ON THE RIEMANNIAN PENROSE INEQUALITY IN DIMENSIONS LESS THAN EIGHT

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Abstract

The positive mass theorem states that a complete asymptotically flat manifold of nonnegative scalar curvature has nonnegative mass and that equality is achieved only for the Euclidean metric. The Riemannian Penrose inequality provides a sharp lower bound for the mass when black holes are present. More precisely, this lower bound is given in terms of the area of an outermost minimal hypersurface, and equality is achieved only for Schwarzschild metrics. The Riemannian Penrose inequality was first proved in three dimensions in 1997 by G. Huisken and T. Ilmanen for the case of a single black hole (see [HI]). In 1999, Bray extended this result to the general case of multiple black holes using a different technique (see [Br]). In this article, we extend the technique of [Br] to dimensions less than eight. Part of the argument is contained in a companion article by Lee [L]. The equality case of the theorem requires the added assumption that the manifold be spin.

1. Introduction

The Penrose conjecture is a longstanding conjecture in general relativity which provides a lower bound for the mass of an asymptotically flat spacelike slice of spacetime, in terms of the area of the black holes in the spacelike slice. Penrose originally formulated the conjecture as a test for the far more ambitious idea of cosmic censorship (see [P]). In the case where the asymptotically flat spacelike slice is time-symmetric, the Penrose conjecture reduces to a statement in Riemannian geometry, which we call the Riemannian Penrose inequality. In this article, we restrict our attention to the Riemannian Penrose inequality. For more background on the general Penrose conjecture, as well as some physical motivation, see [Br, Section 1] and references cited therein.

The Riemannian Penrose inequality was first proved in three dimensions in 1997 by G. Huisken and T. Ilmanen for the case of a single black hole (see [HI]). In 1999,

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Bray extended this result to the general case of multiple black holes using a different technique (see [Br]). Before we state this theorem, let us review some definitions.

Definition

Let $n \geq 3$. A Riemannian manifold (M^n, g) is said to be *asymptotically flat** if there is a compact set $K \subset M$ such that $M \setminus K$ is a disjoint union of *ends*, E_k, such that each end is diffeomorphic to $\mathbb{R}^n \setminus B_1(0)$, and in each of these coordinate charts, the metric g_{ij} satisfies

$$\begin{aligned} g_{ij} &= \delta_{ij} + O(|x|^{-p}), \\ g_{ij,k} &= O(|x|^{-p-1}), \\ g_{ij,kl} &= O(|x|^{-p-2}), \\ R_g &= O(|x|^{-q}), \end{aligned}$$

for some $p > (n-2)/2$ and some $q > n$, where the commas denote partial derivatives in the coordinate chart, and R_g is the scalar curvature of g.

In this case, in each end E_k, the limit

$$m(E_k, g) = \frac{1}{2(n-1)\omega_{n-1}} \lim_{\sigma \to \infty} \int_{S_\sigma} (g_{ij,i} - g_{ii,j})\nu_j \, d\mu$$

exists (see, e.g., [S, Section 4]), where ω_{n-1} is the area of the standard unit $(n-1)$-sphere, S_σ is the coordinate sphere in E_k of radius σ, ν is its outward unit normal, and $d\mu$ is the Euclidean area element on S_σ. We call the quantity $m(E_k, g)$, first considered by Arnowitt, Deser, and Misner (see, e.g., [ADM]), the *ADM mass* of the end (E_k, g), or when the context is clear, we simply call it the *mass*, $m(g)$. R. Bartnik showed that the ADM mass is a Riemannian invariant, independent of choice of asymptotically flat coordinates (see [B, Section 4]).

The Riemannian Penrose inequality may be thought of as a refinement of the celebrated positive mass theorem when black holes are present. Indeed, we need to use the positive mass theorem for our proof.

THEOREM 1.1 ((Riemannian) positive mass theorem)

Let (M^n, g) be a complete asymptotically flat manifold with nonnegative scalar curvature. If $n < 8$ or if M is spin, then the mass of each end is nonnegative. Moreover, if any of the ends has zero mass, then (M^n, g) is isometric to Euclidean space.

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*Note that there are various inequivalent definitions of asymptotic flatness in the literature, but they are all similar in spirit. This one is taken from [S, Section 4].

The ($n < 8$)-case was proved by R. Schoen and S.-T. Yau using minimal surface techniques (see [SY]; see also [S, Section 4]), and later E. Witten proved the spin case using a Bochner-type argument (see [Wi]; see also [B]).

Now, fix a particular end of M^n. Define $\mathcal{S}$ to be the collection of hypersurfaces that are smooth compact boundaries of open sets in M containing all of the other ends. Then each hypersurface in $\mathcal{S}$ defines a meaningful *outside* and *inside*.

Definition
A *horizon* in (M^n, g) is a minimal hypersurface in $\mathcal{S}$. A horizon Σ is *outer-minimizing** if its area minimizes area among all hypersurfaces in $\mathcal{S}$ enclosing Σ.

Definition
The *(Riemannian) Schwarzschild manifold* of dimension n and mass m is $\mathbb{R}^n \setminus \{0\}$ equipped with the metric

$$g_{ij}(x) = \Big(1 + \frac{m}{2}|x|^{2-n}\Big)^{4/(n-2)} \delta_{ij}.$$

Given a mass m, we also define the *Schwarzschild radius* of the mass m to be

$$R_{\mathrm{sc}}(m) = \Big(\frac{m}{2}\Big)^{1/(n-2)}.$$

Note that in a Schwarzschild manifold, the coordinate sphere of radius $R_{\mathrm{sc}}(m)$ is the unique outer-minimizing horizon, and its area A satisfies the equation

$$m = \frac{1}{2}\Big(\frac{A}{\omega_{n-1}}\Big)^{(n-2)/(n-1)}.$$

We can now state the main result of [Br, Theorem 1].

THEOREM 1.2 (Riemannian Penrose inequality in three dimensions)
Let (M^3, g) be a complete asymptotically flat 3-manifold with nonnegative scalar curvature. Fix one end. Let m be the mass of that end, and let A be the area of an outer-minimizing horizon (with one or more components). Then

$$m \geq \sqrt{\frac{A}{16\pi}}$$

with equality if and only if the part of (M, g) outside the horizon is isometric to a Riemannian Schwarzschild manifold outside its unique outer-minimizing horizon.

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*Note that *outermost* implies *outer-minimizing*.

Even though the original motivation from general relativity may have been specific to three dimensions, because of string theory there is a great deal of interest in higher-dimensional black holes. More importantly, from a purely geometric perspective, there appears to be nothing inherently three-dimensional about the Riemannian Penrose inequality, so it is natural to wonder whether the result holds in higher dimensions. The goal of this article is to prove the following generalization.

THEOREM 1.3 (Riemannian Penrose inequality in dimensions less than eight)
Let (M^n, g) *be a complete asymptotically flat manifold with nonnegative scalar curvature, where* $n < 8$. *Fix one end. Let* m *be the mass of that end, and let* A *be the area of an outer-minimizing horizon (with one or more components). Let* ω_{n-1} *be the area of the standard unit* $(n-1)$*-sphere. Then*

$$m \geq \frac{1}{2}\Big(\frac{A}{\omega_{n-1}}\Big)^{(n-2)/(n-1)}.$$

Furthermore, if we also assume that M *is spin, then equality occurs if and only if the part of* (M, g) *outside the horizon is isometric to a Riemannian Schwarzschild manifold outside its unique outer-minimizing horizon.*

Remark. We conjecture that the last statement, the "equality case," holds without the spin assumption. See Theorem 2.8 and Sections 6 and 7 for details on why the spin assumption appears.*

Our proof is limited to dimensions less than eight for the same reason that Schoen and Yau's proof of the positive mass theorem is limited; we need to use regularity of minimal hypersurfaces.

The geometry inside the horizon plays no role at all in the proof of the theorem. Accordingly, our objective is to prove the following theorem.

THEOREM 1.4
Let (M^n, g) *be a complete one-ended asymptotically flat manifold with boundary, where* $n < 8$. *If* (M, g) *has nonnegative scalar curvature, and if the boundary is an outer-minimizing horizon (with one or more components) with total area* A*, then*

$$m \geq \frac{1}{2}\Big(\frac{A}{\omega_{n-1}}\Big)^{(n-2)/(n-1)}.$$

Furthermore, if we also assume that M *is spin, then equality occurs if and only if* (M, g) *is isometric to a Riemannian Schwarzschild manifold outside its unique outer-minimizing horizon.*

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*Sections 6 and 7 are independent of Sections 3, 4, and 5.

We prove this theorem using Bray's conformal flow method in [Br]. One might wonder whether Huisken and Ilmanen's inverse mean curvature flow method in [HI] could also be used for this purpose. Unfortunately, since the Gauss-Bonnet theorem lies at the heart of that method, it would require major new insights to adapt it to higher dimensions. Note that use of the Gauss-Bonnet theorem is also the reason why [HI] does not give the optimal result for multiple black holes.

2. Overview of proof

The vast majority of Bray's proof of the Riemannian Penrose inequality in dimension three applies to dimensions less than eight (see [Br]). In this section, we review the main features of the proof and describe the parts that require modification. Since a large portion of our proof is actually contained in [Br], we try to maintain consistent notation. The main technical tool that we employ is the conformal flow.

Definition

Let M be a manifold with a distinguished end. Let g_t be a family of metrics on M, and let $\Sigma(t)$ be a family of hypersurfaces in $\mathcal{S}$ such that $g_t(x)$ is Lipschitz in t, C^1 in x, and smooth in x outside $\Sigma(t)$. We say that $(M, g_t, \Sigma(t))$ is a *conformal flow* if and only if the following conditions hold for each t:

- (M, g_t) outside $\Sigma(t)$ is a complete asymptotically flat manifold with boundary, and it has nonnegative scalar curvature;
- $\Sigma(t)$ is an outer-minimizing horizon in (M, g_t);
- $\frac{d}{dt}g_t = (4/(n-2))\nu_t g_t$, where $\nu_t(x) = 0$ inside $\Sigma(t)$, and outside $\Sigma(t)$, ν_t is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta_{g_t}\nu_t(x) = 0 & \text{outside } \Sigma(t), \\ \nu_t(x) = 0 & \text{at } \Sigma(t), \\ \lim_{x\to\infty} \nu_t(x) = -1. \end{cases}$$

The formulation of the conformal flow in [Br] is slightly different but defines the same flow. Instead of using the last item in the above definition, we could set $g_t = u_t^{4/(n-2)} g_0$ and demand that

$$\frac{d}{dt}u_t = v_t,$$

where $v_t(x) = 0$ inside $\Sigma(t)$, and outside $\Sigma(t)$, v_t is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta_{g_0}v_t(x) = 0 & \text{outside } \Sigma(t), \\ v_t(x) = 0 & \text{at } \Sigma(t), \\ \lim_{x\to\infty} v_t(x) = -e^{-t}. \end{cases}$$

The fact that these two formulations are equivalent follows from the following simple lemma (see [Br, Appendix A]), which we use repeatedly.

LEMMA 2.1
If g_1 *and* g_2 *are smooth metrics and* ϕ *is a smooth function such that*

$$g_2 = \phi^{4/(n-2)} g_1,$$

then for any smooth function f*,*

$$\Delta_{g_1}(f\phi) = \phi^{(n+2)/(n-2)} \Delta_{g_2} f + f \Delta_{g_1} \phi.$$

THEOREM 2.2
Given initial data $(M^n, g_0, \Sigma(0))$ *satisfying the first two properties of the conformal flow described above, with* $n < 8$*, there exists a conformal flow* $(M, g_t, \Sigma(t))$ *for all* $t \geq 0$*. Moreover,*

- *for all* $t_2 > t_1 \geq 0$*,* $\Sigma(t_2)$ *encloses* $\Sigma(t_1)$ *without touching it;*
- $\Sigma(t)$ *can "jump" at most countably many times; at these jump times, we write* $\Sigma^-(t)$ *and* $\Sigma^+(t)$ *to denote the hypersurface "before" and "after" it jumps, respectively.*[*]

The proof of this theorem in [Br, Theorem 2] is unchanged in higher dimensions, as long as $n < 8$. The basic idea behind the proof is to use a discrete time approximation and then take the limit as the length of the discrete time intervals approaches zero. The $(n < 8)$-hypothesis is required in the proof in order to find smooth outermost minimal area enclosures. The only other place that this dimensional restriction is used again is when we invoke the positive mass theorem.

To prove our main theorem (Theorem 1.4), we use the hypotheses of the theorem as initial data for the conformal flow and prove that the conformal flow has the following properties:

- the area of $\Sigma(t)$ in (M, g_t) is constant in t; call it A;
- the mass of (M, g_t), which we call $m(t)$, is nonincreasing;
- with the right choice of end coordinates, the metric g_t outside $\Sigma(t)$ converges to a Schwarzschild metric;
- the area of the horizon in this Schwarzschild manifold is greater than or equal to A.

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[*] See [Br, Section 4] for a precise statement. With the definition of the conformal flow given above, at a jump time, $\Sigma(t)$ could lie somewhere between $\Sigma^-(t)$ and $\Sigma^+(t)$. However, in the construction of the conformal flow in [Br, Section 4], $\Sigma(t)$ is the outermost horizon in (M, g_t) containing $\Sigma(0)$, and consequently, we have $\Sigma(t) = \Sigma^+(t)$ for $t > 0$.

Once we have established these properties, the main theorem (Theorem 1.4) follows immediately.[*]

LEMMA 2.3
The area of $\Sigma(t)$ in (M, g_t) is constant in t.

The proof of this lemma in [Br, Section 5] is unchanged in higher dimensions. The basic idea behind the proof is that the rate of change of the area $|\Sigma(t)|_{g_t}$ has a contribution from changing $\Sigma(t)$ while leaving g_t fixed and a contribution from changing g_t while leaving $\Sigma(t)$ fixed. The first contribution is zero because $\Sigma(t)$ is minimal, and the second contribution is zero because the metric is not changing at $\Sigma(t)$. (Specifically, $\frac{d}{dt}g_t = (4/(n-2))v_t g_t = 0$ at $\Sigma(t)$.) However, the proof is more subtle than this because $\Sigma(t)$ can jump (see [Br, Section 5] for details).

In order to simplify the rest of our arguments, we use a tool called harmonic flatness.

Definition
A Riemannian manifold (M^n, g) is said to be *harmonically flat at infinity* if there is a compact set $K \subset M$ such that $M \setminus K$ is the disjoint union of *ends*, E_k, such that each end is diffeomorphic to $\mathbb{R}^n \setminus B_{r_k}(0)$, and in each of these coordinate charts, there is a (Euclidean) harmonic function $\mathcal{U}$ such that

$$g_{ij}(x) = \mathcal{U}(x)^{4/(n-2)}\delta_{ij}.$$

In other words, each end is conformally flat with a harmonic conformal factor.

Note that a harmonically flat end necessarily has zero scalar curvature. Expanding $\mathcal{U}$ in spherical harmonics in a particular end E_k, we see that

$$\mathcal{U}(x) = a + b|x|^{2-n} + O(|x|^{1-n})$$

for some constants a and b. Clearly, a manifold that is harmonically flat at infinity is asymptotically flat.[†] A simple computation shows the following.

LEMMA 2.4
In the situation described above, the mass of the end E_k is equal to $2ab$.

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[*] It follows except for the case of equality, which requires an additional simple argument.
[†] However, when $a \neq 1$, it is necessary to change the distinguished coordinate chart.

A short argument of Schoen and Yau [SY] (see also [S, Section 4], [Br, Section 2]) implies the following lemma.

LEMMA 2.5
In order to prove our main theorem (Theorem 1.4, excluding the equality case), we may assume, without loss of generality, that (M^n, g) *is harmonically flat at infinity.*

From now on, we always work in the situation of initial data that is harmonically flat at infinity, and then evolved by the conformal flow. From Lemma 2.1, it is clear that the conformal flow preserves harmonic flatness outside $\Sigma(t)$.

We now consider a third formulation of the conformal flow. By the harmonic flatness assumption, we know that $(g_0)_{ij}(x) = \mathcal{U}_0(x)^{4/(n-2)}\delta_{ij}$ for some harmonic function $\mathcal{U}_0$ on the exterior region $\mathbb{R}^n \setminus B_{R_\mathrm{h}}$ for some R_h. (We adopt the shorthand notation $B_R = B_R(0)$ and $S_R = S_R(0)$.) We choose end coordinates so that $\lim_{x\to\infty}\mathcal{U}_0(x) = 1$, and consequently, we are not allowed to choose the constant R_h arbitrarily. Now, extend $\mathcal{U}_0$ to a positive function on all of M, and define the metric g_flat by

$$g_0 = \mathcal{U}_0^{4/(n-2)} g_\mathrm{flat}.$$

Note that for $|x| > R_\mathrm{h}$, $(g_\mathrm{flat})_{ij}(x) = \delta_{ij}$. We can now reformulate the conformal flow by setting

$$g_t = \mathcal{U}_t^{4/(n-2)} g_\mathrm{flat}$$

and demanding that

$$\frac{d}{dt}\mathcal{U}_t = \mathcal{V}_t,$$

where $\mathcal{V}_t(x) = 0$ inside $\Sigma(t)$, and outside $\Sigma(t)$, $\mathcal{V}_t$ is the unique solution to the Dirichlet problem*

$$\begin{cases} \Delta_{g_\mathrm{flat}}\mathcal{V}_t - \big(\frac{\Delta_{g_\mathrm{flat}}\mathcal{U}_0}{\mathcal{U}_0}\big)\mathcal{V}_t = 0 & \text{outside } \Sigma(t), \\ \mathcal{V}_t(x) = 0 & \text{at } \Sigma(t), \\ \lim_{x\to\infty}\mathcal{V}_t(x) = -e^{-t}. \end{cases}$$

Note that in the region outside $\Sigma(t)$ with $|x| > R_\mathrm{h}$, both $\mathcal{U}_t$ and $\mathcal{V}_t$ are (Euclidean) harmonic functions. We summarize the relationships between the three formulations

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*We know that there is a unique solution because this formulation is equivalent to the previous ones.

of the conformal flow:

$$\mathcal{U}_t = u_t \mathcal{U}_0,$$

$$\mathcal{V}_t = v_t \mathcal{U}_0,$$

$$\nu_t = \frac{v_t}{u_t} = \frac{\mathcal{V}_t}{\mathcal{U}_t}.$$

LEMMA 2.6
The mass, $m(t)$, is nonincreasing.

The proof of this lemma in [Br, Sections 6, 7] is also unchanged in higher dimensions. However, the main idea used in the proof is central to this article, so we summarize the basic argument.

For each time t, consider the two-ended manifold $(\bar{M}_{\Sigma(t)}, \bar{g}_t)$ obtained by reflecting the manifold (M, g_t) through $\Sigma(t)$. Let ω_t be the g_t-harmonic function that approaches 1 at one end and 0 at the other end. We can use ω_t to conformally close the 0-end by considering the metric $\tilde{g}_t = (\omega_t)^{4/(n-2)}\bar{g}_t$ on $\bar{M}_{\Sigma(t)}$. The result is a new one-ended manifold $(\tilde{M}_{\Sigma(t)} = \bar{M}_{\Sigma(t)} \cup \{\mathrm{pt}\}, \tilde{g}_t)$ with nonnegative scalar curvature.* Similarly, if t is a jump time, then we can construct $(\tilde{M}_{\Sigma^\pm(t)}, \tilde{g}_t^\pm)$ by first reflecting through $\Sigma^\pm(t)$. Lemma 2.6 follows from the following key lemma.

LEMMA 2.7
Let $\tilde{m}(t)$ be the mass of $(\tilde{M}_{\Sigma(t)}, \tilde{g}_t)$. If t is not a jump time, then

$$\frac{d}{dt}m(t) = -2\tilde{m}(t).$$

If t is a jump time, let $\tilde{m}^\pm(t)$ be the mass of $(\tilde{M}_{\Sigma^\pm(t)}, \tilde{g}_t^\pm)$. Then

$$\frac{d}{dt^\pm}m(t) = -2\tilde{m}^\pm(t),$$

where $\frac{d}{dt^\pm}m(t)$ denotes the right and left side limits of $\frac{d}{dt}m(t)$.

Lemma 2.6 almost follows immediately from this lemma because the positive mass theorem should tell us that $\tilde{m}(t) \geq 0$ (and that $\tilde{m}^\pm(t) \geq 0$). However, there is a technical point to deal with here: Since the metric $\tilde{g}_t$ is not smooth along $\Sigma(t)$, where the gluing took place, the standard version of the positive mass theorem does not immediately apply. Instead, we apply the following extension of the positive mass

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*One can show that the singularity at pt is removeable.

theorem. The statement has been culled from the work of Bray [Br, Section 6], P. Miao [M], and Y. Shi and L.-F. Tam [ST, Section 3], with some added precision.

THEOREM 2.8
Suppose that $n < 8$, *or suppose that* M *is spin. Let* K *("inside") be a compact subset of a manifold* M *such that* $M \setminus K$ *("outside") is diffeomorphic to an exterior region of* $\mathbb{R}^n$, *and* $\partial K = \Sigma$ *is a smooth hypersurface of* M. *Let* g *be an asymptotically flat metric on* M *which is smooth away from* Σ *and is* C^2 up to Σ *from each side. Assume that* g *has nonnegative scalar curvature away from* Σ, *and assume that* $H_{\text{out}} = H_{\text{in}}$, *where* H_{out} *(resp.,* H_{in}*) is the mean curvature of* Σ *as computed by the outside (resp., inside) metric. Then the mass of* g *is nonnegative.*

Furthermore, if we also assume that M *is spin, then the mass of* g *is zero if and only if* (M, g) *is Euclidean space, or more precisely, there exists a bijective coordinate chart** $M \longrightarrow \mathbb{R}^n$ *such that* $g_{ij}(x) = \delta_{ij}$.

Note that $(\tilde{M}_{\Sigma(t)}, \tilde{g}_t)$ satisfies the hypotheses of Theorem 2.8 since $\Sigma(t)$ is minimal in (M, g_t). Thus $\tilde{m}(t) \geq 0$ (and, similarly, $\tilde{m}^{\pm}(t) \geq 0$), proving Lemma 2.6.

Remark. For the $(n < 8)$-case, the first paragraph of Theorem 2.8 appears in [M] (following [Br]), and the proof does not use spinors. Unfortunately, the "equality case" was not established by [M]. On the other hand, [ST] used spinors to prove all of Theorem 2.8, including the equality case, under the assumption that M is spin.

Remark. We conjecture that the equality case of Theorem 2.8 holds without the spin assumption. This would allow us to remove the spin assumption from the equality case of Theorem 1.4 (see Sections 6 and 7 for details).

Proof of Lemma 2.7
For ease of notation, let us assume that t is not a jump time. (The proof for jump times is the same but with ± superscripts everywhere.) By symmetry, we know that the function ω_t used in the construction of $\tilde{g}_t$ must be $(1/2)(1 - \nu_t)$ on one end (and $(1/2)(1 + \nu_t)$ on the end to be closed up). Therefore, in the one end of $\tilde{M}_{\Sigma(t)}$, for $|x| > R_{\mathrm{h}}$,

$$\begin{aligned}(\tilde{g}_t)_{ij}(x) &= \Big[\frac{1}{2}\big(1 - \nu_t(x)\big)\Big]^{4/(n-2)}(g_t)_{ij}(x)\\ &= \Big[\frac{1}{2}\big(1 - \nu_t(x)\big)\mathcal{U}_t(x)\Big]^{4/(n-2)}\delta_{ij}\\ &= \Big[\frac{1}{2}\big(\mathcal{U}_t(x) - \mathcal{V}_t(x)\big)\Big]^{4/(n-2)}\delta_{ij}.\end{aligned}$$

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*These coordinates might be only $C^{1,1}$ across Σ, but they are smooth elsewhere.

We now compute $\tilde{m}(t)$ by expanding $(1/2)(\mathcal{U}_t(x) - \mathcal{V}_t(x))$. For $|x| > R_{\mathrm{h}}$, $\mathcal{U}_t(x)$ is harmonic, and thus we can expand it as

$$\mathcal{U}_t(x) = A(t) + B(t)|x|^{2-n} + O(|x|^{1-n}).$$

Therefore

$$\mathcal{V}_t(x) = A'(t) + B'(t)|x|^{2-n} + O(|x|^{1-n}).$$

We know that $A(0) = 1$ and $A'(t) = \lim_{x\to\infty} \mathcal{V}_t(x) = -e^{-t}$, so we can write

$$\mathcal{U}_t(x) = e^{-t} + \frac{1}{2}e^t m(t)|x|^{2-n} + O(|x|^{1-n}),$$

$$\mathcal{V}_t(x) = -e^{-t} + \frac{1}{2}e^t\big(m(t) + m'(t)\big)|x|^{2-n} + O(|x|^{1-n}),$$

$$\frac{1}{2}\big(\mathcal{U}_t(x) - \mathcal{V}_t(x)\big) = e^{-t} - \frac{1}{4}e^t m'(t)|x|^{2-n} + O(|x|^{1-n}).$$

Thus $\tilde{m}(t) = -(1/2)m'(t)$. □

Therefore, in order to prove Theorem 1.4, the only part of [Br] that needs to be modified is the part that deals with the convergence to Schwarzschild. The basic idea here is that since $m(t)$ is nonincreasing and bounded below by zero (by positive mass theorem), we might hope that its derivative, $-2\tilde{m}(t)$, converges to zero. Indeed, that turns out to be the case (see Lemma 3.1). The equality case of the positive mass theorem states that the only complete asymptotically flat manifold of nonnegative scalar curvature and zero mass is Euclidean space. Therefore we might also hope that since $\tilde{m}(t)$ is converging to zero, $\tilde{g}_t$ must converge to the flat metric at infinity in some sense. In order to establish this fact, we need a strengthened version of the equality case of the positive mass theorem (see Theorem 3.4), which is proved in a separate article (see [L, Theorem 1.4]). Then it is not hard to see that (with the right choice of end coordinates) g_t must converge to a Schwarzschild metric outside $\Sigma(t)$.*

In order to make this basic argument work, we need to control $\Sigma(t)$. (Specifically, we need Lemma 3.3.) In [Br], this control was obtained using curvature estimates by way of the Gauss-Bonnet theorem, together with a Harnack-type inequality from [BI] which is applicable only in three dimensions. It is this part of the proof that needs to be completely reworked for application to higher dimensions. Even though our new

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*This is a refined version of the fact that the only asymptotically flat manifolds that are scalar-flat and conformal to Euclidean $\mathbb{R}^n \setminus \{0\}$ are Schwarzschild manifolds.

proof is more general, it is actually more elementary and straightforward than the original proof. This content appears in Section 5 of this article.

Section 3 of this article serves as a replacement for [Br, Sections 8 – 12], although there is a fair amount of overlap. We summarize the differences. First, using the three-dimensional curvature estimates described above, it was proved in [Br] that $\Sigma(t)$ eventually encloses any bounded region, and consequently, one can then assume that M is an exterior domain of $\mathbb{R}^3$. It turns out that this simplification is not actually needed for our proof, but it means that we have to be a bit more careful than in [Br]. Second, the strengthened version of the equality case of the positive mass theorem (Theorem 3.4) mentioned above was proved in [Br, Corollary 8] using spinors. We need a different proof here since higher-dimensional manifolds need not be spin; the proof is given in [L, Theorem 1.4]. Third, with the benefit of hindsight, we are able to simplify and streamline many aspects of the original proof.

3. Convergence to Schwarzschild

As mentioned earlier, we want to show that $\tilde{m}(t)$ converges to zero as $t \to \infty$.

LEMMA 3.1
We have

$$\lim_{t\to\infty} \tilde{m}(t) = 0.$$

Proof
We have the following.

CLAIM
The quantity $e^{2t}(m(t) + m'(t))$ *is nondecreasing in* t.

Proof
Recall that for $t_2 > t_1$, $\Sigma(t_2)$ encloses $\Sigma(t_1)$. By the maximum principle and the definition of v_t, it is evident that $e^t v_t(x)$ is nondecreasing in t for any fixed x. Therefore $e^t \mathcal{V}_t(x)$ is also nondecreasing in t for any fixed x. Recall from the proof of Lemma 2.7 that

$$e^t \mathcal{V}_t(x) = -1 + \frac{1}{2} e^{2t}\big(m(t) + m'(t)\big)|x|^{2-n} + O(|x|^{1-n}).$$

The claim follows. □

For now, assume that $m(t)$ is smooth.

CLAIM
We have

$$\tilde{m}'(t) \leq m(0).$$

Proof
Differentiating the monotone quantity from the previous claim,

$$0 \leq \frac{d}{dt}\left[e^{2t}\big(m(t)+m'(t)\big)\right] = e^{2t}\big(m''(t)+3m'(t)+2m(t)\big).$$

Since $m'(t) = -2\tilde{m}(t) \leq 0$, we have

$$0 \leq m''(t) + 2m(t) \leq -2\tilde{m}'(t) + 2m(0),$$

proving the claim. □

Since $\tilde{m}(t)$ is a nonnegative function with finite integral and derivative bounded above, it follows that $\lim_{t\to\infty} \tilde{m}(t) = 0$. Of course, $m(t)$ is not necessarily smooth, but it is a simple exercise to show that the result still holds. □

Since $m(t)$ is nonincreasing and bounded below by zero, it must have a limit.

LEMMA 3.2
Let $M = \lim_{t\to\infty} m(t)$. *Then* $M > 0$.

We postpone the proof of this lemma until the next section so as not to interrupt the flow of the main argument.

Let $r_0 < (1/2)R_{\mathrm{sc}}(M)$, and choose a diffeomorphism $\mathbb{R}^n \setminus B_{r_0} \cong M \setminus K$. That is, we choose coordinates in $\mathbb{R}^n \setminus B_{r_0}$ for the end. Recall that since we chose the normalization $\lim_{x\to\infty} \mathcal{U}_0 = 1$, we *cannot* say that $\mathcal{U}_0$ is harmonic in $\mathbb{R}^n \setminus B_{r_0}$ without losing generality. We can say only that $\mathcal{U}_0$ is harmonic in $\mathbb{R}^n \setminus B_{R_{\mathrm{h}}}$ for some possibly large R_{h}.

We want to talk about convergence of our Riemannian manifold as $t \to \infty$, but we see that with respect to a *fixed* coordinate system at infinity, $\Sigma(t)$ runs off to infinity. Consequently, the relevant part of g_t (the part outside $\Sigma(t)$) disappears in the limit. Therefore we need to change our choice of coordinates as t changes. One way to do this is to introduce a one-parameter group of diffeomorphisms. Choose a smooth vector field X on M such that

$$X = \frac{2}{n-2} r \frac{\partial}{\partial r}$$

on $\mathbb{R}^n \setminus B_{r_0}$, where $r = |x|$ is the radial coordinate on $\mathbb{R}^n \setminus B_{r_0}$. (We extend X inside K so that it is smooth.) Let Φ_t be the one-parameter group of diffeomorphisms of M generated by X.

Definition
Given our conformal flow $(M, g_t, \Sigma(t))$, we define the *normalized conformal flow* $(M, G_t, \Sigma^*(t))$ by

$$G_t = \Phi_t^* g_t,$$
$$\Sigma^*(t) = \Phi_t^{-1}\big(\Sigma(t)\big).$$

Define new functions

$$U_t(x) = e^t \mathcal{U}_t\big(\Phi_t(x)\big),$$
$$V_t(x) = e^t \mathcal{V}_t\big(\Phi_t(x)\big),$$

and a new metric

$$(G_{\text{flat}})_t = e^{-4t/(n-2)} \Phi_t^* g_{\text{flat}}.$$

Note that $G_t = U_t^{4/(n-2)}(G_{\text{flat}})_t$. Also, note that $V_t(x) = 0$ inside $\Sigma^*(t)$, and outside $\Sigma^*(t)$, V_t is the unique solution to the Dirichlet problem

$$\begin{cases} \Delta_{(G_{\text{flat}})_t} V_t - \left(\frac{\Delta_{(G_{\text{flat}})_t} U_0}{U_0}\right) V_t = 0 & \text{outside } \Sigma^*(t), \\ V_t(x) = 0 & \text{at } \Sigma^*(t), \\ \lim_{x\to\infty} V_t(x) = -1. \end{cases}$$

Differentiating the definition of U_t, we see that

$$\frac{d}{dt} U_t = V_t + U_t + X U_t.$$

For all $t > t_0 = ((n-2)/4)\log(\frac{R_{\text{h}}}{r_0})$ and $|x| > r_0$, we see that $((G_{\text{flat}})_t)_{ij}(x) = \delta_{ij}$ and

$$\frac{d}{dt} U_t = V_t + U_t + \frac{2}{n-2} r \frac{\partial}{\partial r} U_t. \tag{1}$$

Since we are concerned with what happens for large t, from now on we always assume that $t > t_0$.

Let $W_t = (1/2)(U_t - V_t)$ outside of $\Sigma^*(t)$. Now, define

$$\tilde{G}_t = W_t^{4/(n-2)}(G_{\text{flat}})_t$$

on the exterior of $\Sigma^*(t)$. Observe that $\tilde{G}_t$ is isometric to $\tilde{g}_t$; consequently, it has mass equal to $\tilde{m}(t)$. Note that for $t > t_0$ and x outside $\Sigma^*(t)$ with $|x| > r_0$,

$$\frac{d}{dt}U_t = 2\Big[U_t - W_t + \frac{1}{n-2}r\frac{\partial}{\partial r}U_t\Big]. \tag{2}$$

As mentioned earlier, in order to make this argument work, we need to obtain control on $\Sigma(t)$. We postpone the proof of this lemma so as not to interrupt the flow of the main argument.

LEMMA 3.3
There exists some $R_{\max} > 0$ such that $\Sigma^*(t)$ is always enclosed by the coordinate sphere of radius $R_{\max}$.

As mentioned earlier, we need to use a strengthened version of the equality case of the positive mass theorem. Essentially, we want to say that a sequence of asymptotically flat manifolds of nonnegative scalar curvature becomes flatter as the mass approaches zero. The proof of this theorem is the subject of a separate article (see [L, Theorem 1.4]).

THEOREM 3.4
For all $n \in \mathbb{N}$, $\alpha > 1$, and $\epsilon > 0$, there exists $\delta > 0$ with the following property.

Let M^n be a manifold on which the positive mass theorem holds, and let g be a complete asymptotically flat metric of nonnegative scalar curvature on M, with coordinates in some end satisfying

$$g_{ij}(x) = U(x)^{4/(n-2)}\delta_{ij}$$

for $|x| > R$, for some positive harmonic function U on $\mathbb{R}^n \setminus \bar{B}_R$ approaching 1 at infinity.

$$\textit{If } m(g) < \delta R^{n-2}, \textit{ then for all } |x| \geq \alpha R,\ |U(x) - 1| < \epsilon\Big(\frac{R}{|x|}\Big)^{n-2}.$$

Now, observe that $(\tilde{G}_t)_{ij}(x) = W_t(x)^{4/(n-2)}\delta_{ij}$ in $\mathbb{R}^n \setminus B_{R_{\max}}$, W_t is harmonic on $\mathbb{R}^n \setminus B_{R_{\max}}$, and $\lim_{t\to\infty}\tilde{m}(t) = 0$. Also, observe that since $\tilde{G}_t$ is isometric to $\tilde{g}_t$, it is a limit of metrics that extend to complete metrics of nonnegative scalar curvature (see [M]). In short, we may apply Theorem 3.4 to $\tilde{G}_t$ in order to conclude that $\lim_{t\to\infty} W_t(x) = 1$ uniformly for $|x| > 2R_{\max}$.

LEMMA 3.5

The following limits hold uniformly over all $|x| \geq 4R_{\max}$*:*

$$\begin{aligned}
&\lim_{t\to\infty} U_t(x) = 1 + \frac{M}{2}|x|^{2-n},\\
&\lim_{t\to\infty} V_t(x) = -1 + \frac{M}{2}|x|^{2-n},\\
&\lim_{t\to\infty} W_t(x) = 1.
\end{aligned}$$

Proof

Let

$$\bar{U}_t(x) = U_t(x) - \Big(1 + \frac{m(t)}{2}|x|^{2-n}\Big),$$

and let

$$\bar{W}_t(x) = W_t(x) - \Big(1 + \frac{\tilde{m}(t)}{2}|x|^{2-n}\Big).$$

Therefore, by equation (2) and Lemma 2.7, we know that for $x \in \mathbb{R}^n \setminus B_{r_0}$ and outside $\Sigma^*(t)$,

$$\frac{d}{dt}\bar{U}_t = 2\Big[\bar{U}_t - \bar{W}_t + \frac{1}{n-2}r\frac{\partial}{\partial r}\bar{U}_t\Big]. \tag{3}$$

Let $\epsilon > 0$. By Theorem 3.4 and the discussion following it, the third equality in the statement of the lemma follows immediately. In other words, we know that for large enough t, $\tilde{m}(t)$ is small enough so that

$$\sup_{x\in S_{2R_{\max}}} |W_t(x) - 1| < \epsilon.$$

So

$$\sup_{x\in S_{2R_{\max}}} |\bar{W}_t(x)| < \epsilon + \frac{\tilde{m}(t)}{2}(2R_{\max})^{2-n} < 2\epsilon$$

for large enough t. Since $\bar{W}_t$ is harmonic and has no constant or $|x|^{2-n}$-terms in its expansion, it follows from the maximum principle and a gradient estimate that for all $|x| > 4R_{\max}$,

$$|\bar{W}_t(x)| < C\epsilon|x|^{1-n}$$

for some constant C independent of ϵ. Analyzing equation (3), we conclude that for large enough t,

$$|\bar{U}_t(x)| < 3C\epsilon|x|^{1-n}.$$

The first equation in the statement of the lemma now follows from the definition of $\bar{U}_t$ and the fact that $\lim_{t\to\infty} m(t) = M$. The second equation in the statement of the lemma follows from the other two. □

LEMMA 3.6

For $X \subset \mathbb{R}^n$, $\epsilon > 0$, *let* $(X)_\epsilon$ *denote the* ϵ*-neighborhood of* X*, that is, the set of points that are distance less than* ϵ *away from* X*. For all* $\epsilon > 0$*, there exists some large* t *such that*

$$\Sigma^*(t) \subset (S_{R_{\mathrm{sc}}(M)})_\epsilon,$$

where $S_{R_{\mathrm{sc}}(M)}$ *is the sphere of radius* $R_{\mathrm{sc}}(M) = (M/2)^{1/(n-2)}$ *in* $\mathbb{R}^n$*.*[*]

Proof

Using maximum principle arguments and Lemma 3.3, one can prove uniform upper and lower bounds on $U_t(x)$ on $\mathbb{R}^n \setminus B_{r_0}$.

Since the area of $\Sigma^*(t)$ with respect to G_t is constant, and since there is a uniform lower bound on $U_t(x)$, we have a uniform upper bound on the Euclidean area of $\Sigma^*(t) \cap (\mathbb{R}^n \setminus B_{r_0})$. Therefore we can show that for some sequence t_i, the part of $\Sigma^*(t_i)$ in $\mathbb{R}^n \setminus B_{r_0}$ weakly converges to some Σ_∞. But moreover, using the uniform bounds on $U_t(x)$, one can argue the stronger statement that $\Sigma^*(t_i)$ converges to Σ_∞ in Hausdorff distance (see [Br, Section 12, Appendix E] for details; one can also argue directly using Proposition 5.1).

Since the V_t's are harmonic and uniformly bounded, we can choose a subsequence such that V_{t_i} converges uniformly on compact subsets of the exterior of Σ_∞ in $\mathbb{R}^n \setminus B_{r_0}$. Since the limit must be a harmonic function, it follows from Lemma 3.5 that the limit is $V_\infty(x) = -1 + (M/2)|x|^{2-n}$. More precisely, given $\epsilon > 0$, for large enough i, we know that $\Sigma_{t_i} \cap (\mathbb{R}^n \setminus B_{r_0}(0)) \subset (\Sigma_\infty)_\epsilon$ and that $|V_{t_i}(x) - V_\infty(x)| < \epsilon$ for all $|x| > r_0$ outside $(\Sigma_\infty)_\epsilon$. Our goal is to show that Σ_∞ is just the sphere of radius $R_{\mathrm{sc}}(M)$, and then the result follows from the Hausdorff convergence.

Suppose that part of Σ_∞ lies inside the sphere of radius $R_{\mathrm{sc}}(M)$. Then we can find some t_i and some point x_0 such that x_0 is outside $\Sigma^*(t_i)$, and yet $V_{t_i}(x_0) > 0$, which is a contradiction.

We now come to the critical part of the proof. Suppose that part of Σ_∞ lies outside the sphere of radius $R_{\mathrm{sc}}(M)$. Then for some $x_0 \in \Sigma_\infty$ and some $r > 0$, the

---

[*] Actually, it is necessary only to show that $\Sigma^*(t)$ lies within the sphere of radius $R_{\mathrm{sc}}(M) + \epsilon$.

ball $B_{2r}(x_0)$ lies completely outside the sphere of radius $R_{\mathrm{sc}}(M)$. The basic intuitive argument is as follows. We know that V_{t_i} is zero at $\Sigma^*(t_i)$, but in $B_r(x_0)$ we know that V_∞ is significantly smaller than zero. The only way this can happen is if the gradient of V_{t_i} is blowing up. In fact, we show that it blows up badly enough that the energy of V_{t_i} blows up, which is a contradiction since we have a bound on energy (described below).

Consider the unique harmonic function f that approaches -1 at infinity and is zero at the sphere $S_{R_{\max}}$. Since $\Sigma^*(t_i)$ is contained in $S_{R_{\max}}$, we can deduce from the energy-minimizing property of harmonic functions that the energy of V_{t_i} in the exterior of $\Sigma^*(t_i)$ is less than the energy of f in the exterior of $S_{R_{\max}}$, namely, $(n-2)\omega_{n-1}R_{\max}^{n-2}$. Let Ω be the region outside $\Sigma^*(t_i)$, let $L_z = \{x \in B_r(x_0) \mid V_{t_i}(x) = z\}$, and let dA_z be the induced measure on L_z. (Note that we suppress the dependence on i in the notation.) Then by the coarea formula and the Hölder inequality,

$$\begin{aligned}(n-2)\omega_{n-1}R_{\max}^{n-2} &\geq \int_\Omega |\nabla V_{t_i}|^2\, dV\\ &\geq \int_{\Omega\cap B_r(x_0)} |\nabla V_{t_i}|^2\, dV\\ &= \int_{-1}^{0}\Big(\int_{L_z} |\nabla V_{t_i}|\, dA_z\Big)\, dz\\ &\geq \int_{-1}^{0} |L_z|^2 \Big(\int_{L_z} |\nabla V_{t_i}|^{-1}\, dA_z\Big)^{-1} dz.\end{aligned}$$

Let $\mu(z) = |\{x \in B_r(x_0) \mid V_{t_i}(x) > z\}|$. Then $\mu'(z) = \int_{L_z} |\nabla V_{t_i}|^{-1}\, dA_z$, and we have

$$(n-2)\omega_{n-1}R_{\max}^{n-2} \geq \int_{-1}^{0} |L_z|^2 \mu'(z)^{-1}\, dz. \tag{4}$$

On the other hand, we know that for some nonzero constant $c < 0$, we have $V_\infty(x) < 2c$ in $B_r(x_0)$. Now, let $\epsilon > 0$, and choose i large enough so that $V_{t_i}(x) < c < 0$ for all $x \in B_r(x_0)$ lying outside $(\Sigma_\infty)_\epsilon$. Therefore

$$\big\{x \in B_r(x_0) \,\big|\, V_{t_i}(x) > c\big\} \subset (\Sigma_\infty)_\epsilon \cap B_r(x_0),$$

and it follows that

$$\lim_{i\to\infty} \mu(c) = 0. \tag{5}$$

Since $\mu(c) = \int_c^0 \mu'(z)\, dz$, we can choose i large enough so that $\mu'(z) < \sqrt{\mu(c)}$ on a set of measure at least $-c/2$. We also know that for $0 > z > c$, $L_z \subset (\Sigma_\infty)_\epsilon \cap B_r(x_0)$, and consequently, these L_z's are Hausdorff converging to $\Sigma_\infty \cap B_r(x_0)$. In particular, for $0 > z > c$, $|L_z|$ is uniformly bounded below by some constant α. Plugging this into our energy bound (4), we see that

$$(n-2)\omega_{n-1}R_{\max}^{n-2} \geq \int_{-c}^{0} |L_z|^2 \mu'(z)^{-1}\, dz \geq -\frac{c}{2}\alpha^2 \mu(c)^{-1/2},$$

which contradicts equation (5). □

The main theorem (Theorem 1.4, excluding the equality case) follows easily from Lemma 3.6. Let $\epsilon > 0$. Since $\Sigma^*(t)$ is outer-minimizing with respect to G_t, we see that A is less than or equal to the area of the sphere of radius $R_{\mathrm{sc}}(M)+\epsilon$ with respect to G_t. Also, the argument in Lemma 3.5 shows that U_t converges to $1+(M/2)|x|^{2-n}$ uniformly on $S_{R_{\mathrm{sc}}(M)+\epsilon}$. So for large enough t, we have

$$\begin{aligned} A &\leq |S_{R_{\mathrm{sc}}(M)+\epsilon}|_{G_t} \\ &= \int_{S_{R_{\mathrm{sc}}(M)+\epsilon}} U_t^{2(n-1)/(n-2)}\, d\mu \\ &\leq \int_{S_{R_{\mathrm{sc}}(M)+\epsilon}} \Big(1+\frac{M}{2}|x|^{2-n}+\epsilon\Big)^{2(n-1)/(n-2)} d\mu, \end{aligned}$$

which converges to $\omega_{n-1}(2M)^{(n-1)/(n-2)}$ as $\epsilon \to 0$, proving our main theorem (Theorem 1.4).

The rest of the article deals with the proofs that were skipped, namely, the proofs of Lemma 3.2, Lemma 3.3 (the primary technical lemma of this article), and the equality case of Theorem 1.4.

4. Proof of Lemma 3.2

Suppose that $M = 0$. We want to argue that this is not possible. We do this by following the same argument we gave in the ($M \neq 0$)-case. We can no longer choose $r_0 < (1/2)R_{\mathrm{sc}}(M)$, but we can still choose some small $r_0 > 0$. All of the arguments are valid up until we reach the proof of Lemma 3.6. In the proof of Lemma 3.6, we argued that for some sequence of t_i's, $\Sigma^*(t_i) \cap (\mathbb{R}^n \setminus B_{r_0})$ Hausdorff converges to some Σ_∞. A priori, Σ_∞ could be empty. However, the following lemma shows that, for a judicious choice of r_0 and t_i's, Σ_∞ is nonempty.

LEMMA 4.1
Suppose that $M = 0$. Let R be any constant such that $r_0 < R < (A/\omega_{n-1})^{1/(n-1)}$. There exists an unbounded sequence of times t_i such that $\Sigma^*(t_i)$ always contains a point x with $|x| > R$.

A more general version of this fact is given in [Br, Section 9].

Proof
Suppose, to the contrary, that there exists some $\tilde{t} > t_0$ such that for all $t > \tilde{t}$, $\Sigma^*(t)$ is contained in S_R, the sphere of radius R. Choose R' between R and $(A/\omega_{n-1})^{1/(n-1)}$. Since the U_t's are harmonic and uniformly bounded, we know that some subsequence converges uniformly on compact subsets of $\mathbb{R}^n \setminus \overline{B_R}$, and by Lemma 3.5 and the fact that $M = 0$, we know that the limit function is 1. In particular, we see that $\lim_{t\to\infty} U_t(x) = 1$ uniformly on $S_{R'}$. Therefore, as $t \to \infty$, $|S_{R'}|_{G_t} = \int_{S_{R'}} U_t(x)^{2(n-1)/(n-2)}\, d\mu \to \omega_{n-1}(C')^{n-1} < A$. This contradicts the fact that $|S_{R'}|_{G_t} \geq A$. □

So as long as we take $r_0 < (1/2)(A/\omega_{n-1})^{1/(n-1)}$ and restrict our attention to times in the sequence described by Lemma 4.1, we know that Σ_∞ is nonempty. Furthermore, for some $x_0 \in \Sigma_\infty$ and some $r > 0$, $B_{2r}(x_0)$ lies completely outside the sphere of radius r_0. The energy argument given in Lemma 3.6 now gives us a contradiction. □

5. Proof of Lemma 3.3

Here, we introduce a technical tool that allows us to locally control the area of $\Sigma(t)$. We describe this tool in the language of integral currents, but this is not actually necessary for the application in this article.

Definition
For any $\gamma \geq 1$, an integral current S in $\mathbb{R}^n$ is said to be *γ-almost area-minimizing* if, for any ball B with $B \cap \operatorname{spt} \partial S = \emptyset$ and any integral current T with $\partial T = \partial(S\lfloor B)$, $|S\lfloor B| \leq \gamma |T|$, where the absolute value signs denote the area.*

Note that a 1-almost area minimizer is an area minimizer. It is well known that if S is an m-dimensional minimal submanifold of $\mathbb{R}^n$, then for any $x \in S$ and $0 < r < d(x, \partial S)$,

$$|S \cap B_r(x)| \geq \alpha_m r^m,$$

where $\alpha_m = \omega_{m-1}/m$ is the volume of the unit ball in $\mathbb{R}^m$. This lower bound on area is a consequence of monotonicity (see [A]). The following result is possibly well known to experts, but we include its proof for the sake of completeness.

PROPOSITION 5.1
Let $\gamma \geq 1$, and let S be an m-dimensional γ-almost area-minimizing integral current in $\mathbb{R}^n$. Let $x \in$ spt S, and let $0 < r < d(x, \operatorname{spt} \partial S)$. Then

$$|S\lfloor B_r(x)| \geq \gamma^{1-m} \alpha_m r^m.$$

---

*In geometric measure theory, the correct term to use here is "mass" rather than "area." We avoid the term "mass" here for obvious reasons. The notation $S\lfloor B$ denotes the restriction of S to B, which is just $S \cap B$ when S is a submanifold.

Proof
Let $F(r) = |S\lfloor B_r(x)|$. Since F is monotonically increasing, $F'(r)$ exists for almost all r, and by the slicing theorem,

$$F'(r) \geq \big|\partial\big(S\lfloor B_r(x)\big)\big|.$$

Let $G(r)$ be the infimum of all areas bounding $\partial(S\lfloor B_r(x))$. By the sharp* isoperimetric inequality (see [Al]),

$$\big|\partial\big(S\lfloor B_r(x)\big)\big| \geq m\alpha_m^{1/m} G(r)^{(m-1)/m}.$$

Finally, by assumption we know that

$$G(r) \geq \frac{1}{\gamma} F(r).$$

Putting the last three inequalities together, we find that

$$F'(r) \geq m\alpha_m^{1/m}\Big(\frac{F(r)}{\gamma}\Big)^{(m-1)/m}.$$

Thus

$$\frac{d}{dr}\big(F(r)^{1/m}\big) \geq \alpha_m^{1/m}\gamma^{(1-m)/m}.$$

The result now follows from integrating this inequality. □

An equivalent formulation of Lemma 3.3 is the following.

LEMMA 5.2
There exists some $R_{\max} > 0$ *such that* $\Sigma(t)$ *is always enclosed by the coordinate sphere of radius* $R_{\max}e^{2t/(n-2)}$.

Proof
First, choose $R_{\max}$ large enough so that $\Sigma(1)$ is contained in the coordinate sphere of radius $R_{\max}$. Next, we choose $R_{\max}$ large enough so that the following claim holds.

CLAIM
Let $R(t) = R_{\max}e^{2t/(n-2)}$. *Choose any* $T > 0$. *Suppose that* $\Sigma(t)$ *is contained in the sphere of radius* $R(t)$ *for all* $t \in [0, T]$. *Then* $\Sigma(T+1)$ *is contained in the sphere of radius* $R(T+1)$.

---

*Using a nonsharp constant in the isoperimetric inequality would simply have the effect of attaining a worse constant in the statement of our proposition.

Clearly, if we can choose $R_{\max}$ large enough so that the claim is true, then we have proved the lemma.

Proof

We know that for some constant C, $1/C < \mathcal{U}_0(x) < 1 + C|x|^{2-n}$ for all $|x| > R_{\mathrm{h}}$. In all of the computations that follow, assume that $|x| > R_{\mathrm{h}}$. That is, we are interested only in the "harmonic part" of the end. Let $s < T$. Since $\Sigma(s)$ is enclosed by the sphere of $R(s)$, the maximum principle tells us that for all $|x| > R(s)$, $v_s(x)$ is smaller than the unique g_0-harmonic function that is zero at the sphere of radius $R(s)$ and approaches $-e^{-s}$ at infinity. Explicitly,

$$v_s(x) \le \frac{1}{\mathcal{U}_0(x)} e^{-s}\left(\left(\frac{R(s)}{|x|}\right)^{n-2} - 1\right).$$

Consequently,

$$\mathcal{V}_s(x) \le e^{-s}\left(\left(\frac{R(s)}{|x|}\right)^{n-2} - 1\right) \quad \text{for } |x| \ge R(s),$$

while

$$\mathcal{V}_s(x) \le 0 \quad \text{for } |x| < R(s).$$

Therefore, for $|x| \ge R(T)$,

$$\begin{aligned}
\mathcal{U}_{T+1}(x) &\le 1 + C|x|^{2-n} + \int_0^T e^{-s}\left(\left(\frac{R(s)}{|x|}\right)^{n-2} - 1\right) ds \\
&= 1 + C|x|^{2-n} + \left[e^{s}\left(\frac{R_{\max}}{|x|}\right)^{n-2} + e^{-s}\right]_0^T \\
&= 1 + C|x|^{2-n} + \left[(e^T - 1)\left(\frac{R_{\max}}{|x|}\right)^{n-2} + e^{-T} - 1\right] \\
&\le CR(T)^{2-n} + \left[e^T\left(\frac{R_{\max}}{R(T)}\right)^{n-2} + e^{-T}\right] \\
&= Ce^{-2T} + 2e^{-T} \\
&\le (2 + C)e^{-T}.
\end{aligned}$$

On the other hand, since $v_s(x) \ge -e^{-s}$, we know that

$$u_{T+1}(x) \ge 1 + \int_0^{T+1} -e^{-s}\, ds = e^{-(T+1)}.$$

Therefore

$$\mathcal{U}_{T+1}(x) \geq \frac{1}{C} e^{-(T+1)}.$$

Now, suppose that $\Sigma(T+1)$ contains a point p with $|x| > R(T+1)$. We will show that we can choose $R_{\max}$ so that the area of $|\Sigma(T+1)|_{g_{T+1}}$ is bigger than A, which is a contradiction. Consider the coordinate ball B of radius $(1/3)R(T)$ around p. For $n < 8$, this ball B lies outside the sphere of radius $R(T)$. Using the bounds above, we see that for $x \in B$,

$$e^{-(T+1)} \leq \mathcal{U}_{T+1}(x) \leq (2+C)e^{-T}.$$

From this we can conclude, straight from the definitions, that $\Sigma(T+1)$ is γ-almost area minimizing in B with respect to the Euclidean metric, where

$$\gamma = (2e + Ce)^{2(n-1)/(n-2)}.$$

So by Proposition 5.1, $\Sigma(T+1)$ has Euclidean area greater than $\alpha_{n-1}\gamma^{2-n}(R(T)/3)^{n-1}$. Therefore

$$\begin{aligned} |\Sigma(T+1)|_{g_{T+1}} &\geq (e^{-(T+1)})^{2(n-1)/(n-2)} \alpha_{n-1} \gamma^{2-n} \Big(\frac{R(T)}{3}\Big)^{n-1} \\ &= e^{-2(n-1)/(n-2)} \alpha_{n-1} \gamma^{2-n} \Big(\frac{R_{\max}}{3}\Big)^{n-1}, \end{aligned}$$

which is just some constant times $R_{\max}^{n-1}$. We just need to choose $R_{\max}$ large enough so that this number is larger than A. □

Thus Lemma 5.2 is proved. □

Thus Lemma 3.3 is proved. □

6. The equality case of the Penrose inequality

Consider (M, g) satisfying the hypotheses of Theorem 1.4, consider the added hypothesis that M is spin, and assume that

$$m = \frac{1}{2}\Big(\frac{A}{\omega_{n-1}}\Big)^{(n-2)/(n-1)}.$$

Our goal is to prove that (M, g) is isometric to a Riemannian Schwarzschild manifold outside its unique outer-minimizing horizon.

As in the proof of the Penrose inequality, by Theorem 2.2 there exists a conformal flow that has $(M, g_0 = g, \Sigma(0) = \partial M)$ as initial data. By monotonicity of $m(t)$

(Lemma 2.6) and the Penrose inequality (Theorem 1.4), $m(t)$ must be constant.[*] Then Lemma 2.7 implies that $\tilde{m}(0) = 0$. Since $\tilde{M}$ is spin, we may apply the equality case of Theorem 2.8 to conclude that $(\tilde{M}_\Sigma, \tilde{g})$ is Euclidean space. That is, there are coordinates on $\tilde{M} \cong \mathbb{R}^n$ such that $\tilde{g}_{ij}(x) = \delta_{ij}$.

Briefly, recall the construction of $(\tilde{M}_\Sigma, \tilde{g})$. We doubled (M, g) through Σ to obtain $(\bar{M}_\Sigma, \bar{g})$, and then we set $\tilde{g} = \omega^{4/(n-2)}\bar{g}$, where ω is the unique $\bar{g}$-harmonic function approaching 1 at infinity, and 0 at the "other" infinity (which gets compactified by the conformal change). Without loss of generality, assume that the compactified point of $\tilde{M} \cong \mathbb{R}^n$ is 0. Define $\mathcal{U} = \omega^{-1}$. By Lemma 2.1, we know that $\mathcal{U}$ is a (Euclidean) harmonic function on $\mathbb{R}^n \setminus \{0\}$.[†] Since $\bar{g} = \mathcal{U}^{4/(n-2)}\tilde{g}$ has an asymptotically flat end corresponding to the point $0 \in \tilde{M} \cong \mathbb{R}^n$, it follows that $\lim_{x\to 0} |x|^{n-2}\mathcal{U}(x) = C$ for some constant C. Thus $\mathcal{U}(x) - C|x|^{2-n}$ is a harmonic function with a removable singularity at the origin. Since $\lim_{x\to\infty} U(x) = \lim_{x\to\infty} \omega^{-1}(x) = 1$, the maximum principle implies that $\mathcal{U}(x) - C|x|^{2-n}$ is identically 1 on all of $\mathbb{R}^n \setminus \{0\}$. Hence $g_{ij}(x) = \mathcal{U}(x)^{4/(n-2)}\delta_{ij}$ is a Riemannian Schwarzschild metric. Moreover, there is only one horizon in a Riemannian Schwarzschild manifold, so that is where Σ must be.

7. Further directions

We conjecture that the spin assumption in the equality case of Theorem 1.4 can be removed. Reviewing the proof of the equality case in Section 6, it is clear that the problem reduces to that of removing the spin assumption from the equality case of the extension of the positive mass theorem articulated in Theorem 2.8. The original nonspin proof of the equality case of the standard positive mass theorem (see [SY]) utilized a perturbation of the metric (involving Ricci curvature and conformal deformation) to show that the manifold was Euclidean. This technique does not immediately generalize to the setup of Theorem 2.8, in which the metric is singular. However, the authors hope that the technique can still be adapted.

If one can prove that the positive mass theorem holds in arbitrary dimensions, this leads to the natural question of whether Theorem 1.4 can be extended to arbitrary dimensions. Even given the validity of positive mass theorem in arbitrary dimensions, there are a few sticking points to extending the proof in this article to higher dimensions. None of these sticking points seems fatal. Although the conformal flow has been proved to exist only for dimensions less than eight (Theorem 2.2), it is reasonable to think that the conformal flow exists for arbitrary dimensions *if* we allow $\Sigma(t)$ to be a singular hypersurface rather than a smooth one. More precisely, we must allow $\Sigma(t)$ to be an integral current with a codimension seven singular set. Once one establishes

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[*] Lemmas 2.6 and 2.7 still hold without the assumption of harmonic flatness.

[†] Technically, Lemma 2.1 is not valid at Σ, but one can see that $\mathcal{U}$ is harmonic across Σ by showing that it minimizes energy.

such an existence result, the essential problem is to show that Theorem 2.8 holds for Σ with this type of singularity. Or in other words, it is already known that if one doubles a manifold of nonnegative scalar curvature through a *smooth* minimal hypersurface, then the result has nonnegative scalar curvature in some weak sense that is useful for applications; the problem is to prove that an analogous result holds when the smooth minimal hypersurface is replaced by a singular minimizing hypersurface. This problem is related to the more general problem of formulating weak notions of nonnegative scalar curvature.

Finally, it would be interesting to understand whether the conformal flow can be used to prove a version of the Penrose inequality for asymptotically hyperbolic spaces. There are already some known versions of the positive mass theorem for asymptotically hyperbolic spaces, but a Penrose inequality for such spaces remains conjectural (see [W]). We note that A. Neves has recently identified an important issue in the inverse mean curvature flow approach to this problem, which highlights some of the subtleties that arise in the asymptotically hyperbolic setting (see [N]).

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