ASIAN J. MATH.
Vol. 15, No. 4, pp. 557–610, December 2011

005

P. D. E. 'S WHICH IMPLY THE PENROSE CONJECTURE*

HUBERT L. BRAY† AND MARCUS A. KHURI‡

Abstract. In this paper, we show how to reduce the Penrose conjecture to the known Riemannian Penrose inequality case whenever certain geometrically motivated systems of equations can be solved. Whether or not these special systems of equations have general existence theories is therefore an important open problem. The key tool in our method is the derivation of a new identity which we call the generalized Schoen-Yau identity, which is of independent interest. Using a generalized Jang equation, we use this identity to propose canonical embeddings of Cauchy data into corresponding static spacetimes. In addition, we discuss the Carrasco-Mars counterexample to the Penrose conjecture for generalized apparent horizons (added since the first version of this paper was posted on the arXiv) and instead conjecture the Penrose inequality for time-independent apparent horizons, which we define.

Key words. Penrose Inequality, Generalized Jang Equation, Inverse Mean Curvature Flow, Conformal Flow of Metrics.

AMS subject classifications. 83C57, 53C80.

1. Introduction. In addition to their intrinsic geometric appeal, the Penrose conjecture [29] and the positive mass theorem [32] are fundamental tests of general relativity as a physical theory. In physical terms, the positive mass theorem states that the total mass of a spacetime with nonnegative energy density is also nonnegative. The Penrose conjecture, on the other hand, conjectures that the total mass of a spacetime with nonnegative energy density is at least the mass contributed by the black holes in the spacetime. In this section, we will explain how these simple physical motivations translate into beautiful geometric statements.

After special relativity, Einstein sought to explain gravity as a consequence of the curvature of spacetime caused by matter. In contrast to Newtonian physics, gravity is not a force but instead is simply an effect of this curvature. As an analogy, consider a heavy bowling ball placed on a bed which causes a significant dimple in the bed. Now roll a small golf ball off to one side of the bowling ball. Note that the path of the golf ball curves around the bowling ball because of the curvature of the surface of the bed. In this analogy, the bowling ball represents the sun, the golf ball represents the earth, and the surface of the bed represents spacetime. Whereas Newton explained the curvature of the path of the smaller object by asserting an inverse square law force of attraction between the two objects, Einstein declared that the curvature of the smaller object's trajectory was due to the curvature of spacetime itself, and that objects which did not have forces (other than gravity) acting upon them followed geodesics in the spacetime. That is, according to general relativity, the sun and all of the planets are actually following geodesics, curves with zero curvature, in the spacetime.

It should also be noted that general relativity is entirely consistent with large scale experiments, whereas Newtonian physics is not. The most notable example may be the precession of the orbit of Mercury around the Sun. Whereas general relativity predicts the rate at which the elliptical orbit precesses around the Sun to as many

*Received November 4, 2010; accepted for publication February 1, 2011.
†Mathematics Department, Duke University, Box 90320, Durham, NC 27708, USA (bray@math.duke.edu). Supported in part by NSF grant #DMS-0706794.
‡Mathematics Department, Stony Brook University, Stony Brook, NY 11794, USA (khuri@math.sunysb.edu). Supported in part by NSF grant #DMS-1007156 and a Sloan Fellowship.

digits as can be measured, Newtonian physics is off by almost one percent, with all possible excuses for the discrepancy having been eliminated. The question, then, is how to turn the beautiful and experimentally verified idea of matter causing curvature of spacetime, which Einstein called his happiest thought, into a precise mathematical theory.

First, assume that (N^4, g_N) is a Lorentzian manifold, meaning that the metric g_N has signature $(-+++)$ at each point. Note that at each point, time-like vectors (vectors v with $g_N(v,v) < 0$) are split into two connected components, one of which we will call future directed time-like vectors, and the other of which we will call past directed time-like vectors.

Next, define $T(v,w)$ to be the energy density going in the direction of v as measured by an observer going in the direction of w, where v, w are future-directed unit time-like vectors at some point $p \in N$. In addition, suppose that T is linear in both slots so that T is a tensor. Then the physical statement that all observed energy densities are nonnegative translates into

$$T(v,w) \geq 0$$

for all future-directed (or both past-directed) time-like vectors v and w at all points $p \in N$, known as the dominant energy condition.

The goal, then, is to set T, which is called the stress-energy tensor, equal to some curvature tensor. A natural first idea is to consider the Ricci curvature tensor since it is also a covariant 2-tensor. In fact, this was Einstein's first idea. However, the second Bianchi identity on a manifold N with metric tensor g_N implies that

$$\mathrm{div}(G) = 0,$$

where $G = Ric_N - \frac{1}{2}R_N g_N$, Ric_N is the Ricci curvature tensor, and R_N is the scalar curvature. This geometric identity led Einstein to propose

$$G = 8\pi T, \tag{1}$$

known as the Einstein equation, since as an added bonus we automatically get a conservation-type property for T, namely $\mathrm{div}(T) = 0$. Naturally this is a very nice feature of the theory since energy and momentum (the spatial components of the energy vector) are conserved in every day experience.

The next step in pursuing this line of thought is to try to find examples of spacetimes which satisfy the dominant energy condition, the simplest case of which would be spacetimes with $G = 0$ which are naturally called vacuum spacetimes. Taking the trace implies that such spacetimes (in 2+1 dimensions and higher) have zero scalar curvature and therefore zero Ricci curvature as well. The first example (in 3+1 dimensions) is clearly Minkowski space

$$\left(\mathbf{R}^4, -dt^2 + dx^2 + dy^2 + dz^2\right)$$

which has zero Riemann curvature tensor. The second simplest example of a spacetime with $G = 0$,

$$\left(\mathbf{R} \times (\mathbf{R}^3 \setminus B_{m/2}(0)), -\left(\frac{1 - \frac{m}{2r}}{1 + \frac{m}{2r}}\right)^2 dt^2 + \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2)\right), \tag{2}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, is a one parameter family of spacetimes called the Schwarzschild spacetimes. When $m > 0$, these spacetimes represent static black holes in a vacuum spacetime.

While the Schwarzschild spacetime can be covered by a single coordinate chart (see Kruskal coordinates described in section 2), the coordinate chart above only covers the exterior region of the black hole and has a coordinate singularity (not an actual metric singularity) on the coordinate cylinder $r = m/2$. For our purposes, however, we will only be interested in the exterior region of the Schwarzschild spacetime, which physically corresponds to the region where observers have yet to pass into the event horizon of the black hole, which is the point of no return from which not even light can escape back out to infinity.

Spacetimes which may be expressed in the form

$$\left(\mathbf{R} \times M, -\phi(x)^2 dt^2 + g\right),$$

where $t \in \mathbf{R}$, $x \in M$, and g is a Riemannian (positive definite) metric on M, are called static spacetimes. This name is appropriate since we see that the components of the spacetime metric in this coordinate chart do not depend on t but instead are entirely functions of x. Note also that static metrics are defined not to have any time/spatial cross terms. (Spacetimes which allow time/spatial cross terms but where the metric components still only depend on x are called stationary spacetimes.)

An important result, first proved by Bunting and Masood-ul-Alam [7] using a very clever argument involving the positive mass theorem, is that the only complete, asymptotically flat *static vacuum* spacetimes with black hole boundaries (or no boundary) are the two spacetimes that we have listed so far, Minkowski and Schwarzschild. This fact suggests that a thorough understanding of these two spacetimes, including what makes them special as compared to generic spacetimes, may be important for understanding some of the most fundamental properties of general relativity.

In fact, the Minkowski and Schwarzschild spacetimes are the extremal spacetimes for the positive mass theorem and the Penrose conjecture, respectively. That is, the case of equality of the positive mass theorem states that any space-like hypersurface of a spacetime satisfying the hypotheses of the positive mass theorem which has $m = 0$ can be isometrically embedded into the Minkowski spacetime. Similarly, the case of equality of the Penrose conjecture (which, while still a conjecture, has no known counter-examples in spite of much examination) states that any space-like hypersurface of a spacetime satisfying the hypotheses of the Penrose conjecture which has $m = \sqrt{A/16\pi}$ (or to be more precise, the region outside of the outermost minimal area enclosure of the apparent horizons) can be isometrically embedded into the Schwarzschild spacetime.

Before we can state these theorems, though, we need to define a few terms. The basic object of interest in this paper is a space-like hypersurface M^3 of a spacetime N^4, along with the induced metric g on M^3 and its second fundamental form k in the spacetime.

From this point on we will assume that M^3 has a global future directed unit normal vector n_{future} in the spacetime. For convenience, we will abuse terminology slightly and also call

$$k(V, W) = -\langle \nabla_V W, n_{future} \rangle \tag{3}$$

the second fundamental form of M^3, where V, W are any vector fields tangent to M^3 and ∇ is the Levi-Civita connection on the spacetime N^4. In this manner we are defining k to be a real-valued symmetric 2-tensor, where the true second fundamental form, which takes values in the normal bundle to M^3, is $k \cdot n_{future}$.

Definition 1. *The triple (M^3, g, k) is called the Cauchy data of M^3 for any positive definite metric g and any symmetric 2-tensor k.*

This name is appropriate because this is the data required to pose initial value problems for p.d.e.'s such as the vacuum Einstein equation $G = 0$, or the Einstein equation coupled with equations which describe how the matter evolves in the spacetime. Note when M^3 is flowed at unit speed orthogonally into the future that

$$\frac{d}{dt} g_{ij} = 2k_{ij},$$

so that k is in fact the first derivative of g in the time direction (up to a factor).

Curiously, as we will see, the positive mass theorem and the Penrose conjecture reduce to and are fundamentally statements about the Cauchy data of space-like hypersurfaces of spacetimes, not the spacetimes themselves.

Definition 2. *At each point on M^3, define $\mu = T(n_{future}, n_{future})$ to be the energy density and the covector J on M^3 to be the momentum density, where $J(v) = T(n_{future}, v)$, where v is any vector tangent to M.*

By the Einstein equation (equation 1) and the Gauss-Codazzi identities [27], it follows that μ and J can be computed entirely in terms of the Cauchy data (M^3, g, k). In fact,

$$(8\pi)\,\mu \;=\; G(n_{future}, n_{future}) = (R + \mathrm{tr}(k)^2 - \|k\|^2)/2 \tag{4}$$

$$(8\pi)\,J \;=\; G(n_{future}, \cdot) = \mathrm{div}\,(k - \mathrm{tr}(k)g)\,, \tag{5}$$

where R is the scalar curvature of (M^3, g) at each point, and the above traces, norms, and divergences are naturally taken with respect to g and the Levi-Civita connection of g. Then the dominant energy condition on T implies that we must have

$$\mu \geq |J|, \tag{6}$$

which we will call the nonnegative energy density condition on (M^3, g, k), where again the norm is taken with respect to the metric g on M^3.

Equations 4 and 5 are called the constraint equations because they impose constraints on the Cauchy data (M^3, g, k) for each initial value problem. For example, we clearly need to impose $\mu = 0$ and $J = 0$ on any Cauchy data which is meant to serve as initial conditions for solving the vacuum Einstein equation $G = 0$. However, for our purposes throughout the rest of this paper, we will be interested in Cauchy data (M^3, g, k) which only needs to satisfy the nonnegative energy density condition in inequality 6. Since the assumption of nonnegative energy density everywhere is a very common assumption, the theorems we prove will apply in a very broad set of circumstances.

Next we turn our attention to the definition of the total mass of a spacetime. Looking back at the Schwarzschild spacetime, time-like geodesics (which represent test particles) curve in the coordinate chart as if they were accelerating towards the center of the spacetime at a rate asymptotic to m/r^2 in the limit as r goes to infinity. Hence, to be compatible with Newtonian physics (with the universal gravitational constant set to 1) in the low field limit, we must define m to be the total mass of the Schwarzschild spacetime.

More generally, consider any spacetime which is isometric to the Schwarzschild spacetime with total mass m for $r > r_0$ and which is any smooth Lorentzian metric satisfying the dominant energy condition on the interior region. Of course the Schwarzschild spacetime satisfies the dominant energy condition since it has $G = 0$. Then the same argument as in the previous paragraph applies to this spacetime, so its total mass must be m as well. This last example inspires the following definition, which comes from considering the $t = 0$ slice of Schwarzschild spacetimes.

DEFINITION 3. *The Cauchy data* (M^3, g, k) *will be said to be Schwarzschild at infinity if* M^3 *can be written as the disjoint union of a compact set* K *and a finite number of regions* E_i *(called ends), where* $k = 0$ *on each end and each* (E_i, g) *is isometric to* $\left(\mathbf{R}^3 \setminus \bar{B}_{R_i}(0), \left(1 + \frac{m_i}{2r}\right)^4 (dx^2 + dy^2 + dz^2)\right)$ *for some* m_i *and some* $R_i > \max(0, -m_i/2)$. *In addition, the mass of the end* E_i *will be defined to be* m_i.

We refer the reader to [31] and [36] for more general definitions of *asymptotically flat* Cauchy data, but for this paper the special case of being precisely Schwarzschild at infinity is sufficiently interesting.

Typically we will be interested in Cauchy data with only one end. However, sometimes it is convenient to allow for the possibility of multiple ends. Each end represents what we would normally think of as a spatial slice of a universe, and the positive mass theorem and the Penrose conjecture may be applied to each end independently. In fact, since ends can be compactified by adding a point at infinity and then using a very large spherical metric on the end without violating the nonnegative energy density assumption, without loss of generality we may assume that any given Cauchy data has only one end for the problems we will be considering.

THEOREM 1. *(The Positive Mass Theorem, Schoen-Yau, 1981 [31]; Witten, 1981 [37])*
Suppose that the Cauchy data (M^3, g, k) *is complete, satisfies the nonnegative energy density condition* $\mu \geq |J|$, *and is Schwarzschild at infinity with total mass* m. *Then*

$$m \geq 0,$$

and $m = 0$ *if and only if* (M^3, g, k) *is the pullback of the Cauchy data induced on the image of a space-like embedding of* M^3 *into the Minkowski spacetime.*

The above theorem has an important special case when $k = 0$ which is already extremely interesting. Note that the nonnegative energy condition reduces to simply requiring (M^3, g) to have nonnegative scalar curvature.

THEOREM 2. *(The Riemannian Positive Mass Theorem, Schoen-Yau, 1979 [30]; Witten, 1981 [37])*
Suppose that the Riemannian manifold (M^3, g) *is complete, has nonnegative scalar curvature, and is Schwarzschild at infinity with total mass* m. *Then*

$$m \geq 0,$$

and $m = 0$ *if and only if* (M^3, g) *is isometric to the flat metric on* $\mathbf{R}^3$.

The adjective Riemannian was introduced by Huisken-Ilmanen in [18] since the theorem is a statement about Riemannian manifolds as opposed to Cauchy data in the more general case. We remind the reader that Cauchy data (M^3, g, k) is still required to have a positive definite metric g.

Notice that the Riemannian positive mass theorem is a beautiful geometric statement about manifolds with nonnegative scalar curvature. In fact, Schoen-Yau were studying such manifolds [33] for purely geometric reasons when they first realized that they could use minimal surface techniques to prove the Riemannian positive mass theorem. They then observed [31] that theorem 1 (which is quite mysterious from a geometric point of view without the physical motivation) reduced to theorem 2 after solving a certain elliptic p.d.e. on (M^3, g, h) called the Jang equation, named after the theoretical physicist who first introduced the equation in [19].

Witten's proof of the positive mass theorem uses spinors and proves both of the above statements by applying the Lichnerowicz-Weitzenbock formula to a spinor which solves the Dirac equation, and then integrating by parts. This proof has a strong appeal because it computes the total mass as an integral of a nonnegative integrand. However, so far it has not been clear how to generalize this approach to achieve the Penrose conjecture, although very interesting works in this direction include [15] and [23].

Before we can state the Penrose conjecture, we need several more definitions. For convenience, we assume that M^3 does not have any boundary, just asymptotically flat ends, and modify the topology of M^3 by compactifying all of the ends of M^3 except for one chosen end by adding the points $\{\infty_k\}$. (However, the metric will still not be defined on these new points.)

Definition 4. *Define $\mathcal{S}$ to be the collection of surfaces which are smooth compact boundaries of open sets U in M^3, where U contains the points $\{\infty_k\}$ and is bounded in the chosen end.*

All of the surfaces that we will be dealing with in this paper will be in $\mathcal{S}$. Also, we see that all of the surfaces in $\mathcal{S}$ divide M^3 into two regions, an inside (the open set) and an outside (the complement of the open set). Thus, the notion of one surface in $\mathcal{S}$ enclosing another surface in $\mathcal{S}$ is well defined as meaning that the one open set contains the other.

Definition 5. *Given any $\Sigma \in \mathcal{S}$, define $\tilde{\Sigma} \in \mathcal{S}$ to be the outermost minimal area enclosure of Σ.*

That is, in the case that there is more than one minimal area enclosure of the surface Σ, choose the outermost one which encloses all of the others. The fact that an outermost minimal area enclosure exists and is unique roughly follows from the following: if ∂A and ∂B are both minimal area enclosures of some surface, then so are $\partial(A \cup B)$ and $\partial(A \cap B)$ since $|\partial(A \cup B)| + |\partial(A \cap B)| = |\partial A| + |\partial B| = 2A_{min}$ and both have area at least A_{min}. A rigorous proof that the outermost minimal area enclosure of a surface in an asymptotically flat manifold exists and is unique is given in [18].

Definition 6. *Define $\Sigma \in \mathcal{S}$ in (M^3, g, k) to be an apparent horizon if it is one of the following three types of horizons,*

a future apparent horizon if

$$H_\Sigma + \mathrm{tr}_\Sigma(k) = 0 \quad \text{on } \Sigma, \tag{7}$$

a past apparent horizon if

$$H_\Sigma - \mathrm{tr}_\Sigma(k) = 0 \quad \text{on } \Sigma, \tag{8}$$

and a future and past apparent horizon if

$$H_\Sigma = 0 \quad \textit{and} \quad \mathrm{tr}_\Sigma(k) = 0 \quad \textit{on } \Sigma, \tag{9}$$

where H_Σ is the mean curvature of the surface Σ in (M^3, g) (with the sign chosen to be positive for a round sphere in flat $\mathbf{R}^3$) and $\mathrm{tr}_\Sigma(k)$ is the trace of k restricted to the surface Σ.

Note that equation 9 follows from assuming both equations 7 and 8 everywhere on Σ. Also note that Σ is not required to be connected, although from a physical point of view each component of Σ is usually thought of as the apparent horizon of a separate black hole. Finally, observe that all three types of horizons are simply minimal surfaces (surfaces with zero mean curvature) in the important special case when $k = 0$.

Physically, the only relevant apparent horizons for a spacecraft flying around in a spacetime are future apparent horizons, because spacecraft are only concerned about being trapped inside black holes in the future. Mathematically, however, merely changing the choice of global normal vector n_{future} to M^3 in N^4 to $-n_{future}$ changes the sign on k which causes past apparent horizons to become future apparent horizons, and vice versa.

Equations 7, 8, 9 are actually all conditions on the mean curvature vector of Σ in the spacetime. Note that at each point of Σ^2, the normal bundle, of which the mean curvature vector is a section, is a 2-dimensional vector space with signature $(-+)$. Naturally, a basis for this vector space is any outward future null vector along with any outward past null vector. Since Σ^2 bounds a region in M^3, outward is well-defined, and since there exists a global normal vector n_{future} to M^3, the future direction is well-defined.

Geometrically, if one flows a submanifold in the normal directions $\vec{\eta}$, then the rate of change of the area form of the submanifold is given by

$$\frac{d}{dt} dA = -\langle \vec{\eta}, \vec{H} \rangle dA$$

where $\vec{H}$ is the mean curvature vector. It turns out that the mean curvature of a surface Σ^2 contained in a slice with Cauchy data (M^3, g, k) has coordinates $(\mathrm{tr}_\Sigma(k), -H)$, where the first coordinate is in the unit future normal direction to the slice and the second component is in the unit direction outward perpendicular to the surface and tangent to the slice. With this convention, then the vector with components (1,1) is an outward future null vector, and the vector with components (-1,1) is an outward past null vector.

Hence, equation 7 is equivalent to requiring that, at each point of Σ, the dot product of the mean curvature vector with any outward future null vector is zero (which implies that the mean curvature vector is a real multiple of the outward future null direction). Similarly, equation 8 is equivalent to saying that the dot product of the mean curvature vector with any outward past null vector is zero. Thus, future apparent horizons have the property that their areas do not change to first order when flowed in outward future null directions. The same is true for past apparent horizons when flowed in outward past null directions.

We are now able to state the Penrose conjecture. An excellent survey of this conjecture is found in [24].

CONJECTURE 1. *(The Penrose Conjecture, 1973 [29] - Standard Version) Suppose that the Cauchy data (M^3, g, k) is complete, satisfies the nonnegative energy* *density condition $\mu \geq |J|$, and is Schwarzschild at infinity with total mass m in a chosen end. If $\Sigma^2 \in \mathcal{S}$ is a future apparent horizon, then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{10}$$

where A is the area of the outermost minimal area enclosure $\tilde{\Sigma}^2 = \partial U^3$ of Σ^2. Furthermore, equality occurs if and only if $(M^3 \setminus U^3, g, k)$ is the pullback of the Cauchy data induced on the image of a space-like embedding of $M^3 \setminus U^3$ into the exterior region of a Schwarzschild spacetime (which maps $\tilde{\Sigma}^2$ to a future apparent horizon).

Penrose's heuristic argument for a future apparent horizon in this conjecture is described in more detail in [5] and [24] but roughly goes as follows: If, as is generally thought, asymptotically flat spacetimes eventually settle down to a Kerr spacetime [16], then in the distant future inequality 10 will be satisfied since explicit calculation verifies this fact for Kerr spacetimes, where A is the area of the event horizon. Given that some energy may radiate out to infinity, the total mass of these slices of Kerr may be less than the original total mass. Also, by the Hawking area theorem [14] (made more rigorous in [9]), and thus by the cosmic censor conjecture [28] as well, the area of the event horizon is nondecreasing in the spacetime evolution. Hence, this leads us to conjecture inequality 10 in the initial Cauchy data slice, but where A is the total area of the event horizons of all of the black holes. The problem, though, is that, unlike apparent horizons, event horizons are not determined by local geometry but instead are defined in terms of which points in spacetime can eventually escape out to infinity along future directed time-like curves. Thus, in principle, there is no way to know which points this includes without looking at the entire evolution of the spacetime into the future. However, in [29] Penrose argued using the cosmic censor conjecture that future apparent horizons, which are defined in terms of local geometry, must be enclosed by event horizons. Thus, the area of $\tilde{\Sigma}$ serves as a lower bound for the total area of the event horizons [20], [17], and the Penrose conjecture follows. This same argument, but run in the opposite time direction, yields the same conjecture for past apparent horizons as well. Thus, in the conjecture one could replace "future apparent horizon" with simply "apparent horizon."

It is also important to note, as Penrose did originally, that a counterexample to the Penrose conjecture would be a very serious issue for general relativity since it would imply that some part of the above reasoning is false. The consensus among many is that the cosmic censor conjecture is the weakest link in the above argument. If the cosmic censor conjecture turns out to be false, and naked singularities (singularities not enclosed by the event horizons of black holes) do develop in generic spacetimes, then this would present a very interesting challenge to general relativity as a physical theory.

However, like the positive mass theorem, setting $k = 0$ yields another beautiful geometric statement about manifolds with nonnegative scalar curvature, which is known to be true.

THEOREM 3. *(The Riemannian Penrose Inequality, Bray, 2001 [3]) Suppose that the Riemannian manifold (M^3, g) is complete, has nonnegative scalar curvature, and is Schwarzschild at infinity with total mass m in a chosen end. If $\Sigma^2 \in \mathcal{S}$ is a zero mean curvature surface, then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{11}$$

where A is the area of the outermost minimal area enclosure $\tilde{\Sigma}^2 = \partial U^3$ of Σ^2. Furthermore, equality occurs if and only if $(M^3 \setminus U^3, g)$ is isometric to the Schwarzschild metric $\left(\mathbf{R}^3 \setminus B_{m/2}(0), \left(1 + \frac{m}{2r}\right)^4 (dx^2 + dy^2 + dz^2)\right)$.

In 1997, Huisken-Ilmanen proved a slightly weaker version of the above result with the modification that A is the area of the *largest connected component* of the outermost minimal area enclosure of Σ^2 and with the additional assumption that $H_2(M^3) = 0$. (This last topological condition can be replaced by assuming that Σ^2 is already a connected component of the outermost minimal area surface of (M^3, g) by Meeks-Simon-Yau [26].) Their method of proof, first proposed by the theoretical physicists Geroch [13] and Jang-Wald [20], uses a parabolic technique called inverse mean curvature flow. Starting with a connected zero mean curvature surface, Huisken-Ilmanen found a weak definition of inverse mean curvature flow, where the surface is flowed out at each point in (M^3, g) with speed equal to the reciprocal of the mean curvature of the surface at that point, for almost every surface in the flow. Then they showed that the Hawking mass of the surface is nondecreasing under this flow, equals the right hand side of the Riemannian Penrose inequality initially, and limits to the left hand side of the Riemannian Penrose inequality as the surface flows out to large round spheres going to infinity. Both the physicists' insight into proposing this idea and the mathematicians' cleverness at generalizing the argument to something which could be made rigorous are remarkably beautiful.

Bray's proof also involves a flow, but of the Riemannian manifold (M^3, g). The flow of metrics stays inside the conformal class of the original metric and eventually flows to a Schwarzschild metric (shown as the case of equality metric). The conformal flow of metrics is chosen so as to keep the area of the outermost minimal area enclosure of Σ constant. Also, the total mass of the Riemannian manifold is nonincreasing by a clever argument (first used by Bunting and Masood-ul-Alam in [7]) using the positive mass theorem after a reflection of the manifold along a zero mean curvature surface and a conformal compactification of one of the resulting two ends. Then since the Schwarzschild metric gives equality in inequality 11, the inequality follows for the original Riemannian manifold (M^3, g).

All three systems of equations discussed in this paper which imply the Penrose conjecture are based on a new geometric identity which we call the generalized Schoen-Yau identity. The identity is proved in section 6, but with the lengthy computations relegated to the appendices for readability. This new identity is a generalization of equation 2.25 in Schoen-Yau's paper [31]. The original Schoen-Yau identity was used to reduce the positive mass theorem to the Riemannian positive mass theorem by solving a p.d.e. called the Jang equation. For all three systems, our technique will involve a generalization of the Jang equation to solving a system of two equations, the first of which is a generalized Jang equation in all three cases. Rather than spending time explaining the Jang equation, we will go straight to our proof since the Jang equation appears as a special case of our method (which for future reference is the case $\phi = 1$).

As a final comment, the Penrose conjecture can be generalized to a statement about Cauchy data on n-manifolds motivated by considering $(n+1)$-dimensional spacetimes, where $n \geq 3$. In fact, the positive mass theorem was proved by Schoen-Yau in dimensions $n \leq 7$ and by Witten in any number of dimensions, but with the additional assumption that M^n is spin. The Riemannian Penrose inequality was proved by Bray [3] in dimension 3 using a proof which that author and Dan Lee [6] have generalized to manifolds in dimensions $n \leq 7$, and in a slightly weaker form by

Huisken-Ilmanen [18] in dimension 3. Since we will be reducing the general case of the Penrose conjecture to the Riemannian Penrose inequality, the techniques presented here have, at a minimum, the potential to address the Penrose conjecture for manifolds with dimensions $n \leq 7$. However, we will focus on $n = 3$ for simplicity.

2. The case of equality. In this section we carefully study the case of equality of the Penrose conjecture since all of the estimates used to prove the conjecture must give equality in these cases. We refer the reader to [27] for a discussion of the Schwarzschild spacetime in Kruskal coordinates and follow those conventions (except for the names of the two functions α and β defined in a moment). Understanding the Schwarzschild spacetime in Kruskal coordinates is essential since this is the simplest global coordinate chart for the spacetime. In Kruskal coordinates, the entire Schwarzschild spacetime is expressed as the subset $uv > -2m/e$ of $\mathbf{R}^2 \times S^2$ with coordinates $(u, v) \in \mathbf{R}^2, \sigma \in S^2$ and line element

$$2\beta(r)dudv + r^2 d\sigma^2, \tag{12}$$

where $d\sigma^2$ is the standard round unit sphere metric on S^2, $r > 0$ is a function of u, v determined by

$$uv = \alpha(r) = (r - 2m)e^{(r/2m)-1},$$

and

$$\beta(r) = (8m^2/r)e^{1-(r/2m)}.$$

The first quadrant region described by $u, v > 0$ is defined to be an exterior region and is isometric to

$$\left(\mathbf{R} \times (\mathbf{R}^3 \setminus B_{2m}(0)), -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2\right) \tag{13}$$

under the isometry

$$u = \sqrt{\alpha(r)}e^{-t/4m}, \qquad v = \sqrt{\alpha(r)}e^{t/4m},$$

which we leave as an exercise for the interested reader to check. Note that we have now defined three different coordinate chart representations for the Schwarzschild spacetime, the two above in equations 12 and 13, and our original one in equation 2.

A key point is that two of these coordinate chart representations of the exterior region of the Schwarzschild spacetime are written in the form of a static spacetime. For example, using the coordinates in equation 13, the exterior region can be expressed as

$$\left(\mathbf{R} \times M^3, -\phi^2 dt^2 + g\right), \tag{14}$$

where

$$\phi^2 = 1 - \frac{2m}{r},$$

which of course gives us

$$r = \frac{2m}{1 - \phi^2} \quad \text{and} \quad r - 2m = \frac{2m\phi^2}{1 - \phi^2}.$$

Hence, if we think of a slice of the static spacetime expressed as the graph of $t = f(x)$ in the static spacetime, $x \in M^3$, we see that

$$f = 2m \log(v/u), \tag{15}$$

$$\frac{2m\phi^2}{1-\phi^2} \exp\left(\frac{\phi^2}{1-\phi^2}\right) = \alpha(r) = uv. \tag{16}$$

The reason that these last two equations are important is that they allow us to understand the behavior of f and ϕ as they approach the boundary of the exterior region $\{x \mid u > 0, v > 0\}$ of the Schwarzschild spacetime. Our slice (intersected with the exterior region of the Schwarzschild spacetime) has a future apparent horizon boundary if $u = 0$ everywhere on the boundary, a past apparent horizon boundary if $v = 0$ everywhere on the boundary, and a future and past apparent horizon boundary if $u = v = 0$ everywhere on the boundary.

The mixed case where $u = 0$ on part of the boundary and $v = 0$ on the rest of the boundary does not represent a traditional apparent horizon boundary. However, we note that whenever this boundary is area-outerminimizing, we are in fact in a case of equality of the Penrose conjecture. This observation helps motivate the definition of a generalized apparent horizon in the next section.

Also, while the $u = 0$ level set and the $v = 0$ level set on M^3 are both smooth (since the gradients of u and v on M^3 are never zero since M^3 is space-like), the boundary of $\{x \in M^3 \mid u(x) > 0, v(x) > 0\}$ in M^3 need not be smooth since the zero level sets of u and v do not need to intersect smoothly. However, when the boundary has corners it is never area outerminimizing and thus not a case of equality of the Penrose conjecture.

We also note that apparent horizons outside of the exterior region, say with $u = 0$ but with $v < 0$ on part of the apparent horizon, are not area-outerminimizing since they have negative mean curvature at some points. Consequently, these last apparent horizons are enclosed by surfaces with less area and are therefore not cases of equality of the Penrose conjecture either.

Kruskal coordinates reveals that the Schwarzschild spacetime is smooth on the boundary of the exterior region and certainly does not have any singularities there. However, static coordinate representations of the Schwarzschild spacetime have coordinate chart singularities there (which do not represent anything geometric or physical). Hence, while Kruskal coordinates u and v are smooth on any slice, even up to the apparent horizon boundary, f and ϕ are not necessarily.

In fact, we see that f goes to $\pm\infty$ logarithmically at the apparent horizon boundary typically (when u or v goes to zero and the other stays positive). Also, ϕ^2 vanishes on the apparent horizon boundary only linearly if either u or v is strictly positive, which means that the derivative of ϕ is going to ∞. However, in the future and past apparent horizon boundary case where u, v both go to zero, then ϕ^2 vanishes quadratically and ϕ is smooth up to the boundary. It is also true that f is smooth up to the boundary in this case by L'Hopital's rule since u and v, which equal zero on the future and past apparent horizon, have nonzero derivatives there (since the hypersurface is space-like). These observations are helpful since we will be dealing with slices of the exterior region of the Schwarzschild spacetime viewed in static coordinates for the rest of this paper.

3. Generalized apparent horizons.

REMARK. *After the first version of this paper appeared on the arXiv and conjectures 4 and 5 were made known, a counterexample was found by Carrasco and Mars [8]. We have updated this section accordingly.*

In this section we describe a more general version of the Penrose conjecture (now known to be too strong by the counterexample of Carrasco and Mars [8]). We refer the reader to [24] for more discussion on other versions of the Penrose conjecture which are still in the running. Naturally it is very important to determine which versions of the Penrose conjecture are true as this may provide a significant hint as to how to approach the conjecture. Even the fact that the Penrose conjecture is *not* true for all generalized apparent horizons, defined below, is quite interesting.

DEFINITION 7. *Define the smooth surface $\Sigma^2 \in \mathcal{S}$ in (M^3, g, k) to be a generalized apparent horizon if*

$$H_\Sigma = |\operatorname{tr}_\Sigma(k)| \tag{17}$$

and a generalized trapped surface if

$$H_\Sigma \leq |\operatorname{tr}_\Sigma(k)|. \tag{18}$$

In terms of the mean curvature vector $\vec{H}$ of Σ^2 in the spacetime, a generalized trapped surface is one where $\vec{H}$ is not strictly inward space-like anywhere on Σ. Also note that this definition of a generalized apparent horizon does not need a globally defined future directed unit normal to M^3 since the definition is unaffected by a change of sign of the second fundamental form k. A related class of surfaces, referred to as "$*$-surfaces", appears in a different context in [34].

Referring to the previous section, note that any smooth slice M^3 of the Schwarzschild spacetime which smoothly intersects (which is often not the case) with the boundary of the first quadrant $\{u \geq 0\ ,\ v \geq 0\}$ of Kruskal coordinates intersects in a generalized apparent horizon. These generalized apparent horizons also give equality in the Penrose conjecture, so it is natural to consider them in a statement of a generalized Penrose conjecture. Also note that traditional apparent horizons, if they are not already generalized apparent horizons, are at least always generalized trapped surfaces.

After a talk on generalized apparent horizons by the first author at the Niels Bohr International Academy's program "Mathematical Aspects of General Relativity" in April 2008, Robert Wald posed the following insightful question: Is it possible for generalized trapped surfaces to exist as boundaries of space-like slices of Minkowski space? (A similar query was posed by Mars and Senovilla in [25].) This question raises the issue of whether or not generalized trapped surfaces always yield a positive contribution to the ADM mass, which of course is a prerequisite for a generalized version of the Penrose conjecture. If one could find a generalized trapped surface which was the boundary of a space-like slice of Minkowski space, then the total mass of the slice would be zero, making a Penrose-type inequality for the surface impossible.

In response to this question, the second author of this paper showed that no such generalized trapped surface in Minkowski space exists [22]. Furthermore, he showed that Witten's proof of the positive mass theorem also works for asymptotically flat manifolds with generalized trapped surface boundary and gives a positive lower bound on the total mass. This result suggests that generalized trapped surfaces and generalized apparent horizons have some physical significance in that such surfaces, along with nonnegative energy density $\mu \geq |J|$ everywhere in the spacetime, always imply that the total mass is positive. Finding the best possible lower bound on the total mass motivates conjecturing a generalized Penrose inequality.

In addition, a discussion between the first author and Tom Ilmanen led to two more conjectures about generalized apparent horizons and generalized trapped surfaces, which are known to be true in the special case $k = 0$. We are pleased that Michael Eichmair [11] has now proved these two conjectures (except for the topological part of conjecture 3) using elliptic techniques (whereas Ilmanen's original ideas used parabolic techniques). We omit the $n = 2$ case in these next two conjectures because they are less relevant for our present purposes, but we understand that Eichmair's results apply there as well.

Conjecture 2. *(Tom Ilmanen, 2006)*
Given complete, asymptotically flat Cauchy data (M^n, g, k), $3 \leq n \leq 7$, with a generalized trapped surface Σ^{n-1}, then there exists a unique outermost generalized trapped surface $\bar{\Sigma}$ which is a generalized apparent horizon.

Conjecture 3. *(Tom Ilmanen, 2006)*
Furthermore, $\bar{\Sigma}$ is strictly area outerminimizing (every other surface which encloses it has larger area), and for $n = 3$, the region exterior to $\bar{\Sigma}$ is diffeomorphic to $\mathbf{R}^3$ minus a finite number of disjoint closed balls.

The above conjecture is a generalization of Meeks-Simon-Yau [26], which is the case when $k = 0$. The topological conclusions of this last conjecture, like the original Meeks-Simon-Yau result, also make this conjecture interesting for its own sake.

All together, these considerations suggest the following generalized Penrose conjecture. Since traditional apparent horizons are always generalized trapped surfaces, this conjecture implies the original Penrose conjecture. However, the following conjecture is now known to be too strong by the counterexample of Carrasco and Mars [8].

Conjecture 4. *Suppose that the Cauchy data (M^3, g, k) is complete, satisfies the nonnegative energy density condition $\mu \geq |J|$, and is Schwarzschild at infinity with total mass m in a chosen end. If $\Sigma^2 \in \mathcal{S}$ is a generalized trapped surface, then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{19}$$

where A is the area of the outermost minimal area enclosure $\tilde{\Sigma}^2 = \partial U^3$ of Σ^2. Furthermore, equality occurs if and only if $(M^3 \setminus U^3, g, k)$ is the pullback of the Cauchy data induced on the image of a space-like embedding of $M^3 \setminus U^3$ into the exterior region of a Schwarzschild spacetime (which maps $\tilde{\Sigma}^2$ to a generalized apparent horizon).

We note that this conjecture is true when $k = 0$ by [3]. In this case, Σ has nonpositive mean curvature and acts as a barrier to imply the existence of an outermost minimal area enclosure of Σ which is minimal.

It is important to note that conjectures 2 and 3 imply that the generalized Penrose conjecture (and hence the original Penrose conjecture) follows from the following conjecture. However, this next conjecture is also now known to be too strong by the counterexample of Carrasco and Mars [8].

Conjecture 5. *Suppose that the Cauchy data (M^3, g, k) is complete, satisfies the nonnegative energy density condition $\mu \geq |J|$, and is Schwarzschild at infinity with total mass m in a chosen end. Suppose also that $\Sigma^2 = \partial U^3 \in \mathcal{S}$ is a strictly area outerminimizing generalized apparent horizon, that no other generalized trapped surfaces enclose it, and that $M^3 \setminus \overline{U^3}$ is diffeomorphic to $\mathbf{R}^3$ minus a finite number of disjoint closed balls. Then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{20}$$

where A is the area of Σ^2. Furthermore, equality occurs if and only if $(M^3 \setminus U^3, g, k)$ is the pullback of the Cauchy data induced on the image of a space-like embedding of $M^3 \setminus U^3$ into the exterior region of a Schwarzschild spacetime (which maps Σ^2 to a generalized apparent horizon).

Hence, we are left with a mixed verdict on generalized apparent horizons. On the one hand, Khuri's theorem [22] proves that generalized apparent horizons always contribute a positive total mass since these surfaces may be used as boundaries for Witten's proof of the positive mass theorem. Also, Eichmair's results [11] prove that there is always an outermost generalized apparent horizon which has the nice property of being strictly area outerminimizing. But the counterexample of Carrasco and Mars [8] proves that the Penrose conjecture is *not* always true for generalized apparent horizons.

However, for our purposes, the above considerations actually simplify the boundary behavior of the system of p.d.e.s (which we will define later) which imply the Penrose conjecture. With generalized apparent horizons, we were going to have to consider mixed types of blow ups at the boundary. The fact that the Penrose conjecture is not true for generalized apparent horizons suggests the simpler boundary behavior of either blow up (on future apparent horizons) or blow down (on past apparent horizons), which is much more pleasant to contemplate than both blow up and blow down on the same connected component of the boundary.

4. Time-independent apparent horizons.

Remark. *This section did not appear in the first version of this paper.*

Consider the case when a surface with multiple connected components is a future apparent horizon on some connected components and a past apparent horizon on the others. While Penrose's original heuristic argument does not apply to this surface (only future apparent horizons or, by symmetry, past apparent horizons), the techniques that we develop in this paper seem to apply perfectly well (unless something very unexpected were to happen in the existence theory of the systems of p.d.e.'s that we define later). Hence, we are interested in a version of the Penrose conjecture which includes this case as well.

Definition 8. *Define a surface in $\mathcal{S}$ to be a time-independent apparent horizon if each connected component of the surface is an apparent horizon (future, past, or both - see definition 6 - but where each connected component is not required to bound a region).*

Hence, on a time-independent apparent horizon Σ with mean curvature H_Σ,

$$H_\Sigma \pm \mathrm{tr}_\Sigma(k) = 0 \tag{21}$$

everywhere, where the $\pm$ is fixed on each connected component. We also want a corresponding notion of trapped surface.

DEFINITION 9. *Define a surface* Σ *to be future outer trapped if*

$$H_\Sigma + \mathrm{tr}_\Sigma(k) \leq 0 \tag{22}$$

on Σ *and past outer trapped if*

$$H_\Sigma - \mathrm{tr}_\Sigma(k) \leq 0 \tag{23}$$

on Σ*, where* H_Σ *is the mean curvature of* Σ*.*

DEFINITION 10. *Define a surface in* $\mathcal{S}$ *to be time-independent outer trapped if each connected component of the surface is either future outer trapped or past outer trapped.*

Hence, on a time-independent outer trapped surface Σ with mean curvature H_Σ,

$$H_\Sigma \pm \mathrm{tr}_\Sigma(k) \leq 0 \tag{24}$$

everywhere, where the $\pm$ is fixed on each connected component. We now state the Penrose conjecture for time-independent outer trapped surfaces.

CONJECTURE 6. *(The Time-Independent Penrose Conjecture)*
Suppose that the Cauchy data (M^3, g, k) *is complete, satisfies the nonnegative energy density condition* $\mu \geq |J|$*, and is Schwarzschild at infinity with total mass* m *in a chosen end. If* $\Sigma^2 \in \mathcal{S}$ *is a time-independent outer trapped surface, then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{25}$$

where A *is the area of the outermost minimal area enclosure* $\tilde{\Sigma}^2 = \partial U^3$ *of* Σ^2*. Furthermore, equality occurs if and only if* $(M^3 \setminus U^3, g, k)$ *is the pullback of the Cauchy data induced on the image of a space-like embedding of* $M^3 \setminus U^3$ *into the exterior region of a Schwarzschild spacetime (which maps* $\tilde{\Sigma}^2$ *to an apparent horizon).*

We note that this conjecture is true when $k = 0$ by [3]. In this case, Σ has non-positive mean curvature and acts as a barrier to imply the existence of an outermost minimal area enclosure of Σ which is minimal.

Furthermore, the works of Andersson, Eichmair, and Metzger [2, 10, 1] imply that with out loss of generality, the above conjecture may be reduced to a simpler case. Define an outermost time-independent apparent horizon to be a time-independent apparent horizon which is not enclosed by any other. The works of Andersson, Eichmair, and Metzger [2, 10, 1] imply that given any time-independent outer trapped surface, there always exists an outermost time-independent apparent horizon which encloses it (though not necessarily uniquely).

We note that it is actually a two-step process to apply the results of Andersson, Eichmair, and Metzger to conclude our desired statement about outermost time-independent apparent horizons. The authors would like to thank Andersson and Eichmair for personally explaining the following argument to us, which may or may not actually appear in their papers out so far, although it is definitely well known to them. In our terminology, given a time-independent outer trapped surface, express it as a union of a future outer trapped surface Σ^+ and a past outer trapped surface Σ^-. Then let $\tilde{\Sigma}^+$ be the outermost future outer trapped surface enclosing Σ^+, where Σ^- acts as a barrier (since after the reversal of the outward direction it becomes a future outer untrapped surface (at least weakly). Next, let $\tilde{\Sigma}^-$ be the outermost past outer trapped surface enclosing Σ^-, where now $\tilde{\Sigma}^+$ acts as a barrier (since after the reversal of its outward direction it becomes a past outer weakly untrapped surface). It follows from the theorems of Andersson, Eichmair and Metzger that $\tilde{\Sigma}^+$ and $\tilde{\Sigma}^-$ exist, are future apparent horizons and past apparent horizons respectively, and are not enclosed by any other apparent horizons, making $\tilde{\Sigma}^+ \cup \tilde{\Sigma}^-$ an outermost time independent apparent horizon. We understand that this argument will be explained in more detail in an upcoming paper by Eichmair and Metzger [12].

Hence, by the above arguments, the following conjecture implies the previous one.

Conjecture 7. *(The Time-Independent Penrose Conjecture - Outermost Case) Suppose that the Cauchy data (M^3, g, k) is complete, satisfies the nonnegative energy density condition $\mu \geq |J|$, and is Schwarzschild at infinity with total mass m in a chosen end. If $\Sigma^2 \in \mathcal{S}$ is an outermost time-independent apparent horizon, then*

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{26}$$

where A is the area of the outermost minimal area enclosure $\tilde{\Sigma}^2 = \partial U^3$ of Σ^2. Furthermore, equality occurs if and only if $(M^3 \setminus U^3, g, k)$ is the pullback of the Cauchy data induced on the image of a space-like embedding of $M^3 \setminus U^3$ into the exterior region of a Schwarzschild spacetime (which maps $\tilde{\Sigma}^2$ to an apparent horizon).

The remainder of this paper will focus on proving Conjecture 7, which by the works of Andersson, Eichmair, and Metzger [2, 10, 1] implies conjecture 6, which trivially implies the standard version of the Penrose conjecture, conjecture 1, since future apparent horizons are time-independent outer trapped.

5. Proof of the Penrose conjecture in a special case. In this section we will prove the Penrose conjecture, conjecture 1, with two extra assumptions, and show how the conjecture follows from the Riemannian Penrose inequality, theorem 3. This special case, where a correct approach is quite clear, will help us motivate the general case which is not so obvious.

A major hint in the statement of the Penrose conjecture is the case of equality. Since the Penrose conjecture is an equality for any slice (space-like hypersurface) of the exterior region of the Schwarzschild spacetime with an apparent horizon boundary, we know that all of our techniques must preserve this equality in every estimate we derive.

On the other hand, if we are given some Cauchy data (M^3, g, k) which comes from a slice of the Schwarzschild spacetime, it may be difficult to recognize it as such. However, our techniques must absolutely be able to recognize these Cauchy data as the instances where we get equality in all of our inequalities.

More generally, suppose (M^3, g, k) comes from a slice of the static spacetime

$$\left(\mathbf{R} \times M^3, -\phi^2 dt^2 + \bar{g}\right), \tag{27}$$

where ϕ is a real-valued function on M and $\bar{g}$ is some other Riemannian metric on M. Notice that the Schwarzschild spacetime can be expressed in the form of equation 27. However, while the Schwarzschild spacetime is vacuum (meaning it has zero Einstein curvature and consequently zero Ricci curvature), we are making no such requirement on $\big(\mathbf{R} \times M^3, -\phi^2 dt^2 + \bar{g}\big)$.

Given a real-valued function f on M, define the graph map

$$F : M \mapsto \mathbf{R} \times M \tag{28}$$

where $F(x) = (f(x), x)$. Then a short calculation reveals that the pullback of the induced metric on the image of F in a coordinate chart is $\bar{g}_{ij} - \phi^2 f_i f_j$, so setting

$$\bar{g}_{ij} = g_{ij} + \phi^2 f_i f_j,$$

guarantees that the pullback of the induced metric on the image of the graph map F is precisely g. A similar type of calculation (but which is much longer and so is carried out in the appendices) yields that the pullback of the second fundamental form of the image of the graph map F in the static spacetime to (M^3, g) is

$$h_{ij} = \frac{\phi Hess_{ij} f + \phi_i f_j + f_i \phi_j}{\left(1 + \phi^2 |df|_g^2\right)^{1/2}}, \tag{29}$$

where subscripts on f and ϕ represent coordinate chart partial derivatives and the Hessian of f is taken with respect to the metric g (or the Levi-Civita connection of g if one prefers). These considerations lead us to the following special case of the Penrose conjecture which has an elegant and relatively short proof using the Gauss-Codazzi identities and the Riemannian Penrose inequality.

Theorem 4. *The Penrose conjecture as stated in conjecture 1 follows for (M^3, g, k) if there exist two smooth functions f and ϕ on M^3 such that*

$$k_{ij} = h_{ij} = \frac{\phi Hess_{ij} f + \phi_i f_j + f_i \phi_j}{\left(1 + \phi^2 |df|_g^2\right)^{1/2}} \quad \textit{outside of } \Sigma \tag{30}$$

and

$$\phi = 0 \textit{ on } \Sigma, \tag{31}$$

where $\phi > 0$ outside of Σ and f has compact support.

Proof. We will reduce the Penrose conjecture on (M^3, g, k) to the Riemannian Penrose inequality on $(M^3, \bar{g})$. To do this we need to show that

- the scalar curvature $\bar{R}$ of $\bar{g}$ is nonnegative and that
- ◦ Σ has zero mean curvature $\bar{H}$ in $(M^3, \bar{g})$.

Then the fact that $\bar{g}$ measures areas to be at least as large as g does implies that the area of any surface in $(M^3, \bar{g})$ is at least as large as the area of that same surface in (M^3, g). Thus,

$$\bar{A} := |\tilde{\Sigma}_{\bar{g}}|_{\bar{g}} \geq |\tilde{\Sigma}_{\bar{g}}|_g \geq |\tilde{\Sigma}_g|_g =: A,$$

where $\tilde{\Sigma}_{\bar{g}}$ and $\tilde{\Sigma}_g$ are the outermost minimal area enclosures of Σ in $(M^3, \bar{g})$ and (M^3, g), respectively. Since f has compact support, the masses of the two manifolds are the same. Then by the Riemannian Penrose inequality

$$m = \bar{m} \geq \sqrt{\frac{\bar{A}}{16\pi}} \geq \sqrt{\frac{A}{16\pi}},$$

which proves that Penrose conjecture on (M^3, g, k). Thus, all that is left to prove are the two bullet points (•) and (∘).

Proof of (∘). Since $\bar{g}_{ij} = g_{ij} + \phi^2 f_i f_j$ and $\phi = 0$ on Σ and ϕ and f are smooth, the two metrics are the same up to first order on Σ. But the mean curvature of a surface, which is the main term in the first variation of area formula, only depends on the metric and the first derivatives of the metric. Hence, $\bar{H} = H$.

Since Σ is an apparent horizon, $H = \pm\text{tr}_\Sigma(k)$. But since $\phi = 0$ on Σ, derivatives along Σ of ϕ are zero as well, so our assumption on the special form of k in equation 30 implies that $\text{tr}_\Sigma(k) = 0$. Hence,

$$\bar{H} = H = \text{tr}_\Sigma(k) = 0.$$

Proof of (•). Working inside of the static spacetime in equation 27, let n be the future pointing normal vector to the image of M^3 under the graph map F from equation 28 and let $\bar{n}$ be the future pointing normal vector to M^3 viewed as the $t = 0$ slice of the static spacetime. Then these two vector fields on hypersurfaces can be extended to the entire spacetime by requiring that these extended vector fields are invariant under translation in the time coordinate (which is an isometry of the spacetime).

The trick is to compute $G(n, \bar{n})$ using the Gauss-Codazzi identities, but in two different ways. We are given the nonnegative energy density condition on (M^3, g, k) that $\mu \geq |J|$. Since we are in the very special case that k actually equals the second fundamental form h of the graph, (M^3, g, h) has $\mu \geq |J|$ too. This is equivalent to saying that $G(n, w) \geq 0$ for all future time-like vectors w in the spacetime. Letting $w = \bar{n}$ thus implies that

$$G(n, \bar{n}) \geq 0. \tag{32}$$

On the other hand, applying the Gauss-Codazzi identities to the $t = 0$ slice of the static spacetime gives us

$$\begin{aligned} (8\pi)\ \bar{\mu} &= G(\bar{n}, \bar{n}) = (\bar{R} + \text{tr}(\bar{p})^2 - \|\bar{p}\|^2)/2 \\ (8\pi)\ \bar{J} &= \ G(\bar{n}, \cdot)\ = \text{div}\,(\bar{p} - \text{tr}(\bar{p})\bar{g}) \end{aligned}$$

where $\bar{p}$ is the second fundamental form of the $t = 0$ slice, which of course is zero by the time symmetry of the spacetime. Hence, $(8\pi)\bar{\mu} = \bar{R}/2$ and $\bar{J} = 0$. Thus, if we let

$$n = \alpha\bar{n} + (\text{vector tangent to } t = 0 \text{ slice}),$$

where α is a positive function on M, we have that

$$G(\bar{n}, n) = \alpha G(\bar{n}, \bar{n}) = \alpha\bar{R}/2. \tag{33}$$

But G is symmetric, so by inequality 32, $\bar{R} \geq 0$, which completes the proof of (•) and the proof of the Penrose inequality in this special case.

The case of equality of the above theorem would follow from conjecture 9 in section 8. We refer the reader to that section for discussion on the case of equality since the main purpose of this section was to motivate the identities computed in the next section.

6. The generalized Schoen-Yau identity. The proof of the Penrose conjecture in the special case presented above suggests how the Gauss-Codazzi identities can be used to compute a formula for the scalar curvature $\bar{R}$ of $\bar{g} = g + \phi^2 df^2$ in terms of the scalar curvature R of g, the graph function f, and the warping factor ϕ. In this section we will derive this formula and then show how this formula leads to an identity central to our approach to the Penrose conjecture.

¿From this point on we will abuse terminology slightly and always refer to the image of the graph map $F(M)$ simply as M and the $t = 0$ slice of the constructed spacetime as $\bar{M}$. This notation is convenient since then (M, g) and $(\bar{M}, \bar{g})$ are spacelike hypersurfaces of the spacetime $\left(\mathbf{R} \times M^3, -\phi^2 dt^2 + \bar{g}\right)$. Let $\pi : M \mapsto \bar{M}$ be the projection map $\pi(f(x), x) = (0, x)$ to the $t = 0$ slice of the spacetime.

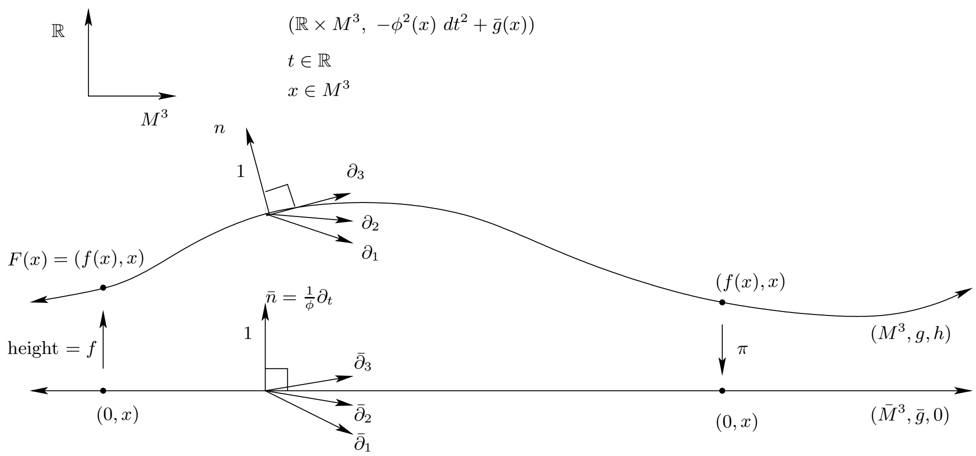

FIG. 1. *Schematic diagram of the constructed static spacetime*

Establishing some notation, let $\bar{\partial}_0 = \partial_t$ and $\{\bar{\partial}_i\}$ be coordinate vectors tangent to $\bar{M}$. Define

$$\partial_i = \bar{\partial}_i + f_i \bar{\partial}_0 \tag{34}$$

to be the corresponding coordinate vectors tangent to M so that $\pi_*(\partial_i) = \bar{\partial}_i$. Then in this coordinate chart, we have that

$$g_{ij} = \bar{g}_{ij} - \phi^2 f_i f_j. \tag{35}$$

It is convenient to write

$$g^{ij} = \bar{g}^{ij} + v^i v^j, \tag{36}$$

where

$$v^i = \frac{\phi f^{\bar{i}}}{(1 - \phi^2 |df|_{\bar{g}}^2)^{1/2}} = \frac{\phi f^i}{(1 + \phi^2 |df|_g^2)^{1/2}}. \tag{37}$$

We also define

$$\bar{v} = v^i \bar{\partial}_i \qquad \text{and} \qquad v = v^i \partial_i \tag{38}$$

so that $\pi_*(v) = \bar{v}$, and observe the useful identity

$$(1 - \phi^2 |df|^2_{\bar{g}}) \cdot (1 + \phi^2 |df|^2_g) = 1, \tag{39}$$

which is evident by looking at the ratios of the volume forms. See appendix C for more discussion on these calculations.

In this paper we use the convention that a barred index (as in $f^{\bar{i}}$ above) denotes an index raised (or lowered) by $\bar{g}$ as opposed to g. That is, $f^{\bar{i}} = \bar{g}^{ij} f_j$, where as usual $f_j = \partial f / \partial x_j$ in the coordinate chart. In general, barred quantities will be associated with the $t = 0$ slice $(\bar{M}, \bar{g})$ and unbarred quantities will be associated with the graph slice (M, g).

In appendix D we compute that the second fundamental form of the graph slice (M, g) in our constructed static spacetime is

$$h_{ij} = \frac{\phi \overline{\text{Hess}}_{ij} f + (f_i \phi_j + \phi_i f_j) - \phi^2 \langle df, d\phi \rangle_{\bar{g}} f_i f_j}{(1 - \phi^2 |df|^2_{\bar{g}})^{1/2}} \tag{40}$$

$$= \frac{\phi \text{Hess}_{ij} f + (f_i \phi_j + \phi_i f_j)}{(1 + \phi^2 |df|^2_g)^{1/2}}, \tag{41}$$

which we list now for future reference.

Finally, we extend h and k trivially in our constructed static spacetime so that $h(\partial_t, \cdot) = 0 = k(\partial_t, \cdot)$ and such that these extended 2-tensors equal the original 2-tensors when restricted to M. Note that this gives $h(\partial_i, \partial_j) = h(\bar{\partial}_i, \bar{\partial}_j)$, so we can call this term h_{ij} without ambiguity. The same is true for k_{ij} and components of 1-forms like f_i and ϕ_i. However, we remind the reader that the Hessian of a function, which is the covariant derivative of the differential of a function, depends on the connection and hence the metric since we will always be using the respective Levi-Civita connections on (M, g) and $(\bar{M}, \bar{g})$.

Now we are ready to proceed to compute a formula for $\bar{R}$. It is a short calculation to verify that, in the constructed spacetime,

$$\langle n, \bar{n} \rangle = -(1 - \phi^2 |df|^2_{\bar{g}})^{-1/2} = -(1 + \phi^2 |df|^2_g)^{1/2} \tag{42}$$

Thus,

$$\bar{n} = (1 + \phi^2 |df|^2_g)^{1/2} n + \tan_{\text{graph}}(\bar{n}) \tag{43}$$

where another short calculation reveals that

$$\tan_{\text{graph}}(\bar{n}) = -\phi f^j \partial_j = -\phi \nabla f. \tag{44}$$

As in the previous section, the trick is to compute $G(n, \bar{n})$ two different ways using the Gauss-Codazzi identities. As before, applying these identities to the $t = 0$ slice $(\bar{M}, \bar{g})$ of the constructed spacetime gives us

$$\begin{aligned} G(n, \bar{n}) &= (1 + \phi^2 |df|^2_g)^{1/2} G(\bar{n}, \bar{n}) \\ &= (1 + \phi^2 |df|^2_g)^{1/2} \cdot \bar{R}/2 \end{aligned}$$

since the $t = 0$ slice has zero second fundamental form. On the other hand, applying the Gauss-Codazzi identities to the graph slice (M, g) yields

$$\begin{aligned} G(n, \bar{n}) &= (1 + \phi^2 |df|^2_g)^{1/2} G(n, n) + G(n, \tan_{\text{graph}}(\bar{n})) \\ &= (1 + \phi^2 |df|^2_g)^{1/2} [R + (\text{tr}_g h)^2 - \|h\|^2_g]/2 \\ &\quad + \text{div}(h - (\text{tr}_g h) g)(-\phi \nabla f). \end{aligned}$$

Combining the two previous equations, we get our first desired result

$$\bar{R} = R + (\mathrm{tr}_g h)^2 - \|h\|_g^2 + 2(d(\mathrm{tr}_g h) - \mathrm{div}(h))(v). \tag{45}$$

Of course, what we are given in the hypotheses of the Penrose conjecture is that $\mu \geq |J|_g$, where

$$\begin{aligned} (8\pi)\ \mu \ &= \ G(n,n) = (R + \mathrm{tr}_g(k)^2 - \|k\|^2)/2 \\ (8\pi)\ J \ &= \ G(n,\cdot) \ = \mathrm{div}(k) - d(\mathrm{tr}_g(k)), \end{aligned}$$

for some symmetric 2-tensor k. Hence,

$$\begin{aligned} \bar{R} = 16\pi(\mu - J(v)) &+ (\mathrm{tr}_g h)^2 - (\mathrm{tr}_g k)^2 - \|h\|_g^2 + \|k\|_g^2 \\ &+2\, v(\mathrm{tr}_g h) - 2\, v(\mathrm{tr}_g k) - 2\, \mathrm{div}(h)(v) + 2\, \mathrm{div}(k)(v). \end{aligned} \tag{46}$$

Note that $\mu - J(v) \geq 0$ since $|v|_g \leq 1$. Hence, as we saw in the previous section, if we can choose a ϕ and an f so that $h = k$, then we immediately get that $\bar{R} \geq 0$. However, we are interested in investigating if a more general relationship between h and k can give a similar result.

Our procedure is to convert our formula for $\bar{R}$ to an expression in terms of the $\bar{g}$ metric. Arguably $\bar{g}$ is more natural than g since it is the metric induced on the $t = 0$ slice of the static spacetime. To perform the conversion, we need several identities for arbitrary symmetric 2-tensors k which are proven in appendix D and which we list here.

Identity 1.

$$(tr_g(k))^2 - \|k\|_g^2 = (tr_{\bar{g}} k)^2 - \|k\|_{\bar{g}}^2 + 2k(\bar{v},\bar{v}) tr_{\bar{g}} k - 2|k(\bar{v},\cdot)|_{\bar{g}}^2$$

Identity 2.

$$v(tr_g k) = \bar{v}(tr_{\bar{g}} k + k(\bar{v},\bar{v}))$$

Identity 3.

$$\overline{\Gamma}_{ij}^k - \Gamma_{ij}^k = h_{ij} v^k - \phi f_i f_j \phi^{\bar{k}}$$

Identity 4.

$$\begin{aligned} div(k)(v) = \overline{div}(k)(\bar{v}) + (\overline{\nabla}_{\bar{v}} k)(\bar{v},\bar{v}) - 2|\bar{v}|_{\bar{g}}^2\, k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right) \\ + \langle h(\bar{v},\cdot), k(\bar{v},\cdot)\rangle_{\bar{g}} + 2h(\bar{v},\bar{v})k(\bar{v},\bar{v}) + (tr_{\bar{g}} h) k(\bar{v},\bar{v}) \end{aligned}$$

Identity 5.

$$v_{\bar{\imath};j} = h_{ij} + v_{\bar{\imath}} h(\bar{v},\cdot)_j - \frac{\phi_i v_{\bar{\jmath}}}{\phi}$$

Identity 6.

$$\overline{div}(k)(\bar{v}) = \overline{div}(k(\bar{v},\cdot)) - \langle h,k\rangle_{\bar{g}} - \langle h(\bar{v},\cdot),k(\bar{v},\cdot)\rangle_{\bar{g}} + k\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right)$$

Identity 7.

$$(\overline{\nabla}_{\bar{v}}k)(\bar{v},\bar{v}) = \bar{v}(k(\bar{v},\bar{v})) - 2\langle h(\bar{v},\cdot),k(\bar{v},\cdot)\rangle_{\bar{g}} - 2h(\bar{v},\bar{v})k(\bar{v},\bar{v}) + 2|\bar{v}|^2_{\bar{g}}\, k\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right)$$

Identity 8.

$$\begin{aligned} div(k)(v) &= \overline{div}(k(\bar{v},\cdot)) + \bar{v}(k(\bar{v},\bar{v})) + k\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right) \\ &\quad -\langle h,k\rangle_{\bar{g}} - 2\langle h(\bar{v},\cdot),k(\bar{v},\cdot)\rangle_{\bar{g}} + (tr_{\bar{g}}h)k(\bar{v},\bar{v}) \end{aligned}$$

Identities 1 and 2 are short calculations. Identity 3 is used in the proof of identity 4. Plugging identities 6 and 7 (which are proved using identity 5) into identity 4 results in identity 8. Finally, plugging identities 1, 2, and 8 into our formula for $\bar{R}$ results in the main identity of this paper.

Identity 9. ***(The Generalized Schoen-Yau Identity)***

$$\begin{aligned} \bar{R} &= 16\pi(\mu - J(v)) + \|h-k\|^2_{\bar{g}} + 2|q|^2_{\bar{g}} - \frac{2}{\phi}\overline{div}(\phi q) \\ &\quad +(tr_{\bar{g}}h)^2 - (tr_{\bar{g}}k)^2 + 2\bar{v}(tr_{\bar{g}}h - tr_{\bar{g}}k) + 2k(\bar{v},\bar{v})(tr_{\bar{g}}h - tr_{\bar{g}}k) \end{aligned}$$

where

$$q = h(\bar{v},\cdot) - k(\bar{v},\cdot) = h(v,\cdot) - k(v,\cdot)\ .$$

Note that the two definitions of q exist on the entire constructed static spacetime and are equal since both h and k are extended trivially in the constructed static spacetime. We also observe that

$$\frac{1}{\phi}\overline{\mathrm{div}}(\phi q) = \mathrm{div}_{ST}(q),$$

where div_{ST} is the divergence operator in the constructed static spacetime.

In the special case that $\phi = 1$, the above identity was derived by a different method by Schoen-Yau as equation 2.25 of [31] (in fact the procedure in [31] may also be used to obtain identity 9 and will be presented in a future paper). In that paper, they used the Jang equation,

$$0 = \mathrm{tr}_{\bar{g}}(h-k)$$

to reduce the positive mass theorem to the Riemannian positive mass theorem. While imposing the Jang equation in the special case that $\phi = 1$ does not imply that $\bar{R} \geq 0$ as would be most desirable, $\bar{R} \geq 2|q|^2_{\bar{g}} - 2\,\overline{\mathrm{div}}(q)$ implies that there exists a conformal factor on $\bar{g}$ such that the conformal metric has nonnegative scalar curvature and total mass less than or equal to that of $\bar{g}$ and g. Then the Riemannian positive mass theorem applied to the metric conformal to $\bar{g}$ implies the positive mass theorem on (M,g). This approach does not quite work for the Penrose conjecture because the conformal factor needed to achieve nonnegative scalar curvature changes the area of the horizon in a way which is difficult to control.

7. The generalized Jang equation. All of the approaches to the Penrose conjecture that we consider in this paper use the generalized Schoen-Yau identity. This identity plays a central role in the remainder of our discussions because it directly relates the nonnegative energy condition on (M^3, g, k) (which implies that $\mu \geq J(v)$ since $|v|_g < 1$) to the scalar curvature of $(M^3, \bar{g})$.

Furthermore, this generalized Schoen-Yau identity strongly motivates the generalized Jang equation,

$$0 = \mathrm{tr}_{\bar{g}}(h - k), \tag{47}$$

which on the original manifold (M^3, g) with Cauchy data (M^3, g, k) is the equation

$$0 = \left(g^{ij} - \frac{\phi^2 f^i f^j}{1 + \phi^2 |df|_g^2} \right) \left(\frac{\phi Hess_{ij} f + \phi_i f_j + f_i \phi_j}{\left(1 + \phi^2 |df|_g^2\right)^{1/2}} - k_{ij} \right) \tag{48}$$

when one substitutes the formulas for h and $\bar{g}^{ij}$ in a coordinate chart. (In this paper we adopt Einstein's convention that whenever there are both raised and lowered indices, summation is implied, so the above formula is a summation over i, j both ranging from 1 to 3.)

Of course the original Jang equation, which again is the special case $\phi(x) = 1$, only had one free function, f, whereas the generalized Jang equation has two free functions, f and ϕ. Hence, to get a determined system of equations, we need to specify one more equation. Later in the paper we will propose various choices for this second equation, but our choice for the first equation will always be the generalized Jang equation above.

Once the generalized Jang equation is specified, the generalized Schoen-Yau identity simplifies greatly to

$$\bar{R} = 16\pi(\mu - J(v)) + \|h - k\|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2 - \frac{2}{\phi}\overline{\mathrm{div}}(\phi q). \tag{49}$$

It is important to note that the first three terms of the right hand side of the above equation are all nonnegative since $\mu \geq |J|_g$ and $|v|_g < 1$.

7.1. Boundary conditions. As discussed in the next subsection, blowups and blowdowns of the generalized Jang equation may occur on future or past apparent horizons. However, as discussed in section 4, there are no future or past apparent horizons in the Cauchy data (M^3, g, k) if we assume that the interior boundary of our Cauchy data is an outermost time-independent apparent horizon, which we do. As also explained in that section, proving the Penrose conjecture for time-independent apparent horizons implies the standard version of the Penrose conjecture as well.

Each connected component of a time-independent apparent horizon is either a future apparent horizon, a past apparent horizon, or both. Based on our study of the case of equality in section 2, we generally expect blowup on future apparent horizons and blowdown on past apparent horizons. Hence, this is a reasonable guess for the boundary behavior we may want to impose on solutions to the generalized Jang equation.

On the other hand, some apparent horizons are both future and past apparent horizons. Furthermore, as explained in section 2, there are case of equality slices of Schwarzschild where the boundary is both a future and a past apparent horizon where the graph function f stays bounded. Hence, bounded boundary behavior may be appropriate for such horizons.

In fact, there are even case of equality slices of Schwarzschild whose boundary is a future apparent horizon where the graph function f blows up on part of the boundary and stays bounded on the rest of the boundary. These examples are compatible with the following boundary conditions:

Possible boundary conditions for the generalized Jang equation:
Given a boundary surface Σ where each connected component is either a future apparent horizon or a past apparent horizon with outward unit normal ν in (M^3, g), we require that $\phi = 0$ on Σ and

$$\langle \nu, v \rangle_g = sign(\mathrm{tr}^g_\Sigma(k)) \tag{50}$$

on Σ, where as usual

$$v = \frac{\phi \nabla f}{(1 + \phi^2 |df|^2_g)^{1/2}}$$

and v is extended to the boundary Σ by continuity.

Note that these boundary conditions are consistent with f blowing up to $+\infty$ where $\mathrm{tr}^g_\Sigma(k) < 0$, blowing down to $-\infty$ where $\mathrm{tr}^g_\Sigma(k) > 0$, and f staying bounded where $\mathrm{tr}^g_\Sigma(k) = 0$ on Σ.

Ultimately, existence theories for the systems of equations that we propose in this paper will have to be understood before we can confidently formulate the correct boundary conditions. The goal is to find boundary conditions which imply that Σ becomes a minimal surface with

$$\bar{H}_\Sigma = 0$$

in $(M^3, \bar{g})$. We discuss the general case of this question in appendix E (arXiv version of this paper). For now, we observe two important special cases.

The first important special case is when Σ is a traditional apparent horizon, either future or past, and $H_\Sigma > 0$. If we also assume that f goes to $\pm\infty$ on each connected component of Σ in a reasonable fashion, then the level sets of f converge to Σ. The formula for the mean curvature of the level sets of f in the new metric $\bar{g}$ is

$$\begin{aligned} \bar{H} &= (1 + \phi^2 |df|^2)^{-1/2} H \\ &= (1 - |v|^2_g)^{1/2} H \end{aligned}$$

since $\bar{g} = g + \phi^2 df^2$ does not change the metric on the level sets of f, stretches lengths perpendicular to the level sets of f by a factor of $(1 + \phi^2 |df|^2)^{1/2}$, and by the first variation formula for area. Then if we assume that f and ϕ behave similarly to the case of equality slices of Schwarzschild, we get the following lemma.

LEMMA 1. *Suppose that (M^3, g, k) has a smooth interior boundary Σ which is a future [past] apparent horizon with $H_\Sigma > 0$. Then if f blows up [blows down] logarithmically, $|df|$ blows up asymptotic to $1/s$, and ϕ^2 goes to zero asymptotic to s (where s is the distance to Σ in (M^3, g)), then the limit of the mean curvatures $\bar{H}$ of the level sets of f in $(M^3, \bar{g})$ is zero.*

The second important special case is the case where the boundary is a future and past apparent horizon. In this case, based on the case of equality slices of

Schwarzschild, we expect f to stay bounded and smooth and ϕ to stay smooth as well.

LEMMA 2. *Suppose that (M^3, g, k) has a smooth interior boundary Σ which is a future and past apparent horizon (which by definition has $H_\Sigma = 0$ and $\mathrm{tr}_\Sigma^g(k) = 0$). Then if f is bounded and smooth and ϕ is smooth and equals zero on Σ, then $\bar{H}_\Sigma = H_\Sigma = 0$.*

The proof of this lemma appeared in this paper already in section 5. The point is that since $\bar{g} = g + \phi^2 df^2$, both metrics $\bar{g}$ and g are the same up to first order on Σ since f and ϕ are smooth and $\phi = 0$ on Σ. Then since the mean curvature is only a function of the metric and first derivatives of the metric, the two mean curvatures are equal, and since $H_\Sigma = 0$, both are zero.

7.2. Blowups, blowdowns, and outermost horizons. One phenomenon of the original Jang equation ($\phi = 1$) is that f can blowup to ∞ on future apparent horizons or blowdown to $-\infty$ on past apparent horizons, and this feature is still present in the generalized Jang equation (given plausible assumptions about the behavior of f and ϕ). More importantly, according to [31], blowups and blowdowns of f with the original Jang equation can *only* occur on apparent horizons.

An important question, then, is to understand when blowups can occur with the generalized Jang equation. Certainly blowups and blowdowns can still occur on traditional apparent horizons. A reasonable conjecture is that the blowup properties of the generalized Jang equation are the same as the original Jang equation as long as ϕ is smooth and strictly positive. However, when ϕ goes to zero, as it does on the boundary by our boundary conditions, there could be some additional subtleties to consider.

A relevant calculation which is useful for studying the question of when f can blowup or blowdown is the following. Recall the standard identity

$$\Delta f = \mathrm{Hess} f(\nu, \nu) + H_\Sigma \cdot \nu(f) + \Delta_\Sigma f$$

for the Laplacian of a function in terms of the Laplacian of that function restricted to a hypersurface with mean curvature H and outward unit normal ν. If we let Σ be any level set of f, then we get that

$$\mathrm{tr}_\Sigma^g(\mathrm{Hess} f) = \mp |df|_g H_\Sigma$$

for blowup and blowdown respectively. Thus, the generalized Jang equation implies that

$$\begin{aligned}
0 &= \mathrm{tr}_{\bar{g}}(h-k) \\
&= \bar{g}^{ij}(h_{ij}-k_{ij}) \\
&= \left[(g^{ij}-\nu^i\nu^j)+(\nu^i\nu^j-v^iv^j)\right](h_{ij}-k_{ij}) \\
&= \mathrm{tr}^g_\Sigma(h-k)+\frac{(h-k)(\nu,\nu)}{1+\phi^2|df|_g^2} \\
&= \frac{\phi\ \mathrm{tr}^g_\Sigma(\mathrm{Hess}f)}{(1+\phi^2|df|_g^2)^{1/2}}-\mathrm{tr}^g_\Sigma(k)+\frac{(h-k)(\nu,\nu)}{1+\phi^2|df|_g^2} \\
&= \mp\frac{\phi|df|_g}{(1+\phi^2|df|_g^2)^{1/2}}H_\Sigma-\mathrm{tr}^g_\Sigma(k)+\frac{(h-k)(\nu,\nu)}{1+\phi^2|df|_g^2} \\
&= \langle\nu,v\rangle_g H_\Sigma-\mathrm{tr}^g_\Sigma(k)+\frac{(h-k)(\nu,\nu)}{1+\phi^2|df|_g^2}
\end{aligned}$$

on level sets of f, where we have used the facts that v and ν are collinear,

$$|v|_g^2 = 1-\frac{1}{1+\phi^2|df|_g^2},$$

and the formulas for $\bar{g}^{ij}$ and h_{ij} from section 6.

Lemma 3. *When f is blowing up or blowing down, the term $h(\nu,\nu)$ is bounded if $\phi^2 df$ is assumed to be smooth and nonzero in the limit up to the boundary and $\phi=0$ on the boundary, which is true in case of equality slices of Schwarzschild.*

Proof. Referring back to section 2, in a smooth slice of Schwarzschild $\phi^2 df$ can be expressed in terms of the smooth Kruskal coordinate variables u, v. Since by equation 15

$$df = 2m\left(\frac{dv}{v}-\frac{du}{u}\right)$$

and by equation 16

$$\phi^2 = uv\gamma(uv)$$

for some smooth function $\gamma \neq 0$ for $\phi^2 < 1$, we have that

$$\phi^2 df = 2m\gamma(uv)\left(udv - vdu\right).$$

The fact that our case of equality slices of Schwarzschild are spacelike implies that $|du|_g \neq 0 \neq |dv|_g$, and the fact that we are assuming blowup or blowdown implies that exactly one of u, v is going to zero on the boundary. Hence, not only is $\phi^2 df$ smooth up to the boundary, it is also nonzero in the limit up to the boundary.

Then the fact that $h(\nu,\nu)$ is bounded up to the boundary assuming this smoothness follows from the short calculation that

$$h(\nu,\nu) = \frac{\nabla_\nu(\phi^2 df)(\nu)}{(\phi^2+|\phi^2 df|_g^2)^{1/2}}, \tag{51}$$

which completes the proof of the lemma.

Thus, referring back to our calculation before the lemma, given blowup or blowdown with behavior on f and ϕ as seen in the case of equality slices of Schwarzschild in section 2, ϕ^2 goes to zero linearly, $|df|_g$ goes to infinity like $1/s$ so that $\phi^2|df|_g^2$ goes to infinity like $1/s$, and $h(\nu,\nu)$ stays bounded. Then since k is given to be smooth and therefore bounded, we conclude that the generalized Jang equation implies that

$$0 = \mp H_\Sigma - \mathrm{tr}^g_\Sigma(k)$$

on surfaces with this type of blowup or blowdown of f, which of course are the equations for future and past apparent horizons, respectively.

8. The Jang - zero divergence equations. Looking at equation 49, the most direct way to get $\bar{R} \geq 0$ is to set $\overline{\mathrm{div}}(\phi q) = 0$. We will call the resulting system of equations, equations 52 and 53, the Jang - zero divergence equations. The following existence conjecture for these equations implies the outermost case of the time-independent Penrose conjecture, conjecture 7, using the Riemannian Penrose inequality, and is therefore an important open problem.

CONJECTURE 8. *Given asymptotically flat Cauchy data (M,g,k) with an outermost time-independent apparent horizon boundary Σ, there exists a solution (f,ϕ) to the system of equations*

$$0 = tr_{\bar{g}}(h-k) \tag{52}$$

$$0 = \overline{div}(\phi q), \tag{53}$$

with $\lim_{x\to\infty} f(x) = 0$, $\phi^2|\nabla f|^2 = o(r^{-1})$, $\nabla(\phi^2|\nabla f|^2) = o(r^{-2})$, *and* $\lim_{x\to\infty}\phi(x) = 1$, *where* $\bar{g} = g + \phi^2 df^2$,

$$h = \frac{\phi Hessf + (df\otimes d\phi + d\phi\otimes df)}{(1+\phi^2|df|_g^2)^{1/2}}\ , \tag{54}$$

$q = h(v,\cdot) - k(v,\cdot)$, *and* $v = \phi\nabla f/(1+\phi^2|df|_g^2)^{1/2}$, *such that* Σ *has zero mean curvature in the* $\bar{g}$ *metric.*

The boundary conditions on f which may lead to Σ having zero mean curvature in $(M^3,\bar{g})$ are discussed in the previous section and in appendix E (arXiv version of this paper).

Also, we comment that while equation 53 is third order in f, subtracting derivatives of equation 52 can remove the third order terms of f in favor of Ricci curvature terms. While the resulting system has quadratic second order terms in f in the second equation, the system is degenerate elliptic.

The above system may also be reduced to a system of 1st order equations by introducing new variables. If we let $\alpha = df$, then the above system has a solution whenever the first order system with variables ϕ (a 0-form), α (a 1-form), and β (a 2-form)

$$0 = d\alpha \tag{55}$$

$$0 = d\beta \tag{56}$$

$$0 = \mathrm{tr}_{\bar{g}}(h-k) \tag{57}$$

$$\phi q = d^{\bar{*}}\beta \tag{58}$$

has a solution, where $d^{\bar{*}}$ is the d star operator with respect to the $\bar{g}$ metric which sends 2-forms to 1-forms. In these variables, $\bar{g} = g + \phi^2\alpha^2$,

$$h = \frac{\phi\nabla\alpha + (\alpha \otimes d\phi + d\phi \otimes \alpha)}{(1+\phi^2|\alpha|_g^2)^{1/2}}\ , \tag{59}$$

$q = h(v,\cdot) - k(v,\cdot)$, and $v = \phi\vec{\alpha}/(1+\phi^2|\alpha|_g^2)^{1/2}$, where $\vec{\alpha}$ is the dual vector to α with respect to g.

THEOREM 5. *Conjectures 8 and 9 (defined below to handle the case of equality) imply conjecture 7, the outermost case of the time-independent Penrose conjecture.*

Proof. As was discussed in the previous section, the point of requiring Σ to be outermost in the above conjecture is so that f does not blowup or blowdown on the interior of M. Then the method of proof assuming conjecture 8 is basically the same as the proof of the Penrose conjecture in the special case in section 5. The total mass of (M^3, g) is the same as the total mass of $(M^3, \bar{g})$ since the total mass is defined in terms of the $1/r$ rate of decay of the metrics which are equal since $||\bar{g} - g||_g = \phi^2|df|_g^2$. Also, since $\bar{g}$ measures lengths, areas, etc. to be greater than or equal to that measured by g,

$$\bar{A} = |\tilde{\Sigma}_{\bar{g}}|_{\bar{g}} \geq |\tilde{\Sigma}_{\bar{g}}|_g \geq |\tilde{\Sigma}_g|_g = A. \tag{60}$$

Hence, the Penrose conjecture on (M^3, g, k) follows from the Riemannian Penrose inequality on $(M^3, \bar{g})$.

Thus, all that remains is to show that the Riemannian Penrose inequality can be applied to (M^3, g). The existence theorem already gives us that $\bar{H} = 0$, so the last thing to check is that $\bar{R} \geq 0$, which follows directly from the generalized Schoen-Yau identity and equation 49. This proves the inequality part of the Penrose conjecture on (M^3, g, k).

In the case of equality of the Penrose conjecture, clearly we must have equality in all of our inequalities. Since the case of equality of the Riemannian Penrose inequality is solely when $\bar{g}$ is the Schwarzschild metric which has zero scalar curvature $\bar{R}$, equation 49 gives us

$$0 = 16\pi(\mu - J(v)) + \|h - k\|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2. \tag{61}$$

Since each of these three terms is nonnegative, each must be zero. Hence, $k = h$. If we could argue that $\phi = \phi_0$, where ϕ_0 is the warping factor from the Schwarzschild spacetime, then we would have that $k = h$ is the second fundamental form and $g = \bar{g} - \phi^2 df^2$ is the induced metric of a slice of a Schwarzschild spacetime, as desired.

However, there is a delicate point here. In fact, ϕ does not have to equal ϕ_0 for (M^3, g, k) to be the Cauchy data from a slice of a Schwarzschild spacetime. If $\phi = c\phi_0$ for some constant $c > 0$, then defining $df_0 = c\, df$ (which can be integrated to recover f_0) implies that (f_0, ϕ_0) and (f, ϕ) produce the same metrics and second fundamental forms. This may seem like a minor point at first, but in fact this statement is still true if (and only if which we leave as an exercise) $dc = 0$ on the open region D where $df \neq 0$. Thus, c may be different constants on each connected component of D. Again, df_0, which is still closed, may be integrated to recover f_0 since the $t = 0$ slice of Schwarzschild is simply connected. Thus, we have the following lemma.

LEMMA 4. *If $\phi = c\phi_0$, where $dc = 0$ on $\{x \mid df \neq 0\}$, then (M^3, g, k) comes from a slice of the Schwarzschild spacetime.*

To prove the case of equality of the Penrose conjecture then, we need to prove the hypotheses of the above lemma. Looking back at equation 61, we see that we must also have

$$0 = \mu - J(v) = (\mu - |J|) + |J|(1 - |v|) + (|J||v| - J(v)),$$

where all norms are with respect to g. Again, since each of the three grouped terms is nonnegative, all must be zero. Since $|v| < 1$, the second term equaling zero implies that $|J| = 0$ so that the first term equalling zero implies that $\mu = 0$.

In the appendices we compute that in the static spacetime $\bar{g} - \phi^2 dt^2$,

$$n = (1 - \phi^2 |df|^2_{\bar{g}})^{1/2} (\bar{n} + \phi \overline{\nabla} f)$$

and that if $\bar{R} = 0$,

$$J = G(n, \cdot) = (1 - \phi^2 |df|^2_{\bar{g}})^{1/2} \left[\overline{\text{Ric}} - \frac{\overline{\text{Hess}}\phi}{\phi} + \frac{\bar{\Delta}\phi}{\phi} \bar{g} \right] (\phi \bar{\nabla} f, \cdot)$$

where $\cdot$ is a tangent vector to the graph slice (M^3, g) in the first instance and its component tangent to the $t = 0$ slice in the second. Then since the Schwarzschild spacetime has $G = 0$, $\bar{R} = 0$, $\bar{\Delta}\phi_0 = 0$, and

$$\overline{\text{Ric}} = \frac{\overline{\text{Hess}}\phi_0}{\phi_0},$$

$J = 0$ in the case of equality implies the overdetermined equation (when $df \neq 0$) for ϕ that

$$0 = \left[\frac{\overline{\text{Hess}}\phi_0}{\phi_0} - \frac{\overline{\text{Hess}}\phi}{\phi} + \left(\frac{\bar{\Delta}\phi}{\phi} - \frac{\bar{\Delta}\phi_0}{\phi_0} \right) \bar{g} \right] (\bar{\nabla} f, \cdot). \tag{62}$$

CONJECTURE 9. *Equation 62 implies the hypotheses of lemma 4.*

Clearly the hypotheses of lemma 4 imply equation 62, but we need the converse to be true as well. Assuming conjecture 8 is true, a proof of conjecture 9 would finish the case of equality part of the outermost case of the time-independent Penrose conjecture.

9. Einstein-Hilbert action methods. Equation 49 is a remarkable equation which deserves very careful consideration. Since we need a lower bound on the scalar curvature $\bar{R}$ of $(M^3, \bar{g})$, the only troublesome term in that equation is the last one, the divergence term. In the previous section, we dealt with this last term by setting it equal to zero. In this section, we make the natural observation that divergence terms can also be dealt with by integrating them.

THEOREM 6. *If $\bar{g} = g + \phi^2 df^2$ on M^3 with boundary Σ^2 and Cauchy data (M^3, g, k) satisfying the nonnegative energy condition $\mu \geq |J|$, the generalized Jang equation*

$$0 = \text{tr}_{\bar{g}}(h - k)$$

is satisfied, and f and ϕ behave at infinity and on the boundary Σ such that equations 66 and 67 are satisfied as expected, then

$$\int_M \bar{R}\phi \, \overline{dV} \geq 0, \tag{63}$$

where $\bar{R}$ is the scalar curvature of $\bar{g}$ and $\overline{dV}$ is the volume form of $\bar{g}$.

In other words, no matter what $\phi(x)$ is (as long as certain boundary conditions are satisfied), the generalized Jang equation by itself already gives a lower bound on the integral of the scalar curvature of $\bar{g}$, weighted by ϕ. Of course the choice of ϕ affects f since ϕ appears in the generalized Jang equation.

In the next couple of sections we discuss two different inequalities of the form

$$\bar{m} - \sqrt{\frac{\bar{A}}{16\pi}} \geq \int_M Q(x)\bar{R}(x)\overline{dV}, \tag{64}$$

for some $Q(x) \geq 0$, where each inequality is based on one of the two proofs of the Riemannian Penrose inequality. The expression for $Q(x)$ differs in the two cases and will be described later. However, if we then choose $\phi(x) = Q(x)$ to be our second equation to be coupled with the generalized Jang equation, then existence of such a system implies

$$m - \sqrt{\frac{A}{16\pi}} \;\geq\; \bar{m} - \sqrt{\frac{\bar{A}}{16\pi}} \;\geq\; \int_M Q(x)\bar{R}(x)\overline{dV} \;=\; \int_M \bar{R}\phi\,\overline{dV} \;\geq\; 0, \tag{65}$$

proving the corresponding form of the Penrose conjecture for the original Cauchy data (M^3, g, k). We will call any method of proof as above an Einstein-Hilbert action method. So far we know of only two methods of this form, the Jang-IMCF equations presented in the next section, and the Jang-CFM equations discussed in the section after that.

Note that any inequality of the form of inequality 64 proves the Riemannian Penrose inequality for $(M^3, \bar{g})$ as a special case since then $\bar{R} \geq 0$ by hypothesis. Thus, unless one finds a new proof of the Riemannian Penrose inequality, the only way to hope to prove an inequality like 64 is either to adapt currently known proofs of the Riemannian Penrose inequality, as we are about to do in this paper, or to use the Riemannian Penrose inequality itself. This last idea deserves additional consideration.

Proof of theorem 6. Applying the divergence theorem to equation 49 gives us that

$$\begin{aligned}\int_M \bar{R}\phi\,\overline{dV} &\geq 2\int_{\Sigma - S_\infty} \phi q(\bar{\nu})\overline{dA} \\ &= 2\int_\Sigma \phi(h-k)(\bar{v},\bar{\nu})\overline{dA},\end{aligned}$$

where k is assumed to converge to zero at infinity (or have compact support), and df is assumed to decay at least as fast as $1/r^2$ at infinity (with reasonable bounds on Hess$f$ as well) so that

$$0 = \lim_{r\to\infty} \int_{S_r} \phi(h-k)(\bar{v},\bar{\nu})\overline{dA}, \tag{66}$$

where $\bar{\nu}$ is the unit outward normal vector to Σ and the sphere at infinity in $(M^3, \bar{g})$.

In appendix E (arXiv version of this paper), we observe that

$$\bar{\nu} = \left(\frac{1+\phi^2|df|_g^2}{1+\phi^2|(df|_\Sigma)|_g^2}\right)^{1/2}(\nu - \langle \nu, v\rangle v)$$

and

$$\overline{dA} = (1 + \phi^2 |(df|_\Sigma)|_g^2)^{1/2} dA.$$

Then since $1 + \phi^2 |df|_g^2 = 1/(1 - |v|_g^2)$, we get that

$$\begin{aligned}\int_M \bar{R}\phi \, \overline{dV} &\geq 2 \int_\Sigma \phi(h-k)(\bar{v}, \bar{\nu}) \overline{dA} \\ &= 2 \int_\Sigma \frac{\phi(h-k)(v, \nu - \langle \nu, v \rangle v)}{(1 - |v|_g^2)^{1/2}} dA \\ &= 0\end{aligned}$$

if we assume that

$$0 = \lim_{\Sigma_\epsilon \to \Sigma} \int_{\Sigma_\epsilon} \frac{\phi(h-k)(v, \nu - \langle \nu, v \rangle v)}{(1 - |v|_g^2)^{1/2}} dA \tag{67}$$

for some smooth family of surfaces Σ_ϵ converging to Σ, proving the theorem.

In the case that f is blowing up (or down) everywhere on Σ, then choosing Σ_ϵ to be the level sets of f simplifies things even more since then $v = \pm |v| \nu$. Then the integrand becomes $\phi(1 - |v|_g^2)^{1/2}(h-k)(v, \nu)$ and equals zero with the usual boundary behavior since $\phi = 0$ on the boundary, $|v|_g$ is going to one on the boundary, and both $h(\nu, \nu)$ and $k(\nu, \nu)$ are bounded by lemma 3. Equation 67 is also clearly satisfied in the case of a future and past apparent horizon where we assume that f and ϕ stay smooth and bounded, since all of the terms in the integrand will be bounded, and $\phi = 0$ on the boundary Σ.

As a final comment on theorem 6 before moving on, we state again that a better understanding of the boundary behavior of f and ϕ is needed. This better understanding will hopefully be achieved when the existence theories for the equations we are proposing are discovered.

Before we get into applications of this theorem, it is worth noting that

$$E(\bar{g}, \phi) = \int_M \bar{R}\phi \, \overline{dV}$$

is, up to a boundary term, the Einstein-Hilbert action of the quotiented static spacetime

$$\left(S^1 \times M, -\phi^2 dt^2 + \bar{g}\right),$$

where we have turned the usual $\mathbf{R}$ time coordinate into an S^1 of length one to get a finite integral. The Einstein-Hilbert action is defined to be the total integral of the scalar curvature R^{ST} of the spacetime. In the appendix we observe that

$$R^{ST} = \bar{R} - 2\frac{\bar{\Delta}\phi}{\phi},$$

and since $dV^{ST} = \phi \, \overline{dV}$, the Einstein Hilbert action of the quotiented spacetime is

$$\begin{aligned}\int_{S^1 \times M} R^{ST} dV^{ST} &= \int_M (\bar{R}\phi - 2\bar{\Delta}\phi) \, \overline{dV} \\ &= \int_M \bar{R}\phi \, \overline{dV} - 2 \int_{\partial M} \langle \bar{\nabla}\phi, \bar{\nu} \rangle_{\bar{g}} \overline{dA}.\end{aligned}$$

We further observe that the boundary term vanishes when ϕ is harmonic on $(M, \bar{g})$, as is the case in the Schwarzschild spacetime.

Finally, the vacuum Einstein equation $G = 0$ is the Euler-Lagrange equation which results from requiring a spacetime to be a critical point of the Einstein-Hilbert action. Since the Minkowski and Schwarzschild spacetimes are the only vacuum static spacetimes (with no boundary or black hole boundary) [7], it follows that they are the only two static spacetimes which are critical points of the Einstein-Hilbert action, or equivalently $E(g, \phi)$, since boundary terms are irrelevant for variations away from the boundary.

10. The Jang-IMCF equations. In this section we show how inverse mean curvature flow in $(M^3, \bar{g})$ may be used to determine a warping factor ϕ for the generalized Jang equation to get a system of equations which, when there are solutions, implies the Penrose conjecture for a single black hole when $H_2(M^3) = 0$. We will call the system of equations we are proposing in this section the Jang-IMCF equations. An important open problem is to find an existence theory for these equations.

Before we state the Jang-IMCF equations, we need to review inverse mean curvature flow. As introduced by Geroch [13] and Jang-Wald [20], a smooth family of surfaces $\Sigma(t)$ in $(M^3, \bar{g})$ is said to satisfy inverse mean curvature flow if the speed in the outward normal direction of the family of surfaces as t increases at each point is equal to $1/\bar{H}$, where $\bar{H} > 0$ is the mean curvature of the surface at that point. This flow has the important and surprising property that the Hawking mass of $\Sigma(t)$ is nondecreasing when $(M^3, \bar{g})$ has nonnegative scalar curvature $\bar{R}$.

To be more precise, define the Hawking mass of a surface Σ in $(M^3, \bar{g})$ to be

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|_{\bar{g}}}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma \bar{H}^2 \overline{dA}\right),$$

where all quantities are computed in $(M^3, \bar{g})$. Then we can compute the rate of change of the Hawking mass of a surface when flowed out orthogonally with speed $\eta = 1/\bar{H}$ in $(M^3, \bar{g})$ by using the first variation formula

$$\frac{d}{dt}(\overline{dA}) = (\eta \bar{H})\overline{dA} = \overline{dA},$$

the second variation formula

$$\frac{d}{dt}\bar{H} = -\Delta\eta - \|\overline{\mathrm{II}}\|_{\bar{g}}^2 \eta - \overline{\mathrm{Ric}}(\bar{\nu}, \bar{\nu})\eta,$$

and the Gauss equation

$$\overline{\mathrm{Ric}}(\bar{\nu}, \bar{\nu}) = \frac{1}{2}\bar{R} - \bar{K} + \frac{1}{2}\bar{H}^2 - \frac{1}{2}\|\overline{\mathrm{II}}\|_{\bar{g}}^2,$$

where $\overline{\mathrm{II}}$ is the second fundamental form of Σ in $(M^3, \bar{g})$, $\overline{\mathrm{Ric}}$ is the Ricci curvature of $(M^3, \bar{g})$, and $\bar{K}$ is the Gauss curvature of Σ, to get

$$\frac{d}{dt}\left(m_H(\Sigma(t))\right) = \sqrt{\frac{|\Sigma(t)|_{\bar{g}}}{16\pi}} \left[\frac{1}{2} + \frac{1}{16\pi}\int_{\Sigma(t)} \frac{2|\overline{\nabla}\bar{H}|_{\bar{g}}^2}{\bar{H}^2} + \bar{R} - 2\bar{K} + \|\overline{\mathrm{II}}\|_{\bar{g}}^2 - \frac{1}{2}\bar{H}^2\right].$$

The essential assumption that $\Sigma(t)$ is connected is used to conclude that

$$\int_{\Sigma(t)} \bar{K}\overline{dA} = 2\pi\chi(\Sigma(t)) \leq 4\pi$$

by the Gauss-Bonnet formula, which, along with

$$\|\overline{\mathrm{II}}\|^2_{\bar{g}} \geq \frac{1}{2}\operatorname{tr}(\overline{\mathrm{II}})^2 = \frac{1}{2}\bar{H}^2$$

allows us to conclude that

$$\frac{d}{dt}\left(m_H(\Sigma(t))\right) \geq \sqrt{\frac{|\Sigma(t)|_{\bar{g}}}{16\pi}} \int_{\Sigma(t)} \frac{\bar{R}}{16\pi} \overline{dA}.$$

If $(M^3, \bar{g})$ has nonnegative scalar curvature, then the above equation implies that the Hawking mass of the smooth family of surfaces determined by inverse mean curvature flow is nondecreasing. However, we get a more general result if we integrate the above equation in t and use the co-area formula

$$\overline{dA}dt = \frac{1}{\eta}\overline{dV} = \bar{H}\overline{dV}$$

to conclude that for a smooth family of surfaces satisfying inverse mean curvature flow which foliates M^3, that

$$m_H(\Sigma(\infty)) - m_H(\Sigma(0)) = \int_M \bar{H}\sqrt{\frac{|\Sigma(t)|_{\bar{g}}}{16\pi}} \left(\frac{\bar{R}}{16\pi}\right)\overline{dV}$$

where at each point $x \in M^3$, $\bar{H}$ is the mean curvature of the surface $\Sigma(t)$ through the point x. Then assuming that $\Sigma(0)$ has $\bar{H} = 0$ and is area outerminimizing and that $(M^3, \bar{g})$ is sufficiently asymptotically flat, we conclude our main result that

$$\bar{m} - \sqrt{\frac{\bar{A}}{16\pi}} \geq \int_M Q(x)\left(\frac{\bar{R}(x)}{16\pi}\right)\overline{dV}(x) \tag{68}$$

where $\bar{A}$ is the area of $\Sigma = \Sigma(0)$, the area outerminimizing minimal boundary of M^3, and

$$Q(x) = \bar{H}\sqrt{\frac{|\Sigma(t)|_{\bar{g}}}{16\pi}}.$$

More generally, Huisken-Ilmanen [18] observed that there exists a weak notion of inverse mean curvature flow in which the surfaces $\Sigma(t)$ jump outward to their outermost minimal area enclosures whenever they are not already that surface. A key step in their approach is to represent the family of surfaces $\Sigma(t)$ as the levels sets of a real-valued function $u(x)$ on M^3 called the level set function. Then if

$$\Sigma(t) = \partial\{x \mid u(x) \leq t\},$$

it follows that $\eta = 1/|\overline{\nabla} u|_{\bar{g}}$ and

$$\bar{H} = \overline{\operatorname{div}}\left(\frac{\overline{\nabla} u}{|\overline{\nabla} u|_{\bar{g}}}\right),$$

so that inverse mean curvature flow on the level sets of $u(x)$ is equivalent to

$$\overline{\operatorname{div}}\left(\frac{\overline{\nabla} u}{|\overline{\nabla} u|_{\bar{g}}}\right) = |\overline{\nabla} u|_{\bar{g}}. \tag{69}$$

Huisken-Ilmanen then proceed to define a notion of weak solutions to the above level set equation using an energy minimization technique. These solutions have "jump regions" where $\overline{\nabla} u = 0$ corresponding to where the family of surfaces $\Sigma(t)$ is not continuously varying but instead "jumps" over these regions. Furthermore, Huisken-Ilmanen, using elliptic regularization, proved that weak solutions of their inverse mean curvature flow always exist. We refer the reader to their beautiful work [18]. However, using their generalized inverse mean curvature flow, we achieve the following theorem.

THEOREM 7. *Given an asymptotically flat* $(M^3, \bar{g})$ *with* $H_2(M^3) = 0$ *and a minimal connected boundary* Σ *which bounds an interior region, then*

$$\bar{m} - \sqrt{\frac{\bar{A}}{16\pi}} \geq \int_M Q(x) \left(\frac{\bar{R}(x)}{16\pi} \right) \overline{dV}(x) \tag{70}$$

where $\bar{A}$ *is the area of the outermost minimal area enclosure* $\tilde{\Sigma} = \partial U^3$ *of* Σ, $\bar{R}$ *is the scalar curvature and* $\overline{dV}$ *is the volume form of* $\bar{g}$, *and*

$$Q = |\overline{\nabla} u|_{\bar{g}} \sqrt{\frac{\bar{A} e^u}{16\pi}}, \tag{71}$$

where $u(x)$ *is a weak solution to Huisken-Ilmanen inverse mean curvature flow equalling zero on* Σ.

Proof. As described in Huisken-Ilmanen's paper, if Σ is not already its own outermost minimal area enclosure, it immediately jumps to it. During this initial jump, but only on this first jump, the area of the surface may decrease. Hence, $\bar{A}$ must be defined to be the area of the outermost minimal area enclosure $\tilde{\Sigma}$ of Σ, which also has zero mean curvature by the maximum principle using Σ as a barrier. Also, since $H_2(M^3) = 0$, it follows that $\tilde{\Sigma}$ is connected. Since each component of $\tilde{\Sigma}$ bounds a region, it follows that if $\tilde{\Sigma}$ did have more than one connected component, all of the components except for one could be removed (by either filling in holes or removing disconnected regions), thereby decreasing the area. Then starting the flow at $\tilde{\Sigma}$, our previous calculations generalize. The condition that $H_2(M^3) = 0$ is also used to guarantee that each $\Sigma(t)$ is connected after each jump and therefore has Euler characteristic ≤ 2 as needed in the computation of the rate of change of the Hawking masses of $\Sigma(t)$.

By the first variation formula mentioned earlier in this section, inverse mean curvature flow grows the area form exponentially. Hence,

$$|\Sigma(t)|_{\bar{g}} = \bar{A} e^t.$$

Thus, we have that

$$Q(x) = \overline{\text{div}} \left(\frac{\overline{\nabla} u}{|\overline{\nabla} u|_{\bar{g}}} \right) \sqrt{\frac{\bar{A} e^u}{16\pi}},$$

which equals the desired result by equation 69.

Theorem 7 deserves careful consideration. In the case that $\bar{R} \geq 0$, we recover a Riemannian Penrose inequality for a single black hole. More generally, however, since $\bar{R}/16\pi$ is energy density, we see that we have a kind of integral of energy density on the right hand side of equation 70, modified by the factor $Q(x)$. On the flat metric on $\mathbf{R}^3$ and starting inverse mean curvature flow at a point, $Q(x) = 1$, and on the Schwarzschild metric, $Q(x)$ is the harmonic function going to one at infinity and equal to zero on the minimal neck. This last fact, which can be verified by direct calculation, will turn out to be important since this harmonic function also equals the warping factor $\phi(x)$ in the Schwarzschild metric.

Theorem 7 and theorem 6 together motivate the system of equations,

$$0 = \mathrm{tr}_{\bar{g}}(h-k) \tag{72}$$

$$|\overline{\nabla} u|_{\bar{g}} = \overline{\mathrm{div}} \left(\frac{\overline{\nabla} u}{|\overline{\nabla} u|_{\bar{g}}} \right) \tag{73}$$

$$Q = |\overline{\nabla} u|_{\bar{g}} \sqrt{\frac{\bar{A} e^u}{16\pi}} \tag{74}$$

$$c\,\phi = Q, \tag{75}$$

where for our later convenience we choose $c = \sqrt{\frac{\bar{A}}{16\pi}}$. The first equation is the generalized Jang equation again. The second equation is the level set formulation of inverse mean curvature flow on $(M^3, \bar{g})$. The third equation is the definition of $Q(x)$ in terms of the inverse mean curvature flow level set function $u(x)$. The new equation, then, is the fourth equation, which sets $\phi(x)$ equal to $Q(x)$, up to a constant. Then by theorems 7 and 6, we conclude that

$$\bar{m} - \sqrt{\frac{\bar{A}}{16\pi}} \geq \int_M Q(x) \left(\frac{\bar{R}(x)}{16\pi} \right) \overline{dV}(x)$$
$$= \frac{c}{16\pi} \int_M \bar{R} \phi \overline{dV} \quad \geq \quad 0.$$

Then recalling that $\bar{g}$ measures areas at least as large as g does, we have that

$$\bar{A} = |\tilde{\Sigma}_{\bar{g}}|_{\bar{g}} \geq |\tilde{\Sigma}_{\bar{g}}|_g \geq |\tilde{\Sigma}_g|_g = A,$$

where again $\tilde{\Sigma}_{\bar{g}}$ is the outermost minimal area enclosure of Σ in $(M^3, \bar{g})$ and $\tilde{\Sigma}_g$ is the outermost minimal area enclosure of Σ in (M^3, g). Recall also that since $H_2(M^3) = 0$, Σ connected (and bounding a region) implies that both $\tilde{\Sigma}_{\bar{g}}$ and $\tilde{\Sigma}_g$ are also connected (and bound a region). Hence, if we can solve the above system with boundary conditions so that Σ has zero mean curvature in $(M^3, \bar{g})$ and so that the total masses of (M^3, g) and $(M^3, \bar{g})$ are the same, then we would be able to conclude that

$$m = \bar{m} \geq \sqrt{\frac{\bar{A}}{16\pi}} \geq \sqrt{\frac{A}{16\pi}},$$

which would prove the Penrose conjecture for a single black hole in the case that $H_2(M^3) = 0$.

In the case of equality in the above inequalities, $(M^3, \bar{g})$ has to be a time symmetric slice of the Schwarzschild spacetime by the original Huisken-Ilmanen result. Thus, inverse mean curvature flow yields precisely the spherically symmetric spheres of Schwarzschild, so u is easy to compute. Direct computation then reveals that $Q(x)$ is the harmonic function in $(M^3, \bar{g})$ which equals zero on $\Sigma(0)$ and goes to one at infinity. Since ϕ equals Q (up to a multiplicative constant, which is irrelevant after a constant rescaling of the time coordinate in what follows), we get that $\left(\mathbf{R} \times M^3, -\phi^2 dt^2 + \bar{g}\right)$ is isometric to a Schwarzschild spacetime. Hence, $g = \bar{g} - \phi^2 df^2$ is the induced metric on a slice of Schwarzschild with graph function $f(x)$. Finally, examining the case of equality of theorem 6 (and that theorem's use of the generalized Schoen-Yau identity) forces $\|h-k\|_{\bar{g}}^2 = 0$, which of course implies that $k_{ij} = h_{ij}$. Hence, the original Cauchy data (M^3, g, k) is the induced Cauchy data on a slice of a Schwarzschild spacetime with graph function $f(x)$.

Thus, understanding this system of equations, and whatever existence theory might be associated with it, is a very interesting and important open problem. A first step is to observe that Q does not need to be defined in the system. Hence, our system is equivalent to

The Jang-inverse mean curvature flow equations

$$0 = \mathrm{tr}_{\bar{g}}(h-k) \tag{76}$$

$$|\overline{\nabla} u|_{\bar{g}} = \overline{\mathrm{div}}\left(\frac{\overline{\nabla} u}{|\overline{\nabla} u|_{\bar{g}}}\right) \tag{77}$$

$$\phi = |\overline{\nabla} u|_{\bar{g}}\, e^{u/2}, \tag{78}$$

where we recall that

$$\bar{g} = g + \phi^2 df^2$$

and

$$h = \frac{\phi \mathrm{Hess} f + (df \otimes d\phi + d\phi \otimes df)}{(1+\phi^2 |df|_g^2)^{1/2}}\ ,$$

which can be thought of as three equations and three free functions f, u, and ϕ on the original Cauchy data (M^3, g, k).

In fact, the third equation, equation 78, can be used to solve for ϕ in terms of u, du, and df. The purpose of this is to recognize that the Jang-IMCF equations may also be thought of as two equations and two free functions f and u once we substitute for ϕ. Since only first derivatives of f and u appear in the expression for ϕ below, the resulting equivalent system is two second order equations in f and ϕ.

Unfortunately, the expression for ϕ in terms of f and u on (M^3, g) is a bit messy, but at least it is explicit. From equation 78, we get

$$\phi = |du|_{\bar{g}}\, e^{u/2} \tag{79}$$

which is simple enough except that ϕ also appears in the expression for $\bar{g}$. Next we note that

$$\begin{aligned} |du|_{\bar{g}}^2 &= \bar{g}^{ij} u_i u_j \\ &= \left(g^{ij} - \frac{\phi^2 f^i f^j}{1+\phi^2 |df|_g^2} \right) u_i u_j \\ &= |du|_g^2 - \frac{\phi^2 \langle df, du\rangle_g^2}{1+\phi^2|df|_g^2}, \end{aligned}$$

which, when combined with equation 79 gives us

$$\phi^2 = e^u \left(|du|_g^2 - \frac{\phi^2 \langle df, du\rangle_g^2}{1+\phi^2|df|_g^2} \right).$$

It follows that ϕ^2 solves the quadratic equation,

$$|df|_g^2 \cdot \phi^4 + B \cdot \phi^2 - e^u |du|_g^2 = 0,$$

where $B = 1 + e^u \left(\langle df, du \rangle_g^2 - |df|_g^2 |du|_g^2 \right)$. Thus,

$$\phi^2 = \frac{-B + \sqrt{B^2 + 4e^u |df|_g^2 |du|_g^2}}{2|df|_g^2}, \tag{80}$$

which is clearly always nonnegative (and where we disregard the negative square root in the quadratic formula since that solution is nonpositive). Thus, an equivalent formulation of the Jang-IMCF equations is

$$0 = \mathrm{tr}_{\bar{g}}(h - k) \tag{81}$$

$$|\overline{\nabla} u|_{\bar{g}} = \overline{\mathrm{div}} \left(\frac{\overline{\nabla} u}{|\overline{\nabla} u|_{\bar{g}}} \right) \tag{82}$$

where

$$\bar{g} = g + \phi^2 df^2, \tag{83}$$

$$h = \frac{\phi \mathrm{Hess} f + (df \otimes d\phi + d\phi \otimes df)}{(1 + \phi^2 |df|_g^2)^{1/2}}, \tag{84}$$

$$\phi = \frac{\sqrt{-\frac{B}{2} + \sqrt{\frac{B^2}{4} + e^u |df|_g^2 |du|_g^2}}}{|df|_g}, \tag{85}$$

and

$$B = 1 + e^u \left(\langle df, du \rangle_g^2 - |df|_g^2 |du|_g^2 \right), \tag{86}$$

which can be thought of as two equations and two free functions f and u on the original Cauchy data (M^3, g, k). (In equation 85, $\phi = |du|_g e^{u/2}$ when $|df|_g = 0$ by equation 79).

We end this section with a general discussion of some of the challenges involved in finding an existence theory for the Jang-IMCF equations. First, note that these equations reduce to the Huisken-Ilmanen IMCF equation on (M^3, g) when $\mathrm{tr}_g(k) = 0$ since then we can choose $f = 0$ (which implies $\bar{g} = g$) to satisfy the generalized Jang equation (equation 81). Thus, clearly a notion of a weak solution to this system of equations is required. Furthermore, the notion of "jumps" must still be involved when there are regions in which $du = 0$. Note that when $du = 0$, then $\phi = 0$. Thus, if f stays smooth and bounded, it would follow that $h = 0$, which means that the generalized Jang equation cannot be solved unless $\mathrm{tr}_g(k) = 0$ in this region as well. If $\mathrm{tr}_g(k) \neq 0$ in this region, then this would suggest that f needs to be unbounded or undefined in this region. Clearly this is an important issue to understand.

Given these and other considerations, one might be tempted to be pessimistic about finding a general existence theory for the Jang-IMCF equations. In fact, it was once the case that most were pessimistic about the original inverse mean curvature flow proposed by Geroch [13], right up until Huisken-Ilmanen [18] found an amazingly beautiful and natural existence theory for a generalized version of inverse mean curvature flow. Thus, there is also precedent for optimism.

11. The Jang-CFM equations. In this section we comment that there is at least one other Einstein-Hilbert action method in addition to the Jang-IMCF equations. So far we have seen how the Penrose conjecture would follow from a general existence theory for the Jang-Zero Divergence equations presented in section 8 or, for a single black hole in dimension three, from a general existence theory for the Jang-IMCF equations presented in section 10. In this section, we briefly discuss a third system of equations whose existence theory would also imply the Penrose conjecture. The precise statement of this third system is a bit laborious and so we do not state it here, but only describe it and the additional considerations it involves.

In section 9, we explained how any inequality of the form

$$\bar{m} - \sqrt{\frac{\bar{A}}{16\pi}} \geq \int_M Q(x)\bar{R}(x)\overline{dV}, \tag{87}$$

for some $Q(x) \geq 0$, leads to a system of equations which implies the Penrose conjecture. The first equation in the system is the generalized Jang equation and the second equation in the system is simply $\phi(x) = Q(x)$ (times a constant if one likes). In section 10 we pursued this approach in detail for the Huisken-Ilmanen inverse mean curvature flow.

Bray's proof [3] of the Riemannian Penrose inequality, when revisited, also yields an inequality of the form of equation 87. This proof of the Riemannian Penrose inequality involves a conformal flow of metrics (CFM) which flows an initial asymptotically flat metric with nonnegative scalar curvature to a Schwarzschild metric in the limit as the flow parameter goes to infinity. Furthermore, the area of the horizon stays constant, and (by the Riemannian positive mass theorem it turns out that) the total mass is nonincreasing during the flow.

To generalize the conformal flow of metrics (CFM) proof to get an inequality as in equation 87, we first need to generalize the positive mass theorem to get an inequality of the form

$$\tilde{m} \geq \int_M Q(x)\tilde{R}(x)d\tilde{V}, \tag{88}$$

for some $Q(x) \geq 0$ on some $(M, \tilde{g})$. Witten's spinor proof [37] of the Riemannian positive mass theorem provides such a result, for example, at least whenever a spinor solution to the Dirac equation exists (since we are not assuming $\tilde{R} \geq 0$ anymore, there is an issue now). Also, a result of this type can be found by multiplying the metric $\tilde{g}$ by a conformal factor to achieve zero scalar curvature globally (when such a factor exists), and then measuring how much the mass changes. This last idea is made precise by Jauregui in [21]. Finally, one can also use inverse mean curvature flow starting from any point to prove an inequality of the above form in equation 88. This third approach currently has the advantage over the first two in that it is known to work in all cases in dimension three by the previous section and the work of Huisken and Ilmanen [18] and Streets [35].

In the conformal flow of metrics (M, g_t) with total masses $m(t)$,

$$m'(t) = -\frac{1}{2}\tilde{m}(t). \tag{89}$$

Hence,

$$m - \sqrt{\frac{A}{16\pi}} \geq \int_0^\infty \frac{1}{2}\tilde{m}(t)dt \tag{90}$$

since the areas $A(t)$ of the horizons of (M, g_t) stay constant and the flow of metrics converges to Schwarzschild where $m - \sqrt{\frac{A}{16\pi}} = 0$. Then plugging equation 88 into the above equation and accounting how the scalar curvature transforms conformally gives a result of the desired form in equation 87. Hence, modulo the existence questions needed to get an equality of the form of equation 88, we get a generalization of the Riemannian Penrose inequality.

One difference between the Jang-CFM equations and the Jang-IMCF equations, however, is that the Jang-CFM equations are not local. That is, $Q(x)$ in this case does not satisfy a local p.d.e. at each point and instead has a more complicated expression. Hence, for the Jang-CFM equations to have an existence theory, the theory would have to work for a wide range of possible Q. On the other hand, the $Q(x)$ from the Jang-CFM equations has the potential to have better regularity than the $Q(x)$ from the Jang-IMCF equations. For the Jang-IMCF equations, $c\phi = Q$ is not necessarily continuous or even positive, and in fact equals zero in jump regions of the inverse mean curvature flow on $(M^3, \bar{g})$, which as discussed at the end of the previous section, introduces additional analytical challenges.

12. Open problems. The two most interesting and important open problems discussed in this paper are finding an existence theory for the Jang-Zero Divergence Equations (which would prove the Penrose conjecture) and finding an existence theory for the Jang-IMCF equations (which would prove the Penrose conjecture for a single black hole when $H_2(M^3) = 0$). Another interesting problem is to find a general existence theory for any Einstein-Hilbert action method as long as the associated $Q(x)$ has certain properties. If the $Q(x)$ from the Jang-CFM equations qualified for such a theory, this would also prove the Penrose conjecture.

Another interesting problem is to find additional Einstein-Hilbert action methods by finding new inequalities of the form of equation 87. Since the special case of $\bar{R} = 0$ implies the Riemannian Penrose inequality, one would either have to find a new proof of the Riemannian Penrose inequality or use the Riemannian Penrose inequality itself to find such a generalization. There may be reasonable ideas to try in this latter approach.

There is also the question of the physical interpretation of inequalities of the form of equation 87. Rewriting the inequality gives us

$$\bar{m} \geq \sqrt{\frac{\bar{A}}{16\pi}} + \int_M Q(x)\bar{R}(x)\overline{dV}, \tag{91}$$

which could be interpreted as saying that the total mass of a time-symmetric slice of a spacetime (not necessarily with nonnegative energy density) is at least equal to the mass contributed by the black holes (the first term) plus a weighted integral of the energy density (the second term), since energy density at each point can be interpreted as $\bar{\mu} = \bar{R}/16\pi$. The purpose of Q can be interpreted as the need to account for potential energy. Also, Q should go to zero (and does in the IMCF and CFM cases) at and inside the horizons of the black holes since matter inside black holes should not affect the total mass.

We also believe that the generalized Schoen-Yau identity and the generalized Jang equation have much potential for many possible applications in the study of general relativity. One point of view, for example, is that the Jang-Zero Divergence equations give a canonical way of embedding Cauchy data (M^3, g, k) into a static spacetime. If one is interested in understanding how the initial Cauchy data evolves under the vacuum Einstein equations, or some other equation coupled with the Einstein equation, then one could compute how the canonical static metrics associated with the evolving Cauchy data slices evolve. One nice property of this approach is that if the initial Cauchy data is a slice of the Schwarzschild spacetime, and we are solving the vacuum Einstein equations for example, then while the Cauchy data is evolving in what might appear to be complicated ways, the associated canonical static spacetime remains the Schwarzschild spacetime. Also, since the generalized Jang equation blows up on horizons, this method could only be used to study the exterior region of spacetimes outside the apparent horizons of black holes. There may be some advantages to this restriction if this becomes a natural way to avoid spacetime singularities.

Appendix A. Introduction to the appendices. The target audience of these appendices are graduate students and other researchers who are interested in entering geometric relativity as a field to study. As such, we have included more detail in these calculations than is typical. We justify this choice in part with the fact that there are so many computations, many people would have a hard time duplicating all of these computations in a reasonable amount of time, even with well chosen hints. We also hope that these appendices will be useful to students and researchers who are interested in practicing their computational skills. We recommend the book "Semi-Riemannian Geometry with Applications to Relativity" by Barrett O'Neill as an excellent introduction to the differential geometry of general relativity, and we mostly follow that book's notation here. Readers should go through the calculations in these appendices in order since notational conventions which are established in one appendix apply to the appendices which follow as well.

The authors would like to thank Alan Parry for helping with the TeXing and Jeff Jauregui for helpful comments improving the readability of these appendices.

Appendix B. Curvature of static spacetimes. In this section we compute the Einstein curvature, Ricci curvature, and scalar curvature of the general static spacetime metric

$$\tilde{g} = -\phi(x)^2 dt^2 + \bar{g}$$

on $\mathbf{R} \times M^3$, where $t \in \mathbf{R}$, $x \in M^3$ and $\bar{g}$ is a positive definite metric on M^3.

First, choose a coordinate chart on $(M^3, \bar{g})$ with coordinates (x^1, x^2, x^3) and let $x^0 = t$. Then $\{\bar{\partial}_\alpha = \frac{\partial}{\partial x^\alpha}\}_{\alpha=0}^3$ is a basis of the tangent plane at each point of the spacetime. Let $\{\bar{\omega}^\alpha\}_{\alpha=0}^3$ be the corresponding dual basis of one forms at each point of the spacetime so that $\bar{\omega}^\alpha(\bar{\partial}_\beta) = \delta_\beta^\alpha$. (We use bars over these bases instead of tildes to be consistent with section 6 and subsequent appendices). Finally, we define the components of $\tilde{g}$ (which are the same as the components of $\bar{g}$ for tangent vectors to M^3) to be

$$\tilde{g}_{\alpha\beta} = \langle \bar{\partial}_\alpha, \bar{\partial}_\beta \rangle_{\tilde{g}} \quad (= \tilde{g}(\bar{\partial}_\alpha, \bar{\partial}_\beta) \text{ by convention})$$

so that $\tilde{g}_{00} = -\phi(x)^2$, $\tilde{g}_{0i} = \tilde{g}_{i0} = 0$, and $\tilde{g}_{ij} = \langle \bar{\partial}_i, \bar{\partial}_j \rangle_{\bar{g}}$ for $1 \le i, j \le 3$.

(Notation: We adopt the convention that Greek indices always range from 0 to 3 and Latin indices always range from 1 to 3. Also, we adopt Einstein's convention that any time an index is both an upper and lower index in an expression, summation over that index is implied.)

(Recall also that $\tilde{g}^{\alpha\beta}$ are the components of the inverse matrix of $\tilde{g}$ expressed as a matrix at each point of the coordinate chart, and that indices of a tensor may be raised or lowered by contracting with $\tilde{g}^{\alpha\beta}$ or $\tilde{g}_{\alpha\beta}$, respectively [27]).

By the Koszul formula [27], the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ can be expressed in terms of its components as

$$\tilde{\nabla}_{\bar{\partial}_\alpha}\bar{\partial}_\beta = \tilde{\Gamma}_{\alpha\beta}^{\ \ \gamma}\bar{\partial}_\gamma,$$

where

$$\tilde{\Gamma}_{\alpha\beta}^{\ \ \gamma} = \frac{1}{2}\tilde{g}^{\gamma\theta}\left(\tilde{g}_{\alpha\theta,\beta} + \tilde{g}_{\beta\theta,\alpha} - \tilde{g}_{\alpha\beta,\theta}\right) \tag{92}$$

are called the Christoffel symbols of $\tilde{\nabla}$.

(Notation: Commas denote differentiation with respect to the coordinate chart so that $\tilde{g}_{\alpha\beta,\theta} = \frac{\partial \tilde{g}_{\alpha\beta}}{\partial x^\theta}$).

Plugging in our expressions for $\tilde{g}_{\alpha\beta}$, short calculations reveal that

$$0 = \tilde{\Gamma}_{00}^{\ \ 0} = \tilde{\Gamma}_{0i}^{\ \ j} = \tilde{\Gamma}_{i0}^{\ \ j} = \tilde{\Gamma}_{ij}^{\ \ 0},$$

$$\tilde{\Gamma}_{0i}^{\ \ 0} = \tilde{\Gamma}_{i0}^{\ \ 0} = \frac{\phi_i}{\phi}, \qquad \text{and that} \qquad \tilde{\Gamma}_{00}^{\ \ i} = \phi \cdot \phi^{\bar{\imath}},$$

where $\phi_i = \phi_{,i} = \frac{\partial\phi(x)}{\partial x^i}$ and, as previously stated, $\phi^{\bar{\imath}} = \bar{g}^{ij}\phi_j = \tilde{g}^{ij}\phi_j = \tilde{g}^{i\alpha}\phi_\alpha = \phi^{\tilde{\imath}}$ since ϕ is only a function of x and does not depend on t. Note that we are using the convention that a raised index with a bar or tilde over it denotes raising the index with $\bar{g}$ or $\tilde{g}$, respectively.

Since the Lie bracket of coordinate vector fields is zero, it follows from the definition of the Riemann curvature tensor that

$$R_{ijk}^{\ \ \ l} = \bar{\omega}^l(\tilde{\nabla}_{\bar{\partial}_i}\tilde{\nabla}_{\bar{\partial}_j}\bar{\partial}_k - \tilde{\nabla}_{\bar{\partial}_j}\tilde{\nabla}_{\bar{\partial}_i}\bar{\partial}_k).$$

Hence,

$$\begin{aligned} R_{ijk}^{\ \ \ l} &= \bar{\omega}^l\left(\tilde{\nabla}_{\bar{\partial}_i}\left(\tilde{\Gamma}_{jk}^{\ \ \alpha}\bar{\partial}_\alpha\right) - \tilde{\nabla}_{\bar{\partial}_j}\left(\tilde{\Gamma}_{ik}^{\ \ \alpha}\bar{\partial}_\alpha\right)\right) \\ &= \bar{\omega}^l\left(\left(\tilde{\Gamma}_{jk}^{\ \ \alpha}\right)_{,i}\bar{\partial}_\alpha + \tilde{\Gamma}_{jk}^{\ \ \alpha}\tilde{\Gamma}_{i\alpha}^{\ \ \beta}\bar{\partial}_\beta - \left(\tilde{\Gamma}_{ik}^{\ \ \alpha}\right)_{,j}\bar{\partial}_\alpha - \tilde{\Gamma}_{ik}^{\ \ \alpha}\tilde{\Gamma}_{j\alpha}^{\ \ \beta}\bar{\partial}_\beta\right) \\ &= \left(\tilde{\Gamma}_{jk}^{\ \ l}\right)_{,i} - \left(\tilde{\Gamma}_{ik}^{\ \ l}\right)_{,j} + \sum_\alpha\left(\tilde{\Gamma}_{jk}^{\ \ \alpha}\tilde{\Gamma}_{i\alpha}^{\ \ l} - \tilde{\Gamma}_{ik}^{\ \ \alpha}\tilde{\Gamma}_{j\alpha}^{\ \ l}\right) \end{aligned}$$

For the beginner, we note that about half of the text books define the Riemann curvature tensor to be the negative of what we used above. However, all texts eventually end up with the same definition of the Ricci curvature (defined in a moment) which is agreed to be a positive multiple of the metric on the standard sphere.

Plugging in our formulas for the Christoffel symbols, we thus compute that

$$\begin{aligned} \tilde{R}_{0jk}^{\ \ \ 0} &= \left(\tilde{\Gamma}_{jk}^{\ \ 0}\right)_{,0} - \left(\tilde{\Gamma}_{0k}^{\ \ 0}\right)_{,j} + \sum_\alpha\left(\tilde{\Gamma}_{jk}^{\ \ \alpha}\tilde{\Gamma}_{0\alpha}^{\ \ 0} - \tilde{\Gamma}_{0k}^{\ \ \alpha}\tilde{\Gamma}_{j\alpha}^{\ \ 0}\right) \\ &= \left(-\frac{\phi_k}{\phi}\right)_{,j} + \sum_m \tilde{\Gamma}_{jk}^{\ \ m}\cdot\frac{\phi_m}{\phi} - \frac{\phi_k}{\phi}\cdot\frac{\phi_j}{\phi} \\ &= -\frac{\overline{\text{Hess}}_{jk}\phi}{\phi} \end{aligned}$$

from which it follow that

$$\tilde{R}_{j00}{}^{j} = \phi\overline{\Delta}\phi$$

where $\overline{\text{Hess}}$ is the Hessian and $\overline{\Delta}$ is the Laplacian on $(M^3, \bar{g})$. The second computation follows from the first by first lowering the raised 0 index (introducing a factor of $-\phi^2$), using the antisymmetry of the Riemann curvature tensor to switch indices, and then taking the trace of the Hessian. We remind the reader that the Latin letters j, k, m range from 1 to 3. The beginning student should review the definition of the Hessian, the Laplacian, and the use of normal coordinates. In this case, we note that we may choose normal coordinates on M^3 such that $\tilde{\Gamma}_{jk}{}^{m} = 0$ at a single point. Similarly, it is straightforward to verify that

$$\tilde{R}_{0j0}{}^{0} = 0 = \tilde{R}_{kj0}{}^{k}.$$

It turns out that the above components of the Riemann curvature tensor are all that we need to compute the Ricci curvature of the spacetime. For example,

$$\tilde{\text{Ric}}_{jk} = \tilde{R}_{\alpha jk}{}^{\alpha} = \tilde{R}_{ajk}{}^{a} + \tilde{R}_{0jk}{}^{0} = \bar{R}_{ajk}{}^{a} + \tilde{R}_{0jk}{}^{0} = \overline{\text{Ric}}_{jk} + \tilde{R}_{0jk}{}^{0}.$$

The first equality is the definition of the Ricci curvature as the trace of the Riemann curvature tensor. For the second equality recall our summation convention for Latin and Greek indices stated above. The third equality is a consequence of the Gauss equation for submanifolds since the $t = 0$ slice of our spacetime has zero second fundamental form by symmetry. The fourth equality is simply the definition of the Ricci curvature $\overline{\text{Ric}}$ of $(M^3, \bar{g})$. Also,

$$\tilde{\text{Ric}}_{00} = \tilde{R}_{\alpha 00}{}^{\alpha} = \tilde{R}_{000}{}^{0} + R_{j00}{}^{j} = \tilde{R}_{j00}{}^{j}$$

by antisymmetry of the Riemann curvature tensor. Finally,

$$\tilde{\text{Ric}}_{j0} = \tilde{R}_{\alpha j0}{}^{\alpha} = \tilde{R}_{0j0}{}^{0} + \tilde{R}_{kj0}{}^{k} = 0.$$

Thus, putting it all together, we have formulas for the components of the Ricci curvature of the static spacetime metric $\tilde{g} = -\phi(x)^2 dt^2 + \bar{g}$ on $\mathbf{R} \times M^3$, namely

$$\begin{aligned} \tilde{\text{Ric}}_{00} &= \phi\overline{\Delta}\phi \\ \tilde{\text{Ric}}_{jk} &= \overline{\text{Ric}}_{jk} - \frac{\overline{\text{Hess}}_{jk}\phi}{\phi} \\ \tilde{\text{Ric}}_{j0} &= \tilde{\text{Ric}}_{0j} = 0 \end{aligned}$$

in terms of the Ricci curvature $\overline{\text{Ric}}$ of $(M^3, \bar{g})$ and the Hessian and Laplacian of ϕ on $(M^3, \bar{g})$.

Next we can compute the scalar curvature of the spacetime by taking the trace of the Ricci curvature,

$$\tilde{R} = \tilde{g}^{jk}\tilde{\text{Ric}}_{jk} = \bar{R} - 2\frac{\overline{\Delta}\phi}{\phi}.$$

Hence, since $\tilde{G} = \tilde{\mathrm{Ric}} - \frac{1}{2}\tilde{R}\tilde{g}$, the components of the Einstein curvature tensor of the static spacetime metric are

$$\begin{aligned}
\tilde{G}_{00} &= \frac{1}{2}\bar{R}\phi^2 \\
\tilde{G}_{jk} &= \overline{\mathrm{Ric}}_{jk} - \frac{\overline{\mathrm{Hess}}_{jk}\phi}{\phi} + \left(\frac{\overline{\Delta}\phi}{\phi} - \frac{\bar{R}}{2}\right)\bar{g} \\
\tilde{G}_{j0} &= \tilde{\mathrm{Ric}}_{0j} = 0
\end{aligned}$$

as desired.

Appendix C. The second fundamental form of the graph. In this section we will compute the second fundamental form of a space-like slice of the static spacetime $\left(\mathbf{R} \times M^3, \tilde{g}\right)$, where

$$\tilde{g} = -\phi^2 dt^2 + \bar{g}, \tag{93}$$

ϕ is a real-valued function on M, and $\bar{g}$ is a Riemannian metric on M. Given a real-valued function f on M, define the graph map

$$F : M \mapsto \mathbf{R} \times M \tag{94}$$

where $F(x) = (f(x), x)$.

As we established in section 6, we will abuse terminology slightly and always refer to the image of the graph map $F(M)$ simply as M and the $t = 0$ slice of the constructed spacetime as $\bar{M}$. This notation is convenient since then (M, g) and $(\bar{M}, \bar{g})$ are space-like hypersurfaces of the spacetime $\left(\mathbf{R} \times M^3, -\phi^2 dt^2 + \bar{g}\right)$ (given appropriate bounds on the gradient of f). Let $\pi : M \mapsto \bar{M}$ be the projection map $\pi(f(x), x) = (0, x)$ to the $t = 0$ slice of the spacetime.

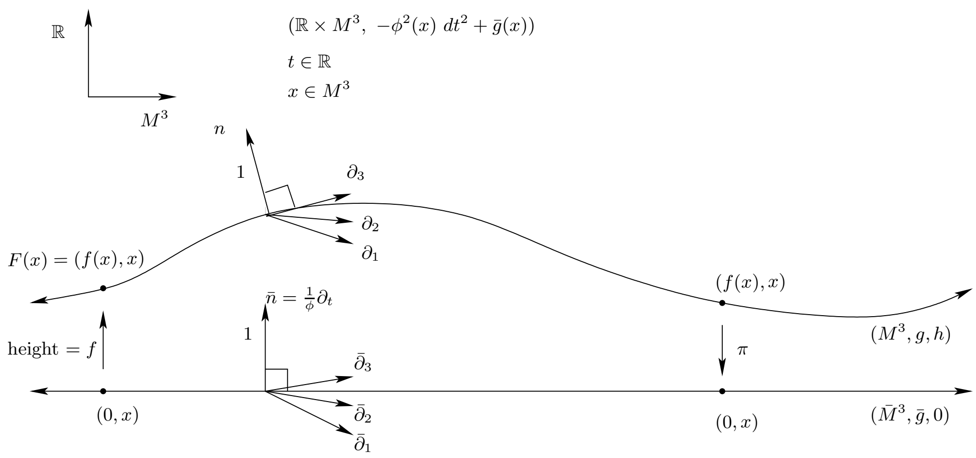

FIG. 2. *Schematic diagram of the constructed static spacetime*

We repeat some notation and definitions from section 6 for clarity. Let $\bar{\partial}_0 = \partial_t$ and $\{\bar{\partial}_i\}$ be coordinate vectors tangent to $\bar{M}$. Define

$$\partial_i = \bar{\partial}_i + f_i \bar{\partial}_0 \tag{95}$$

to be the corresponding coordinate vectors tangent to M so that $\pi_*(\partial_i) = \bar{\partial}_i$. Then in this coordinate chart, the components of the metrics g and $\bar{g}$ induced from the spacetime are

$$g_{ij} = \langle \partial_i, \partial_j \rangle \quad \text{and} \quad \bar{g}_{ij} = \langle \bar{\partial}_i, \bar{\partial}_j \rangle$$

where the angle brackets refer to the spacetime metric. Then it follows immediately that

$$g_{ij} = \bar{g}_{ij} - \phi^2 f_i f_j. \tag{96}$$

We comment here that the reader should think of g, ϕ, and f as the variables that we get to choose which determine $\bar{g}$. The metric g comes from the initial Cauchy data (M, g, k) and ϕ and f are functions which will satisfy a system of equations of our choosing.

The inverse of $\{g_{ij}\}$ turns out to be

$$g^{ij} = \bar{g}^{ij} + v^i v^j, \tag{97}$$

where

$$v^i = \frac{\phi f^{\bar{i}}}{(1 - \phi^2 |df|_{\bar{g}}^2)^{1/2}} = \frac{\phi f^i}{(1 + \phi^2 |df|_g^2)^{1/2}}. \tag{98}$$

The above computation is most easily verified at each point in normal coordinates of $\bar{g}$ at that point, where the gradient of f is assumed to lie in the first coordinate direction. The second part of equation 98 can be computed in the same manner, but where we consider that $\bar{g}_{ij} = g_{ij} + \phi^2 f_i f_j$ and then use normal coordinates as before, but this time for the metric g. (For the beginner, the use of normal coordinates is exemplified in more detail in a moment.)

We also define

$$\bar{v} = v^i \bar{\partial}_i \qquad \text{and} \qquad v = v^i \partial_i \tag{99}$$

so that $\pi_*(v) = \bar{v}$, and observe the useful identity

$$(1 - \phi^2 |df|_{\bar{g}}^2) \cdot (1 + \phi^2 |df|_g^2) = 1, \tag{100}$$

which follows directly from computing the ratio of the volume forms of g and $\bar{g}$ two different ways, namely with respect to g and then $\bar{g}$.

As established in section 6, we use the convention that a barred index (as in $f^{\bar{i}}$ above) denotes an index raised (or lowered) by $\bar{g}$ as opposed to g. That is, $f^{\bar{i}} = \bar{g}^{ij} f_j$, where as usual $f_j = \partial f / \partial x^j$ in the coordinate chart. In general, barred quantities will be associated with the $t = 0$ slice $(\bar{M}, \bar{g})$ and unbarred quantities will be associated with the graph slice (M, g).

Our next step is to compute the unit normal vector n to the graph slice (M, g) defined by f. It is straightforward to verify that

$$n = \frac{\bar{\partial}_0 + \phi^2 f^{\bar{k}} \bar{\partial}_k}{\phi \left(1 - \phi^2 \, |df|_{\bar{g}}^2\right)^{1/2}} \tag{101}$$

has unit length in the spacetime metric, is perpendicular to the tangent vectors $\partial_i = \bar{\partial}_i + f_i \bar{\partial}_0$ to the graph slice, and hence must be the correct expression.

Following our convention for the definition of the second fundamental form defined in equation 3, we thus have that the components of the second fundamental form h are

$$\begin{aligned} h_{ij} &= h(\partial_i, \partial_j) \\ &= -\langle \tilde{\nabla}_{\partial_i}\partial_j, n\rangle \\ &= \langle \tilde{\nabla}_{\partial_i} n, \partial_j\rangle \\ &= \left\langle \tilde{\nabla}_{(\bar{\partial}_i + f_i\bar{\partial}_0)} \left[\frac{\bar{\partial}_0 + \phi^2 f^{\bar{k}}\bar{\partial}_k}{\phi\left(1-\phi^2\,|df|^2_{\bar{g}}\right)^{1/2}} \right],\ \bar{\partial}_j + f_j\bar{\partial}_0 \right\rangle \\ &= \frac{\left\langle \tilde{\nabla}_{(\bar{\partial}_i + f_i\bar{\partial}_0)} \left[\bar{\partial}_0 + \phi^2 f^{\bar{k}}\bar{\partial}_k \right],\ \bar{\partial}_j + f_j\bar{\partial}_0 \right\rangle}{\phi\left(1-\phi^2\,|df|^2_{\bar{g}}\right)^{1/2}} \end{aligned}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on our spacetime $\left(\mathbf{R}\times M^3, \tilde{g}\right)$. The third and fifth equalities follow from the fact that $\langle n, \partial_j\rangle = 0$ on M.

¿From the form of the above expression, we see that the Christoffel symbols of the spacetime, defined and computed in appendix B, are going to come into play. From those computations, it follows that

$$\begin{aligned} \tilde{\nabla}_{\bar{\partial}_0}\bar{\partial}_0 &= \tilde{\Gamma}_{00}{}^{0}\bar{\partial}_0 + \tilde{\Gamma}_{00}{}^{k}\bar{\partial}_k = \phi\phi^{\bar{k}}\bar{\partial}_k \\ \tilde{\nabla}_{\bar{\partial}_0}\bar{\partial}_i &= \tilde{\Gamma}_{0i}{}^{0}\bar{\partial}_0 + \tilde{\Gamma}_{0i}{}^{k}\bar{\partial}_k = \frac{\phi_i}{\phi}\bar{\partial}_0 \\ \tilde{\nabla}_{\bar{\partial}_i}\bar{\partial}_0 &= \tilde{\Gamma}_{i0}{}^{0}\bar{\partial}_0 + \tilde{\Gamma}_{i0}{}^{k}\bar{\partial}_k = \frac{\phi_i}{\phi}\bar{\partial}_0 \end{aligned}$$

where we remind the reader that Latin indices, when summation is implied by one raised and one lowered, only sum from 1 to 3, and a bar over a raised index indicates that the index was raised with $\bar{g}$ as opposed to g.

It now becomes convenient to use normal coordinates on $(\bar{M}, \bar{g})$. Note that these are not normal coordinates on the whole spacetime, just on the $t = 0$ slice of the spacetime $(\bar{M}, \bar{g})$. Since by symmetry this slice has zero second fundamental form, $\tilde{\nabla}_{\bar{\partial}_i}\bar{\partial}_k = \overline{\nabla}_{\bar{\partial}_i}\bar{\partial}_k = 0$ at a single point of our choosing. In normal coordinates, derivatives of the metric components $\bar{g}_{ij}$ and $\bar{g}^{ij}$ are zero at the chosen point, thereby making the Christoffel symbols zero at that point as well. In addition, $(f^{\bar{k}})_i = (f_m\bar{g}^{mk})_i = f_{im}\bar{g}^{mk} = f_i{}^{\bar{k}}$ at this single arbitrary point. Note that $f_{im} = \frac{\partial^2 f}{\partial x^i \partial x^m}$ is simply a coordinate chart second derivative in our notation.

Hence,

$$\begin{aligned} &\tilde{\nabla}_{(\bar{\partial}_i + f_i\bar{\partial}_0)}\left[\bar{\partial}_0 + \phi^2 f^{\bar{k}}\bar{\partial}_k\right] \\ &\qquad = \frac{\phi_i}{\phi}\bar{\partial}_0 + (2\phi\phi_i f^{\bar{k}} + \phi^2 f_i{}^{\bar{k}})\bar{\partial}_k + f_i(\phi\phi^{\bar{k}}\bar{\partial}_k + \phi^2 f^{\bar{k}}\cdot\frac{\phi_k}{\phi}\bar{\partial}_0) \\ &\qquad = \left(\frac{\phi_i}{\phi} + \phi f_i f^{\bar{k}}\phi_k\right)\bar{\partial}_0 + \left(2\phi\phi_i f^{\bar{k}} + \phi^2 f_i{}^{\bar{k}} + \phi f_i\phi^{\bar{k}}\right)\bar{\partial}_k. \end{aligned}$$

Since $\langle \bar{\partial}_0, \bar{\partial}_0\rangle = -\phi^2$ and $\langle \bar{\partial}_k, \bar{\partial}_j\rangle = \bar{g}_{kj}$ (which then lowers indices on other terms), we have computed that

$$h_{ij} = \frac{-\phi^2 f_j \left(\frac{\phi_i}{\phi} + \phi f_i f^{\bar{k}} \phi_k\right) + \left(2\phi\phi_i f^{\bar{k}} + \phi^2 f_i{}^{\bar{k}} + \phi f_i \phi^{\bar{k}}\right) \bar{g}_{kj}}{\phi \left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}}$$

$$= \frac{-\phi_i f_j - \phi^2 f_i f_j f^{\bar{k}} \phi_k + 2\phi_i f_j + \phi f_{ij} + f_i \phi_j}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}}$$

$$= \frac{\phi f_{ij} + f_i \phi_j + \phi_i f_j - \phi^2 \langle df, d\phi\rangle_{\bar{g}} f_i f_j}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}} \tag{102}$$

at the chosen point in normal coordinates for $(\bar{M}, \bar{g})$. Since $\overline{\mathrm{Hess}}_{ij} f = f_{ij}$ as well at the chosen point in these normal coordinates, we have that

$$h_{ij} = \frac{\phi \overline{\mathrm{Hess}}_{ij} f + f_i \phi_j + \phi_i f_j - \phi^2 \langle df, d\phi\rangle_{\bar{g}} f_i f_j}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}} \tag{103}$$

at the chosen point. However, the above equation represents the components of the tensorial equation

$$h = \frac{\phi \overline{\mathrm{Hess}} f + df \otimes d\phi + d\phi \otimes df - \phi^2 \langle df, d\phi\rangle_{\bar{g}} df \otimes df}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}} \tag{104}$$

by which we mean both the left and right sides of the equation are tensors. Since tensorial equations may be verified in any coordinate chart, we conclude that equation 104 is true at our chosen point. Since our chosen point was arbitrary, it follows that equation 104 is true at every point. Thus, equation 103 is true at every point as well, in any coordinate chart.

The beginner differential geometer should take note on how using normal coordinates simplified these computations substantially. However, the computation can also be done straightforwardly without using normal coordinates, just not as elegantly.

As we mentioned originally in section 6, we extend h trivially in our constructed static spacetime so that $h(\partial_t, \cdot) = 0$. Note that this gives $h(\partial_i, \partial_j) = h(\bar{\partial}_i, \bar{\partial}_j)$, so we can call this term h_{ij} without ambiguity. Our next goal is to convert the above formula for h_{ij} expressed with respect to $\bar{g}$ to one expressed with respect to g.

To convert the tensor $\overline{\mathrm{Hess}} f$, we must recall that it is defined to be the covariant derivative $\overline{\nabla}$ of the 1-tensor df. Note that df does not involve any metric, since $df(W) = W(f)$ by definition. However, $\overline{\nabla}$ does involve the metric $\bar{g}$ when applied to tensors. For example, in coordinates,

$$\begin{aligned} \overline{\mathrm{Hess}}_{ij} f =& \overline{\nabla}(df)(\bar{\partial}_i, \bar{\partial}_j) \\ =& \bar{\partial}_i(df(\bar{\partial}_j)) - df(\overline{\nabla}_{\bar{\partial}_i} \bar{\partial}_j) \\ =& \bar{\partial}_i(\bar{\partial}_j(f)) - (\overline{\nabla}_{\bar{\partial}_i} \bar{\partial}_j)(f) \\ =& f_{ij} - \bar{\Gamma}_{ij}{}^k f_k \end{aligned}$$

which involves the metric $\bar{g}$ and its first derivatives by equation 92. Hence,

$$\overline{\mathrm{Hess}}_{ij} f - \mathrm{Hess}_{ij}\, f = -\,(\bar{\Gamma}_{ij}{}^k - \Gamma_{ij}{}^k) f_k. \tag{105}$$

Thus, we need to compute the difference of the Christoffel symbols of $\bar{g}$ and g. By equations 96 and 97,

$$
\begin{aligned}
2\Gamma_{ij}{}^{k} =& g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}) \\
=& \left(\bar{g}^{km} + \frac{\phi^2 f^{\bar{k}} f^{\bar{m}}}{1 - \phi^2 |df|_{\bar{g}}^2}\right) \cdot \\
& \left(\bar{g}_{im,j} + \bar{g}_{jm,i} - \bar{g}_{ij,m} - (\phi^2 f_i f_m)_{,j} - (\phi^2 f_j f_m)_{,i} + (\phi^2 f_i f_j)_{,m}\right) \\
=& \left(\bar{g}^{km} + v^k v^m\right) \left\{\bar{g}_{im,j} + \bar{g}_{jm,i} - \bar{g}_{ij,m} - 2\phi\phi_j f_i f_m - \phi^2 f_m f_{ij} - \phi^2 f_i f_{mj}\right. \\
& \left. -2\phi\phi_i f_j f_m - \phi^2 f_m f_{ji} - \phi^2 f_j f_{mi} + 2\phi\phi_m f_i f_j + \phi^2 f_j f_{im} + \phi^2 f_i f_{jm}\right\} \\
=& \left(\bar{g}^{km} + v^k v^m\right) \left\{\bar{g}_{im,j} + \bar{g}_{jm,i} - \bar{g}_{ij,m} - 2\phi^2 f_m f_{ij}\right. \\
& \left. -2\phi\phi_j f_i f_m - 2\phi\phi_i f_j f_m + 2\phi\phi_m f_i f_j\right\}
\end{aligned} \tag{106}
$$

so that in normal coordinates on $(\bar{M}, \bar{g})$,

$$
\begin{aligned}
\overline{\Gamma}_{ij}{}^{k} - \Gamma_{ij}{}^{k} =& \phi^2 f_{ij} f^{\bar{k}} + \phi f_i \phi_j f^{\bar{k}} + \phi\phi_i f_j f^{\bar{k}} - \phi f_i f_j \phi^{\bar{k}} \\
& + v^k \left[\phi^2 v(f) f_{ij} + \phi v(f)\phi_j f_i + \phi v(f)\phi_i f_j - \phi v(\phi) f_i f_j\right] \\
=& \phi f^{\bar{k}} \left(\phi f_{ij} + f_i \phi_j + \phi_i f_j\right) - \phi f_i f_j \phi^{\bar{k}} \\
& + v^k \phi v(f) \left(\phi f_{ij} + f_i\phi_j + \phi_i f_j\right) - \phi v(\phi) f_i f_j v^k \\
=& \left(\phi f^{\bar{k}} + \frac{\phi^3 |df|_{\bar{g}}^2 f^{\bar{k}}}{1 - \phi^2 |df|_{\bar{g}}^2}\right) \left(\phi f_{ij} + f_i \phi_j + \phi_i f_j\right) \\
& - \frac{\phi^3 \langle df, d\phi\rangle_{\bar{g}} f_i f_j f^{\bar{k}}}{1 - \phi^2 |df|_{\bar{g}}^2} - \phi f_i f_j \phi^{\bar{k}} \\
=& \frac{\phi f^{\bar{k}}}{1 - \phi^2 |df|_{\bar{g}}^2} \left[\phi f_{ij} + f_i\phi_j + \phi_i f_j - \phi^2 \langle df, d\phi\rangle f_i f_j\right] - \phi f_i f_j \phi^{\bar{k}} \\
=& h_{ij} v^k - \phi f_i f_j \phi^{\bar{k}}.
\end{aligned} \tag{107}
$$

(Note that the above equation is a proof of identity 3 from section 6.) Plugging this equation into equation 105 gives us

$$
\overline{\mathrm{Hess}}_{ij} f - \phi f_i f_j \langle d\phi, df\rangle_{\bar{g}} = \mathrm{Hess}_{ij} f - h_{ij} v(f). \tag{108}
$$

Hence, using the above formula and equation 100, we can transform equation 103 to

$$
\begin{aligned}
h_{ij} =& \frac{\phi \overline{\mathrm{Hess}}_{ij} f + f_i \phi_j + \phi_i f_j - \phi^2 \langle df, d\phi\rangle_{\bar{g}} f_i f_j}{\left(1 - \phi^2 |df|_{\bar{g}}^2\right)^{1/2}} \\
=& \left(1 + \phi^2 |df|_g^2\right)^{1/2} \left(\phi\, \mathrm{Hess}_{ij} f - \phi h_{ij} v(f) + f_i \phi_j + \phi_i f_j\right)
\end{aligned}
$$

so that by the definition of v in equation 98

$$
\left(1 + \phi^2 |df|_g^2\right) h_{ij} = \left(1 + \phi^2 |df|_g^2\right)^{1/2} \left(\phi\, \mathrm{Hess}_{ij} f + f_i \phi_j + \phi_i f_j\right).
$$

Thus, we see that the second fundamental form of the graph slice expressed with respect to the metric g is

$$h_{ij} = \frac{\phi \text{Hess}_{ij} f + (f_i \phi_j + \phi_i f_j)}{(1 + \phi^2 |df|_g^2)^{1/2}}, \tag{109}$$

as claimed in section 6. Technically, we've shown that the above equation is true at an arbitrary point in normal coordinates. Again, the above equation represents the components of the tensorial equation

$$h = \frac{\phi \text{Hess} f + (df \otimes d\phi + d\phi \otimes df)}{(1 + \phi^2 |df|_g^2)^{1/2}}, \tag{110}$$

which is therefore true at the arbitrary point, and thus is true everywhere. Hence, equation 109 is true at every point as well, in any coordinate chart.

Appendix D. Derivation of identities. In this appendix we finish the proof of the generalized Schoen-Yau identity sketched out in section 6. We begin with equation 46 derived in that section, which we repeat for clarity:

$$\begin{aligned} \bar{R} = 16\pi(\mu - J(v)) + (\text{tr}_g h)^2 - (\text{tr}_g k)^2 - \|h\|_g^2 + \|k\|_g^2 \\ +2\, v(\text{tr}_g h) - 2\, v(\text{tr}_g k) - 2\, \text{div}(h)(v) + 2\, \text{div}(k)(v). \end{aligned}$$

We will convert this formula for $\bar{R}$ to an expression in terms of the $\bar{g}$ metric. To perform the conversion, we need several identities (originally listed in section 6) for arbitrary symmetric 2-tensors k which we now prove. We continue with the notation established in the previous appendix, which might be thought of as an introduction to this appendix.

Identity 1.

$$(\text{tr}_g(k))^2 - \|k\|_g^2 = (\text{tr}_{\bar{g}} k)^2 - \|k\|_{\bar{g}}^2 + 2k(\bar{v}, \bar{v}) \text{tr}_{\bar{g}} k - 2|k(\bar{v}, \cdot)|_{\bar{g}}^2$$

Proof.

$$\begin{aligned} (\text{tr}_g\, k)^2 - \|k\|_g^2 &= (g^{ij} k_{ij})^2 - g^{ik} g^{jl} k_{ij} k_{kl} \\ &= \left[\left(\bar{g}^{ij} + v^i v^j\right) k_{ij}\right]^2 - \left(\bar{g}^{ik} + v^i v^k\right) \left(\bar{g}^{jl} + v^j v^l\right) k_{ij} k_{kl} \\ &= [\text{tr}_{\bar{g}}\, k + k(\bar{v}, \bar{v})]^2 - \|k\|_{\bar{g}}^2 - k(\bar{v}, \bar{v})^2 - 2\, |k(\bar{v}, \cdot)|_{\bar{g}}^2 \\ &= (\text{tr}_{\bar{g}}\, k)^2 - \|k\|_{\bar{g}}^2 + 2k(\bar{v}, \bar{v})\, \text{tr}_{\bar{g}}\, k - 2\, |k(\bar{v}, \cdot)|_{\bar{g}}^2 . \end{aligned}$$

The first equality is true by definition. The second equality follows from equation 97. For the third equality, remember that k is defined to have zero time-time components and time-spatial components. Hence, $k(v, w) = k(\bar{v}, w)$ for all w since v projects to $\bar{v}$ (and consequently v and $\bar{v}$ are equal except for their time components).

Identity 2.

$$v(\text{tr}_g k) = \bar{v}(\text{tr}_{\bar{g}} k + k(\bar{v}, \bar{v}))$$

Proof.

$$\begin{aligned} v(\text{tr}_g\, k) &= v\left(\text{tr}_{\bar{g}}\, k + k(\bar{v}, \bar{v})\right) \\ &= \bar{v}\left(\text{tr}_{\bar{g}}\, k + k(\bar{v}, \bar{v})\right). \end{aligned}$$

The first equality was shown in the proof of identity 1. The second equality follows since v and $\bar{v}$ only differ by their time components, and the function being differentiated does not depend on the time coordinate, by definition.

Identity 3.

$$\overline{\Gamma}_{ij}^{k} - \Gamma_{ij}^{k} = h_{ij}v^k - \phi f_i f_j \phi^{\bar{k}}$$

Proof. See equation 107.

Identity 4.

$$\begin{aligned}\operatorname{div}(k)(v) = \overline{\operatorname{div}}(k)(\bar{v}) + (\overline{\nabla}_{\bar{v}} k)(\bar{v}, \bar{v}) - 2|\bar{v}|_{\bar{g}}^2 \, k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right) \\ + \langle h(\bar{v}, \cdot), k(\bar{v}, \cdot)\rangle_{\bar{g}} + 2h(\bar{v}, \bar{v})k(\bar{v}, \bar{v}) + (\operatorname{tr}_{\bar{g}} h)k(\bar{v}, \bar{v})\end{aligned}$$

Proof.

$$\begin{aligned}
\operatorname{div}(k)(v) &= g^{ij}(\nabla_{\partial_i} k)(\partial_j, v) \\
&= g^{ij}\left[\partial_i(k(\partial_j, v)) - k(\nabla_{\partial_i}\partial_j, v) - k(\partial_j, \nabla_{\partial_i} v)\right] \\
&= g^{ij}\left[(k_{j\alpha}v^\alpha)_{,i} - \Gamma_{ij}{}^{m} k_{m\alpha} v^\alpha - k\left(\partial_j,\ (v^\alpha)_{,i}\ \partial_\alpha + v^\alpha \nabla_{\partial_i}\partial_\alpha\right)\right] \\
&= g^{ij}\left[(k_{j\alpha}v^\alpha)_{,i} - \Gamma_{ij}{}^{m} k_{m\alpha} v^\alpha - (v^\alpha)_{,i}\ k_{j\alpha} - v^\alpha \Gamma_{i\alpha}{}^{m} k_{jm}\right] \\
&= g^{ij}\left(k_{j\alpha,i} - \Gamma_{ij}{}^{m} k_{m\alpha} - \Gamma_{i\alpha}{}^{m} k_{jm}\right) v^\alpha \\
&= \left(\bar{g}^{ij} + v^i v^j\right)\Big[k_{j\alpha,i} + k_{m\alpha}\left(-\overline{\Gamma}_{ij}{}^{m} + h_{ij}v^m - \phi f_i f_j \phi^{\bar{m}}\right) \\
&\qquad\qquad + k_{jm}\left(-\overline{\Gamma}_{i\alpha}{}^{m} + h_{i\alpha}v^m - \phi f_i f_\alpha \phi^{\bar{m}}\right)\Big] v^\alpha \\
&= \overline{\operatorname{div}}(k)(\bar{v}) + (\operatorname{tr}_{\bar{g}} h)k(\bar{v}, \bar{v}) - \phi\, |df|_{\bar{g}}^2\, k(\overline{\nabla}\phi,\ \bar{v}) + \langle h(\bar{v}, \cdot), k(\bar{v}, \cdot)\rangle_{\bar{g}} \\
&\quad - \phi\ \bar{v}(f)\ k(\overline{\nabla} f, \overline{\nabla}\phi) + (\overline{\nabla}_{\bar{v}} k)(\bar{v}, \bar{v}) + h(\bar{v}, \bar{v})k(\bar{v}, \bar{v}) \\
&\quad - \phi\ \bar{v}(f)^2\ k(\overline{\nabla}\phi,\ \bar{v}) + h(\bar{v}, \bar{v})k(\bar{v}, \bar{v}) - \phi\ \bar{v}(f)^2\ k(\bar{v}, \overline{\nabla}\phi) \\
&= \overline{\operatorname{div}}(k)(\bar{v}) + (\operatorname{tr}_{\bar{g}} h)k(\bar{v}, \bar{v}) + \langle h(\bar{v}, \cdot), k(\bar{v}, \cdot)\rangle_{\bar{g}} + (\overline{\nabla}_{\bar{v}} k)(\bar{v}, \bar{v}) \\
&\quad + 2h(\bar{v}, \bar{v})k(\bar{v}, \bar{v}) - k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right)\left\{2\phi^2\, |df|_{\bar{g}}^2\, |\bar{v}|_{\bar{g}}^2 + 2\phi^2\, |df|_{\bar{g}}^2\right\} \\
&= (\overline{\nabla}\cdot k)(\bar{v}) + (\operatorname{tr}_{\bar{g}} h)k(\bar{v}, \bar{v}) + \langle h(\bar{v}, \cdot), k(\bar{v}, \cdot)\rangle_{\bar{g}} + (\overline{\nabla}_{\bar{v}} k)(\bar{v}, \bar{v}) \\
&\quad + 2h(\bar{v}, \bar{v})k(\bar{v}, \bar{v}) - 2\,|\bar{v}|_{\bar{g}}^2\, k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right).
\end{aligned}$$

The first equality is the definition of divergence. The second equality is the definition of the covariant derivative of a tensor. The third and fourth equalities use Christoffel symbols as defined in the previous appendix. The sixth equality uses equation 97 and then identity 3. The seventh equality is most easily seen by using normal coordinates with respect to $\bar{g}$. The eighth equality combines terms using the fact that $\bar{v}$ is parallel to $\overline{\nabla} f$ in $(M^3, \bar{g})$ by the definition of $\bar{v}$ in equations 98 and 99. The ninth equality is the simplification

$$2\phi^2\, |df|_{\bar{g}}^2\, |\bar{v}|_{\bar{g}}^2 + 2\phi^2\, |df|_{\bar{g}}^2 = 2\phi^2\, |df|_{\bar{g}}^2 \left(|\bar{v}|_{\bar{g}}^2 + 1\right) = \frac{2\phi^2\, |df|_{\bar{g}}^2}{1 - \phi^2\, |df|_{\bar{g}}^2} = 2\,|v|^2.$$

Identity 5.

$$v_{\bar{i};\bar{j}} = h_{ij} + v_{\bar{i}} h(\bar{v},\cdot)_j - \frac{\phi_i v_{\bar{j}}}{\phi}$$

Proof. First, let us clarify our notation. Recall that bars refer to the metric $\bar{g}$. Hence, for example, $v_{\bar{i}} = \bar{g}_{ik} v^k = \langle v, \bar{\partial}_i \rangle_{\bar{g}}$, where v^k is defined in equation 98. As is standard, semicolons refer to covariant differentiation (whereas commas refer to coordinate chart derivatives). Of course in our case, we need to specify with respect to which metric are we performing covariant differentiation. Hence, we place a bar over the semicolon to denote covariant differentiation with respect to $\bar{g}$. Hence, $v_{\bar{i};\bar{j}} = \langle \overline{\nabla}_{\bar{\partial}_j} v, \bar{\partial}_i \rangle_{\bar{g}}$.

All of our computations in the proof of this identity and the two that follow only involve the metric $\bar{g}$, so it is notationally convenient (though not really necessary) to use normal coordinates with respect to this metric. Then at that point,

$$\begin{aligned}
v_{\bar{i};\bar{j}} = v_{\bar{i},j} &= \left(\frac{\phi f_i}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}} \right)_{,j} \\
&= \frac{\phi f_{ij} + f_i \phi_j}{(1 - \phi^2 |df|^2_{\bar{g}})^{1/2}} + \frac{\left(\phi\phi_j |df|^2_{\bar{g}} + \phi^2 f_{\alpha j} f^{\bar{\alpha}}\right)(\phi f_i)}{(1 - \phi^2 |df|^2_{\bar{g}})^{3/2}} \\
&= h_{ij} + \frac{\phi^2 \langle df, d\phi\rangle_{\bar{g}} f_i f_j - \phi_i f_j}{(1 - \phi^2 |df|^2_{\bar{g}})^{1/2}} + \left[\frac{\phi_j}{\phi} |\bar{v}|^2_{\bar{g}} + \frac{\phi f_{\alpha j} v^\alpha}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}} \right] \cdot v_{\bar{i}} \\
&= h_{ij} + |\bar{v}|^2_{\bar{g}} \cdot \frac{v_{\bar{i}} \phi_j}{\phi} + \frac{\phi^2 \langle df, d\phi\rangle_{\bar{g}} f_i f_j - \phi_i f_j + \phi v_{\bar{i}} f_{\alpha j} v^\alpha}{(1 - \phi^2 |df|^2_{\bar{g}})^{1/2}} \\
&= h_{ij} + |\bar{v}|^2_{\bar{g}} \cdot \frac{v_{\bar{i}} \phi_j}{\phi} + v_{\bar{i}} h(\bar{v},\cdot)_j + \left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{-1/2} \cdot \\
&\quad \left\{ v_{\bar{i}} v^\alpha \left(\phi^2 \langle df, d\phi\rangle_{\bar{g}} f_\alpha f_j - f_\alpha \phi_j - \phi_\alpha f_j\right) + \phi^2 \langle df, d\phi\rangle_{\bar{g}} f_i f_j - \phi_i f_j \right\} \\
&= h_{ij} + v_{\bar{i}} h(\bar{v},\cdot)_j - \frac{\phi_i v_{\bar{j}}}{\phi} \\
&\quad + |\bar{v}|^2_{\bar{g}} \frac{v_{\bar{i}} \phi_j}{\phi} - v_{\bar{i}} v^\alpha \left(\frac{v_{\bar{\alpha}} \phi_j}{\phi} + \frac{\phi_\alpha v_{\bar{j}}}{\phi} \right) + \phi \bar{v}(\phi) \left(f_i f_j + |\bar{v}|^2_{\bar{g}} f_i f_j \right) \\
&= h_{ij} + v_{\bar{i}} h(\bar{v},\cdot)_j - \frac{\phi_i v_{\bar{j}}}{\phi} + \frac{\bar{v}(\phi)}{\phi} \left((1 + |\bar{v}|^2_{\bar{g}}) \phi^2 f_i f_j - v_{\bar{i}} v_{\bar{j}} \right) \\
&= h_{ij} + v_{\bar{i}} h(\bar{v},\cdot)_j - \frac{\phi_i v_{\bar{j}}}{\phi}.
\end{aligned}$$

The above calculations follow from our formula for h in equation 102, our definition of $\bar{v}$ in equations 98 and 99, and the substitution

$$v_{\bar{i}} = \frac{\phi f_i}{\left(1 - \phi^2 |df|^2_{\bar{g}}\right)^{1/2}}$$

which we use a number of times.

IDENTITY 6.

$$\overline{\mathrm{div}}(k)(\bar{v}) = \overline{\mathrm{div}}(k(\bar{v},\cdot)) - \langle h, k\rangle_{\bar{g}} - \langle h(\bar{v},\cdot), k(\bar{v},\cdot)\rangle_{\bar{g}} + k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right)$$

Proof. By identity 5,

$$\begin{aligned}\overline{\mathrm{div}}\,(k(\bar{v},\cdot)) &= \overline{\mathrm{div}}(k)(\bar{v}) + \langle k_{ij}, v_{\bar{\imath};\bar{j}}\rangle_{\bar{g}} \\ &= \overline{\mathrm{div}}(k)(\bar{v}) + \langle k, h\rangle_{\bar{g}} + \langle k(\bar{v},\cdot), h(\bar{v},\cdot)\rangle_{\bar{g}} - k\left(\frac{\overline{\nabla}\phi}{\phi}, \bar{v}\right).\end{aligned}$$

IDENTITY 7.

$$(\overline{\nabla}_{\bar{v}}k)(\bar{v},\bar{v}) = \bar{v}(k(\bar{v},\bar{v})) - 2\langle h(\bar{v},\cdot), k(\bar{v},\cdot)\rangle_{\bar{g}} - 2h(\bar{v},\bar{v})k(\bar{v},\bar{v}) + 2|\bar{v}|^2_{\bar{g}}\, k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right)$$

Proof. By identity 5,

$$\left(\overline{\nabla}_{\bar{v}}\bar{v}\right)_{\bar{\imath}} = v^j v_{\bar{\imath};\bar{j}} = h(\bar{v},\cdot)_i + h(\bar{v},\bar{v})v_{\bar{\imath}} - |\bar{v}|^2_{\bar{g}}\frac{\phi_i}{\phi}$$

so that by the definition of covariant differentiation of a symmetric 2-tensor,

$$\begin{aligned}(\overline{\nabla}_{\bar{v}}k)(\bar{v},\bar{v}) &= \bar{v}(k(\bar{v},\bar{v})) - 2k(\bar{v}, \overline{\nabla}_{\bar{v}}\bar{v}) \\ &= \bar{v}(k(\bar{v},\bar{v})) - 2\langle k(\bar{v},\cdot), h(\bar{v},\cdot)\rangle_{\bar{g}} - 2k(\bar{v},\bar{v})h(\bar{v},\bar{v}) + 2\,|\bar{v}|^2_{\bar{g}}\, k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right)\end{aligned}$$

proving the identity.

IDENTITY 8.

$$\begin{aligned}\mathrm{div}(k)(v) = \overline{\mathrm{div}}(k(\bar{v},\cdot)) + \bar{v}(k(\bar{v},\bar{v})) + k\left(\bar{v}, \frac{\overline{\nabla}\phi}{\phi}\right) \\ -\langle h, k\rangle_{\bar{g}} - 2\langle h(\bar{v},\cdot), k(\bar{v},\cdot)\rangle_{\bar{g}} + (\mathrm{tr}_{\bar{g}}h)k(\bar{v},\bar{v})\end{aligned}$$

Proof. Plugging identities 6 and 7 into identity 4 and simplifying proves the identity.

IDENTITY 9. (**The Generalized Schoen-Yau Identity**)

$$\begin{aligned}\bar{R} = 16\pi(\mu - J(v)) + \|h-k\|^2_{\bar{g}} + 2|q|^2_{\bar{g}} - \frac{2}{\phi}\overline{\mathrm{div}}(\phi q) \\ +(\mathrm{tr}_{\bar{g}}h)^2 - (\mathrm{tr}_{\bar{g}}k)^2 + 2\bar{v}(\mathrm{tr}_{\bar{g}}h - \mathrm{tr}_{\bar{g}}k) + 2k(\bar{v},\bar{v})(\mathrm{tr}_{\bar{g}}h - \mathrm{tr}_{\bar{g}}k)\end{aligned}$$

where

$$q = h(\bar{v},\cdot) - k(\bar{v},\cdot) = h(v,\cdot) - k(v,\cdot)\ .$$

Proof. First we recall equation 46 derived in section 6,

$$\begin{aligned}\bar{R} = 16\pi(\mu - J(v)) + (\mathrm{tr}_g h)^2 - (\mathrm{tr}_g k)^2 - \|h\|^2_g + \|k\|^2_g \\ +2\,v(\mathrm{tr}_g h) - 2\,v(\mathrm{tr}_g k) - 2\,\mathrm{div}(h)(v) + 2\,\mathrm{div}(k)(v).\end{aligned}$$

Next, we plug in identities 1, 2, and 8. Note that these identities are true for arbitrary symmetric 2-tensors k and hence are true for h as well. Thus,

$$\begin{aligned}\bar{R} = 16\pi(\mu - J(v))\\
&+(\mathrm{tr}_{\bar{g}}h)^2 - \|h\|_{\bar{g}}^2 + 2h(\bar{v},\bar{v})\mathrm{tr}_{\bar{g}}h - 2|h(\bar{v},\cdot)|_{\bar{g}}^2\\
&-(\mathrm{tr}_{\bar{g}}k)^2 + \|k\|_{\bar{g}}^2 - 2k(\bar{v},\bar{v})\mathrm{tr}_{\bar{g}}k + 2|k(\bar{v},\cdot)|_{\bar{g}}^2\\
&+2\bar{v}(\mathrm{tr}_{\bar{g}}h) + 2\bar{v}(h(\bar{v},\bar{v}))\\
&-2\bar{v}(\mathrm{tr}_{\bar{g}}k) + 2\bar{v}(k(\bar{v},\bar{v}))\\
&-2\overline{\mathrm{div}}(h(\bar{v},\cdot)) - 2\bar{v}(h(\bar{v},\bar{v})) - 2h\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right)\\
&+2\|h\|_{\bar{g}}^2 + 4|h(\bar{v},\cdot)|_{\bar{g}}^2 - 2(\mathrm{tr}_{\bar{g}}h)h(\bar{v},\bar{v})\\
&+2\overline{\mathrm{div}}(k(\bar{v},\cdot)) + 2\bar{v}(k(\bar{v},\bar{v})) + 2k\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right)\\
&-2\langle h,k\rangle_{\bar{g}} - 4\langle h(\bar{v},\cdot),k(\bar{v},\cdot)\rangle_{\bar{g}} + 2(\mathrm{tr}_{\bar{g}}h)k(\bar{v},\bar{v}).\end{aligned}$$

Simplifying and combining terms then gives us that

$$\begin{aligned}\bar{R} = 16\pi(\mu - J(v))\\
&+\|h\|_{\bar{g}}^2 - 2\langle h,k\rangle_{\bar{g}} + \|k\|_{\bar{g}}^2\\
&+2|h(\bar{v},\cdot)|_{\bar{g}}^2 - 4\langle h(\bar{v},\cdot),k(\bar{v},\cdot)\rangle_{\bar{g}} + 2|k(\bar{v},\cdot)|_{\bar{g}}^2\\
&-2\overline{\mathrm{div}}(h(\bar{v},\cdot)) + 2\overline{\mathrm{div}}(k(\bar{v},\cdot)) - 2h\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right) + 2k\left(\bar{v},\frac{\overline{\nabla}\phi}{\phi}\right)\\
&+(\mathrm{tr}_{\bar{g}}h)^2 - (\mathrm{tr}_{\bar{g}}k)^2 + 2\bar{v}(\mathrm{tr}_{\bar{g}}h - \mathrm{tr}_{\bar{g}}k) + 2k(\bar{v},\bar{v})(\mathrm{tr}_{\bar{g}}h - \mathrm{tr}_{\bar{g}}k)\end{aligned}$$

so that

$$\begin{aligned}\bar{R} = 16\pi(\mu - J(v)) + \|h-k\|_{\bar{g}}^2 + 2|q|_{\bar{g}}^2 - 2\overline{\mathrm{div}}(q) - 2q\left(\frac{\overline{\nabla}\phi}{\phi}\right)\\
&+(\mathrm{tr}_{\bar{g}}h)^2 - (\mathrm{tr}_{\bar{g}}k)^2 + 2\bar{v}(\mathrm{tr}_{\bar{g}}h - \mathrm{tr}_{\bar{g}}k) + 2k(\bar{v},\bar{v})(\mathrm{tr}_{\bar{g}}h - \mathrm{tr}_{\bar{g}}k)\end{aligned}$$

where

$$q = h(\bar{v},\cdot) - k(\bar{v},\cdot) = h(v,\cdot) - k(v,\cdot)\ .$$

Note that these two definitions of q exist on the entire constructed static spacetime and are equal since both h and k are extended trivially in the time direction of the constructed static spacetime and v and $\bar{v}$ differ only in their time components. By the product rule the above equation proves the identity.

REFERENCES

[1] L. Andersson, M. Eichmair, and J. Metzger, *Jang's equation and its applications to marginally trapped surfaces*, Proceedings of the Complex Analysis and Dynamical Systems IV Conference (Nahariya, Israel, May 2009), arXiv:1006.4601v1 [gr-qc].

[2] L. Andersson, and J. Metzger, *The area of horizons and the trapped region*, Comm. Math. Phys., 290:3 (2009), pp. 941–972.

[3] H. L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem,* J. Differential Geom., 59 (2001), pp. 177–267.

[4] H. L. Bray, *Black holes, geometric flows, and the Penrose inequality in general relativity*, Notices of the AMS, 49 (2002), pp. 1372–1381.

[5] H. L. Bray and P. T. Chrusciel, *The Penrose inequality* in *The Einstein equations and the large scale behavior of gravitational fields (50 years of the Cauchy problem in general relativity)*, edited by P.T. Chrusciel, H. Friedrich, Birkhaeuser, Basel, 2004, pp. 39–70. ESI preprint 1390, http://arxiv.org/abs/gr-qc/0312047

[6] H. L. Bray and D. Lee, *On the Riemannian Penrose inequality in dimensions less than 8*, preprint, arXiv:0705.1128, 2008, (accepted by the Duke Mathematical Journal).

[7] G. Bunting and A. Masood-ul-Alam, *Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time*, Gen. Relativity Gravitation, 19:2 (1987), pp. 147–154.

[8] A. Carrasco and M. Mars, *A counterexample to a recent version of the Penrose conjecture*, Class. Q. Grav., 27:6 (2010), 062001.

[9] P. T. Chrusciel, E. Delay, G. Galloway, and R. Howard, *Regularity of horizons and the area theorem*, Annales Henri Poincare, 2 (2001), pp. 109–178, gr-qc/0001003.

[10] M. Eichmair, *The Plateau problem for marginally outer trapped surfaces*, J. Differential Geom., 83:3 (2009), pp. 551–583.

[11] M. Eichmair, *Existence, regularity, and properties of generalized apparent horizons*, Comm. Math. Phys., 294:3 (2010), pp. 745–760.

[12] M. Eichamir and J. Metzger, in preparation.

[13] R. Geroch, *Energy extraction*, Ann. New York Acad. Sci., 224 (1973), pp. 108–117.

[14] S. Hawking, and G. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics, Cambridge, 1973.

[15] M. Herzlich, *A Penrose-like inequality for the mass of Riemannian asymptotically flat manifolds*, Commun. Math. Phys., 188 (1997), pp. 121–133.

[16] M. Heusler, *Black hole uniqueness theorems*, Cambridge Lecture Notes in Physics (Cambridge univ. Press) Vol. 6, Cambridge, 1996.

[17] G. T. Horowitz, *The positive energy theorem and its extensions* in *Asymptotic behavior of mass and spacetime geometry*, Springer Lecture Notes in Physics, 202 Ed. F. Flaherty (Springer, New York, 1984), pp. 1–20.

[18] G. Huisken, and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom., 59 (2001), pp. 353–437.

[19] P.-S. Jang, *On the positivity of energy in general relativity*, J. Math. Phys., 19 (1978), pp. 1152–1155.

[20] P.-S. Jang and R. Wald, *The positive energy conjecture and the cosmic censorship hypothesis*, J. Math. Phys., 18 (1977), pp. 41–44.

[21] J. Jauregui, thesis, Duke University (in preparation).

[22] M. Khuri, *Nonexistence of generalized apparent horizons in Minkowski space*, Class. Q. Grav., 26 (2009), 078001.

[23] M. Khuri, *A Penrose-like inequality for general initial data sets*, Comm. Math. Phys., 290:2 (2009), pp. 779–788.

[24] M. Mars, *Present status of the Penrose inequality*, Classical Quantum Gravit, 26:19 (2009), 193001.

[25] M. Mars, and J. M. M. Senovilla, *Trapped surfaces and symmetries*, Class. Q. Grav., 20 (2003), pp. L293-L300.

[26] W. Meeks III, L. Simon, and S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math., 116 (1982), pp. 621–659.

[27] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Pure and Applied Mathematics (Academic Press) Vol. 103, New York, 1983.

[28] R. Penrose, *Gravitational collapse: the role of general relativity*, Rivista del Nuovo Cimento Numero Speciale **1** (1969), pp. 252–276.

[29] R. Penrose, *Naked singularities*, Ann. N. Y. Acad. Sci., 224 (1973), pp. 125–134.

[30] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys., 65 (1979), pp. 45–76.

[31] R. Schoen and S.-T. Yau, *Proof of the positive mass theorem II*, Commun. Math. Phys., 79:2 (1981), pp. 231–260.
[32] R. Schoen and S.-T. Yau, *Positivity of the total mass of a general space-time*, Phys. Rev. Lett., 43 (1979), pp. 1457–1459.
[33] R. Schoen and S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with nonnegative scalar curvature*, Ann. of Math., 110 (1979), pp. 127–142.
[34] J. M. M. Senovilla, *Classification of spacelike surfaces in spacetime*, Class. Q. Grav., 24 (2007), pp. 3091–3124.
[35] J. Streets, *Applications of inverse mean curvature flow to negative mass singularities* (in preparation).
[36] R. Wald, *General relativity*, Univ. Chicago Press, Chicago, 1984.
[37] E. Witten, *A new proof of the positive energy theorem*, Commun. Math. Phys., 80 (1981), pp. 381–402.