my work on *wave dark matter*:

Idea 1: Natural geometric axioms motivate studying the Einstein-Klein-Gordon equations with a cosmological constant. Is the scalar field of the Klein-Gordon equation dark matter?

Idea 2: Wave types of equations, such as the Klein-Gordon equation, naturally form density waves in their matter densities.

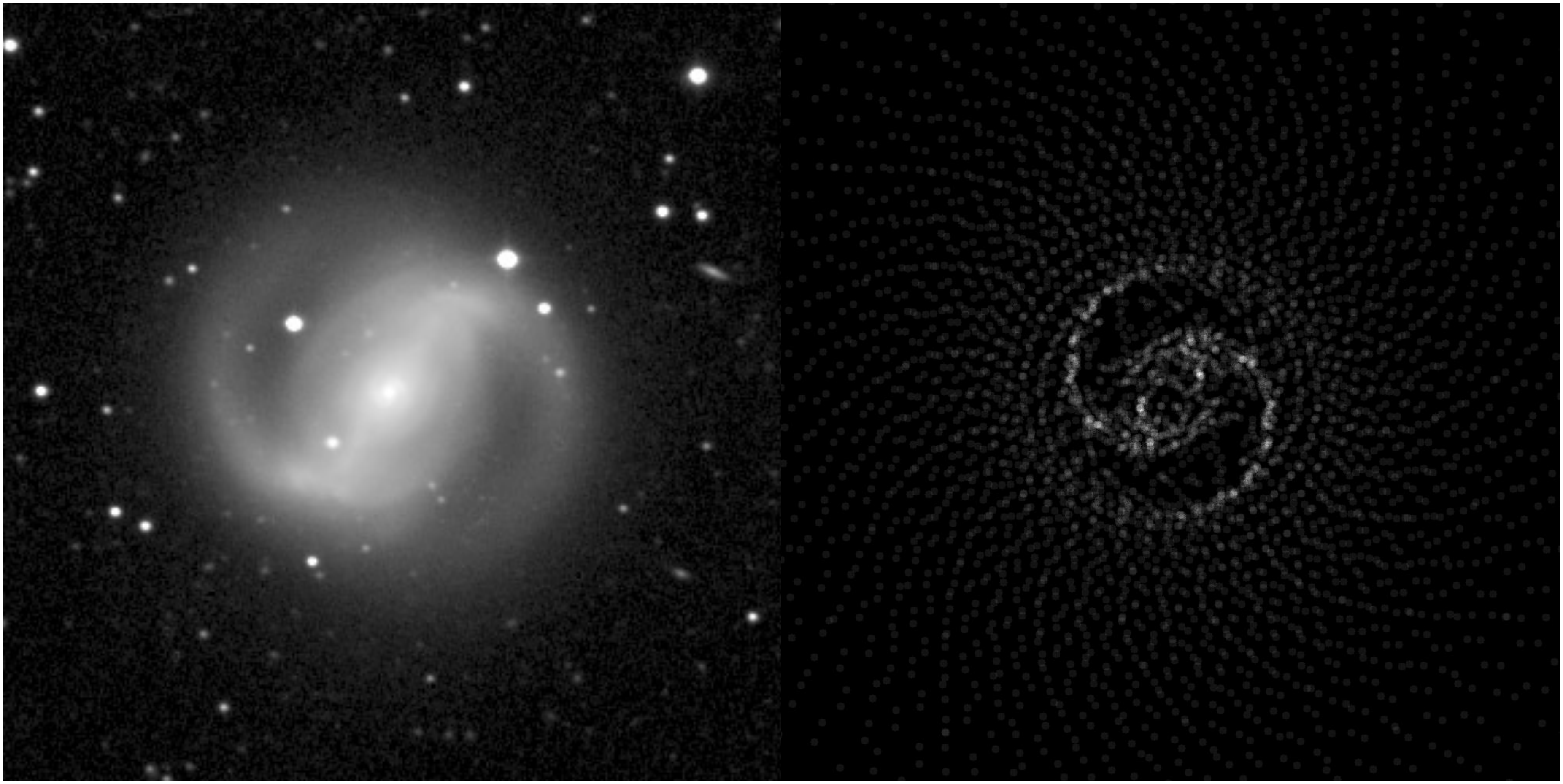
Idea 3: Density waves in dark matter, through gravity, naturally form density waves in the regular baryonic matter. Does this explain the observed spiral density waves in spiral galaxies?

Spiral Galaxy Simulation #1



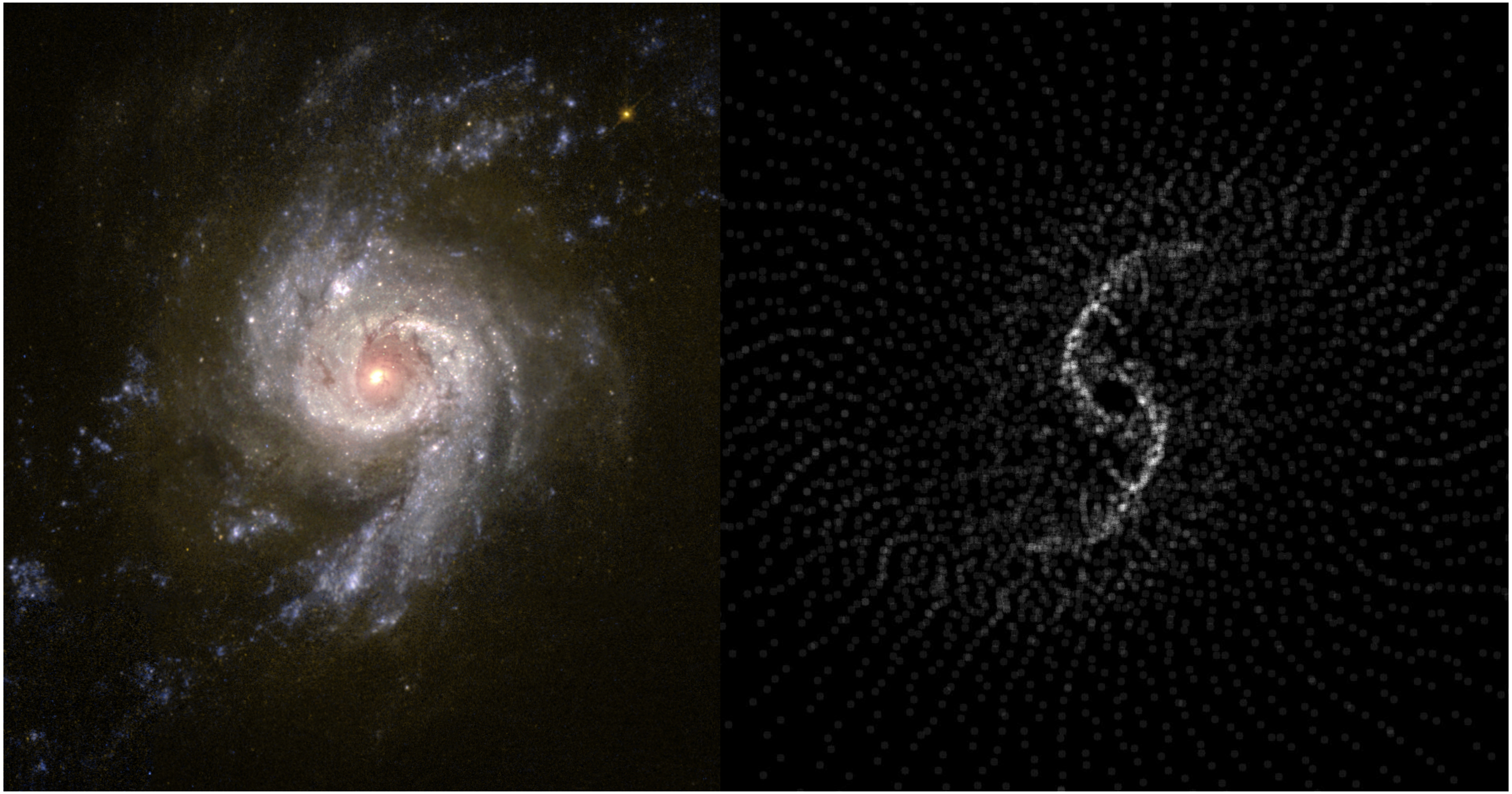
NGC1300 on the left, simulation on the right.

Spiral Galaxy Simulation #2



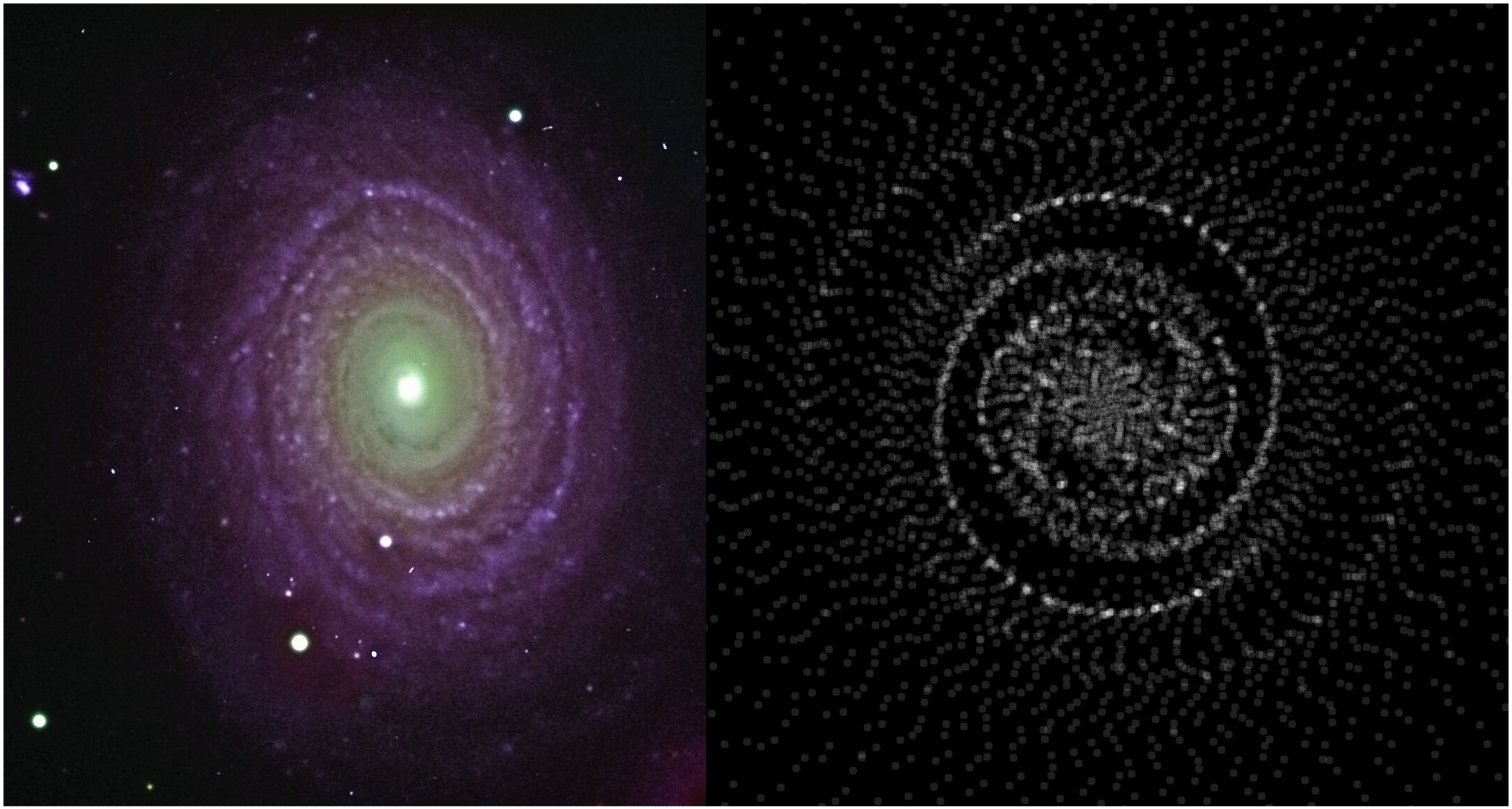
NGC4314 on the left, simulation on the right.

Spiral Galaxy Simulation #3



NGC3310 on the left, simulation on the right.

Spiral Galaxy Simulation #4

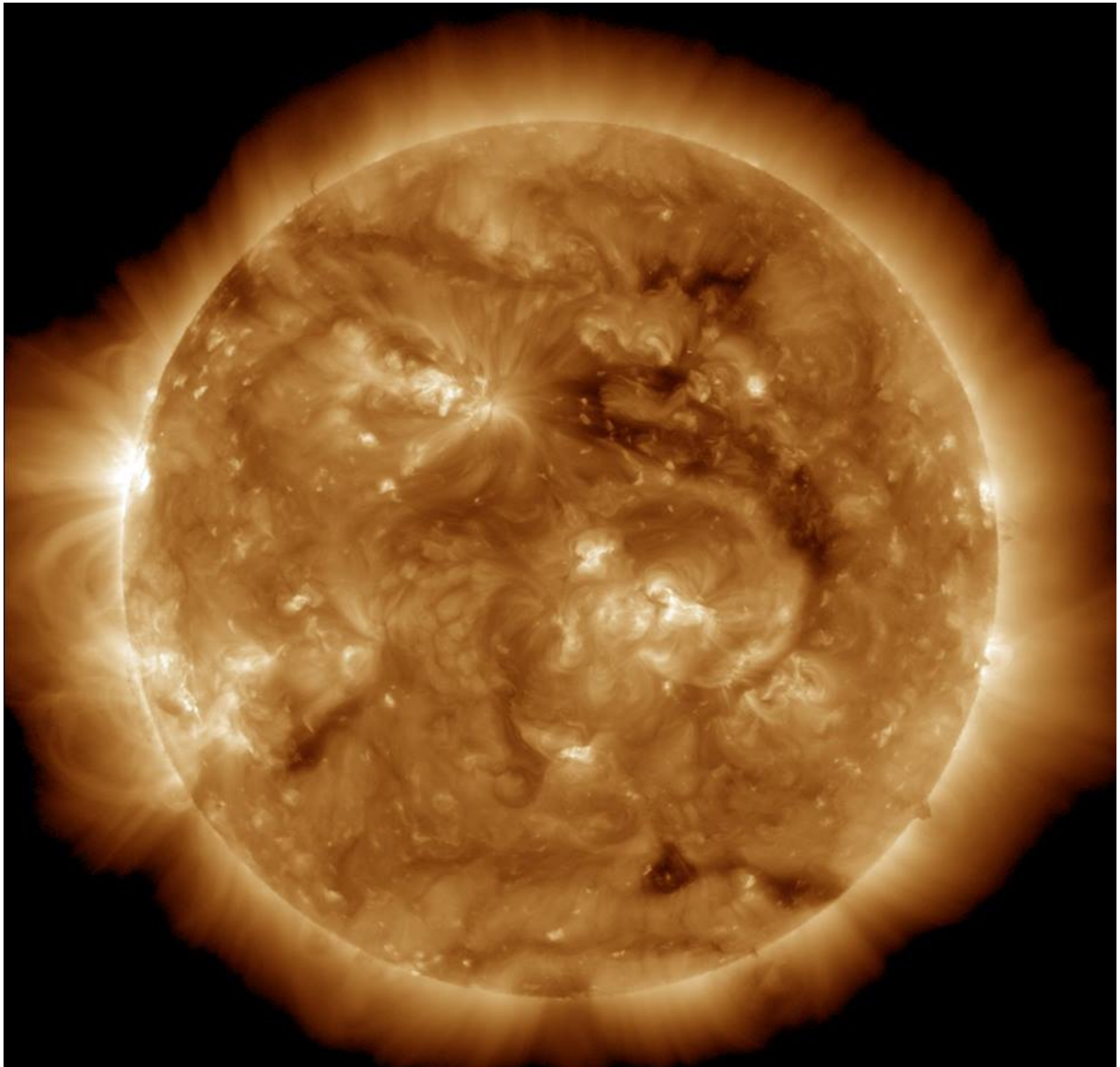


NGC488 on the left, simulation on the right.

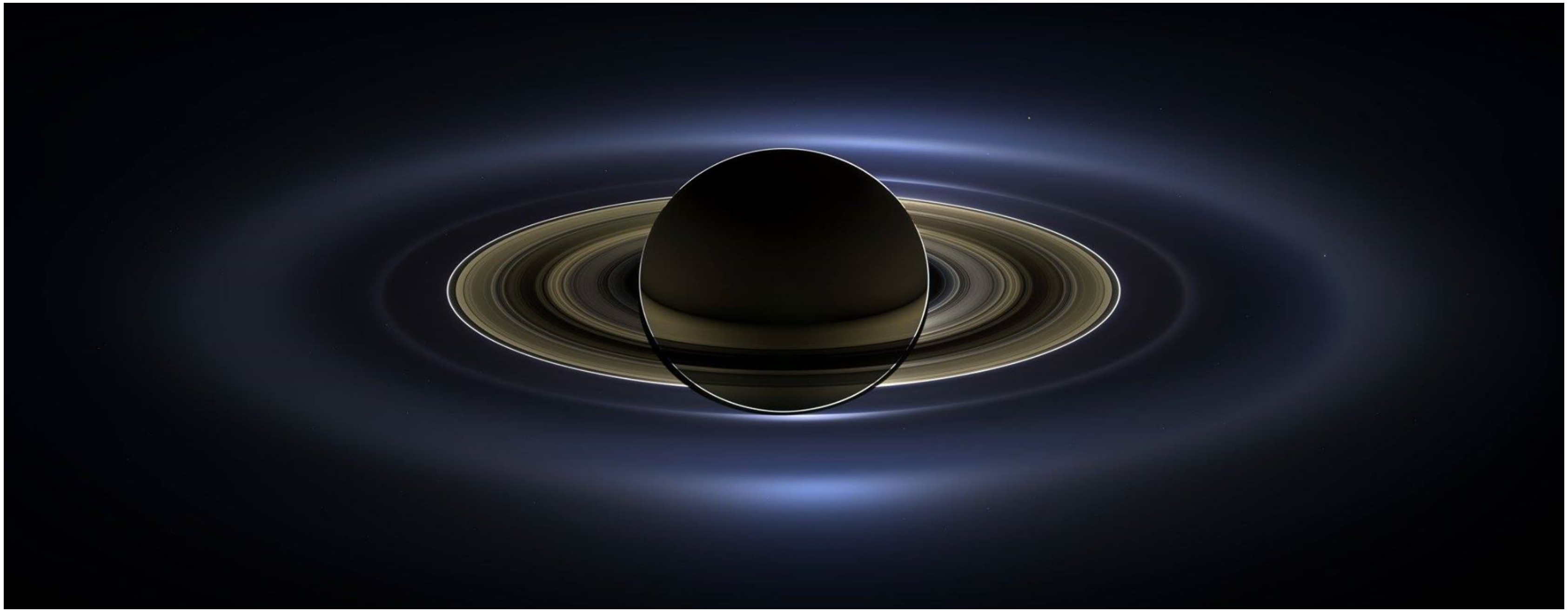
Milky Way Galaxy look-alike Galaxy NGC 6744

The Andromeda Galaxy

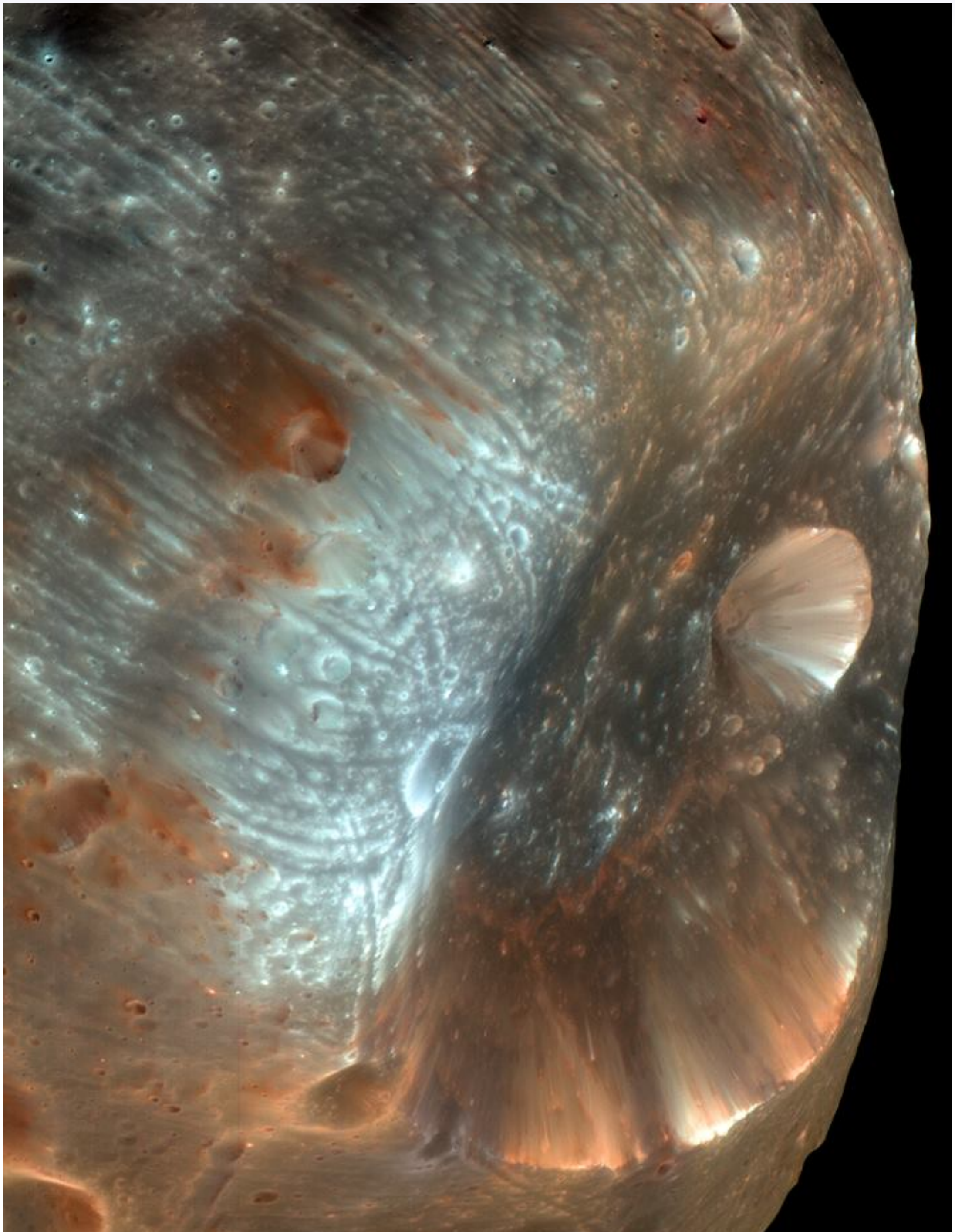
Except for hydrogen, almost all of the atoms in this room were created inside stars like our Sun that later exploded as supernovas. We are all made out of stardust.



Saturn backlit by the Sun



Stickney Crater on Phobos (the larger moon of Mars)



Phobos (the larger moon of Mars)

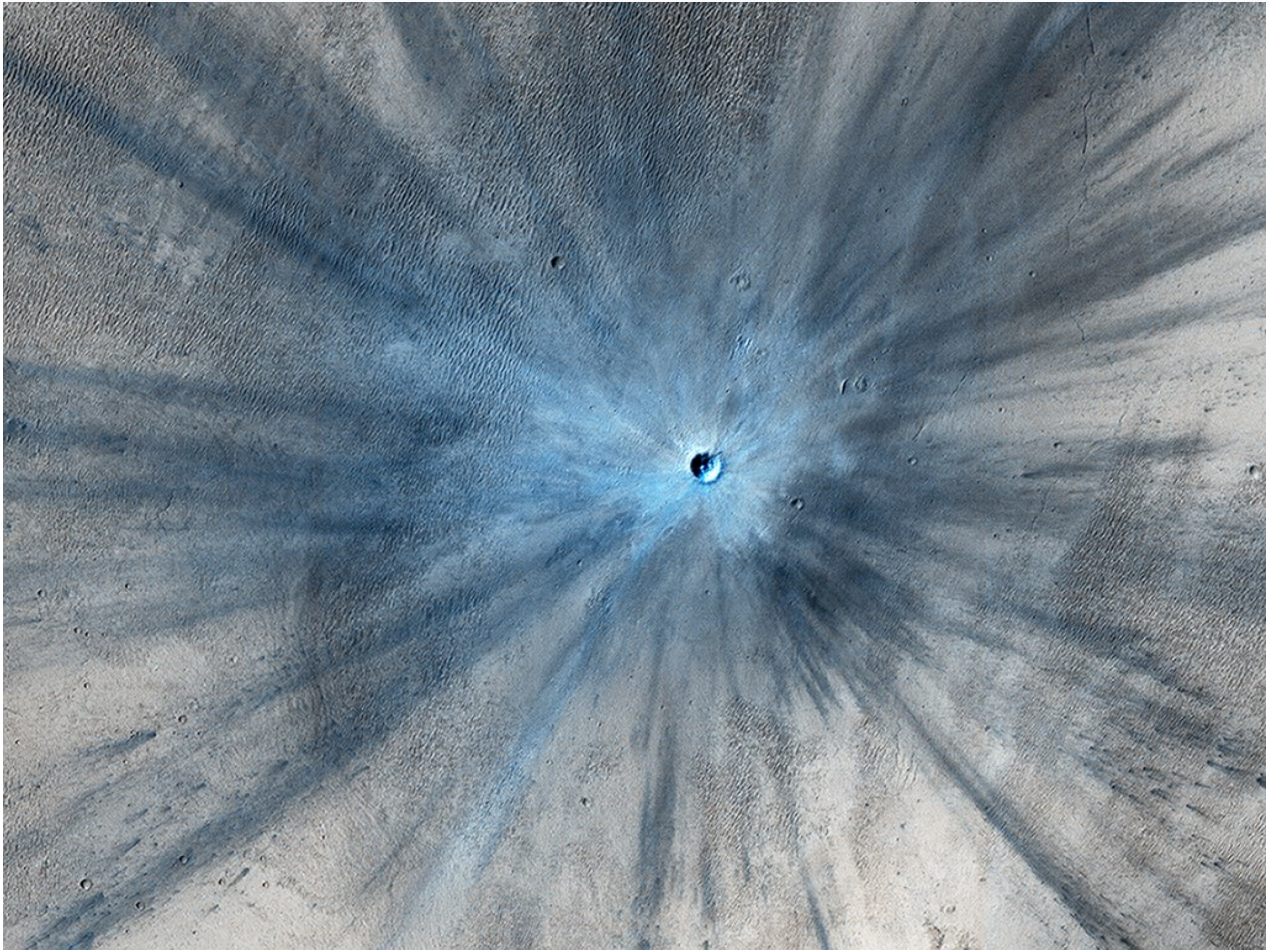
The Earth and the Moon as seen from Mars

Apollo 11 photo of the Earth from the Moon

The Earth

The Moon from the International Space Station

A New Martian Impact Crater (2010 to 2012)



My science fiction book for kids ...
with plenty of real math and science in it.