Part 1 of 5:

A New Monotonic, Clone-Independent, Reversal Symmetric, and Condorcet-Consistent Single-Winner Election Method

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Summary. In recent years, the Pirate Party of Sweden, the Wikimedia Foundation, the Debian project, the "Software in the Public Interest" project, the Gentoo project, and many other private organizations adopted a new single-winner election method for internal elections and referendums. In this paper, we will introduce this method, demonstrate that it satisfies e.g. resolvability, Condorcet, Pareto, reversal symmetry, monotonicity, and independence of clones and present an O(C^3) algorithm to calculate the winner, where C is the number of alternatives.

Keywords and Phrases: Condorcet criterion, independence of clones, monotonicity, Pareto efficiency, reversal symmetry, single-winner election methods, prudent ranking rules

JEL Classification Number: D71

This paper is the first part of a series of papers that can be downloaded here:

http://m-schulze.webhop.net/schulze1.pdf
http://m-schulze.webhop.net/schulze2.pdf
http://m-schulze.webhop.net/schulze3.zip
http://m-schulze.webhop.net/schulze4.pdf
http://m-schulze.webhop.net/schulze5.pdf

Contents

1. Introduction

One important property of a good single-winner election method is that it minimizes the number of “overruled” voters (according to some heuristic). Because of this reason, the Simpson-Kramer method, that always chooses that alternative whose worst pairwise defeat is the weakest, was very popular over a long time. However, in recent years, the Simpson-Kramer method has been criticized by many social choice theorists. Smith (1973) criticizes that this method doesn’t choose from the top-set of alternatives. Tideman (1987) complains that this method is vulnerable to the strategic nomination of a large number of similar alternatives, so-called *clones*. And Saari (1994) rejects this method for violating *reversal symmetry*. A violation of reversal symmetry can lead to strange situations where still the same alternative is chosen when all ballots are reversed, meaning that the same alternative is identified as best one and simultaneously as worst one.

In this paper, we will show that only a slight modification (section 4.8) of the Simpson-Kramer method is needed so that the resulting method satisfies the criteria proposed by Smith (section 4.7), Tideman (section 4.6), and Saari (section 4.4). The resulting method will be called *Schulze method*. Random simulations by Wright (2009) confirmed that, in almost 99% of all instances, the Schulze method conforms with the Simpson-Kramer method (table 9.1). In this paper, we will prove that, nevertheless, the Schulze method still satisfies all important criteria that are also satisfied by the Simpson-Kramer method, like resolvability (section 4.2), Pareto (section 4.3), monotonicity (section 4.5), and prudence (section 4.9). Because of these reasons, already several private organizations have adopted the Schulze method.

1997 – 2006: In 1997, I proposed the Schulze method to a large number of people, who are interested in mathematical aspects of election methods. In January 2003, the “Software in the Public Interest” (SPI) project, a software developer organization with about 500 eligible members, adopted this method. In June 2003, the Debian project, a software developer organization with about 900 eligible members, adopted this method in a referendum with 144 against 16 votes; Debian GNU/Linux is the largest and most popular non-commercial Linux distribution. In May 2005, the Gentoo Foundation, a software developer organization with about 300 eligible members, adopted this method; Gentoo Linux is another wide-spread Linux distribution. In December 2005, the French Wikipedia section (about 1,200,000 registered users) adopted this method for polls among its users.

2007 – 2010: In May 2008, the Wikimedia Foundation, a non-profit charitable organization with about 43,000 eligible members, adopted the proposed method for the election of its Board of Trustees; the Wikimedia Foundation is the umbrella organization e.g. for Wikipedia, Wiktionary, Wikiquote, Wikibooks, Wikisource, Wikinews, Wikiversity, and Wikispecies; it is, therefore, the fifth most important Internet corporation (after Google/YouTube, Facebook, Yahoo!, and Baidu). In October 2009, the “Pirate Party of Sweden” (about 8,000 eligible members) adopted this method. In May 2010, the “Pirate Party of Germany” (about 12,000 eligible members) adopted this method.

Today (September 2011), the proposed method is used by more than 50 organizations with more than 70,000 eligible members in total. Furthermore, the fact that the proposed method is an integral part of Debian’s voting software (*Debian Vote Engine*) means that this method is the standard election method in all Debian user groups and in many other Linux user groups. Therefore, the proposed method is more wide-spread than all other Condorcet-consistent single-winner election methods together.

There has been some debate about an appropriate name for this method. Some people suggested names like “beatpath”, “beatpaths”, “beatpath method”, “beatpath winner”, “beatpath power ranking” (BeatPower), “beatpaths power ranking” (BeatPower), “path method”, “path voting”, “path winner”, “Schwartz sequential dropping” (SSD), and “cloneproof Schwartz sequential dropping” (CSSD or CpSSD). Brearley (1999) suggested names like “descending minimum gross score” (DminGS), “descending minimum augmented gross score” (DminAGS), and “descending minimum doubly augmented gross score” (DminDAGS), depending on how the strength of a pairwise link is measured. Heitzig suggested names like “strong immunity from binary arguments” (SImA) and “sequential dropping towards a spanning tree” (SDST). However, I prefer the name “Schulze method”, not because of academic arrogance, but because the other names do not refer to the method itself but to specific heuristics for implementing it, and so may mislead readers into believing that no other method for implementing it is possible.

In section 2 of this paper, the Schulze method is defined. In section 3, this method is applied to concrete examples. In section 4, this method is analyzed. Short descriptions of this method can also be found in publications by Tideman (2006, page 228–232), Stahl and Johnson (2007, page 119–129), Camps (2008), McCaffrey (2008), and Börgers (2009, page 37–42). This method is also discussed in papers by Yue (2007), Wright (2009), and Rivest and Shen (2010).

2. Definition of the Schulze Method

2.1. Preliminaries

We presume that A is a finite and non-empty set of alternatives. $C \in \mathbb{N}$ with $1 < C < \infty$ is the number of alternatives in A.

A binary relation $\succ$ on A is *asymmetric* if it has the following property:

$\forall\ a,b \in A$, exactly one of the following three statements is valid:

1. $a \succ b$.
2. $b \succ a$.
3. $a \approx b$ (where “$a \approx b$” means “neither $a \succ b$ nor $b \succ a$”).

A binary relation $\succ$ on A is *irreflexive* if it has the following property:

$\forall\ a \in A$: $a \approx a$.

A binary relation $\succ$ on A is *transitive* if it has the following property:

$\forall\ a,b,c \in A$: ($a \succ b$ and $b \succ c \Rightarrow a \succ c$).

A binary relation $\succ$ on A is *negatively transitive* if it has the following property (where “$a \succsim b$” means “not $b \succ a$”):

$\forall\ a,b,c \in A$: ($a \succsim b$ and $b \succsim c \Rightarrow a \succsim c$).

A binary relation $\succ$ on A is *linear* if it has the following property:

$\forall\ a,b \in A$: ($b \in A \setminus \{a\} \Rightarrow a \succ b$ or $b \succ a$).

A *strict partial order* is an asymmetric, irreflexive, and transitive relation. A *strict weak order* is a strict partial order that is also negatively transitive. A *linear order* is a strict weak order that is also linear. A *profile* is a finite and non-empty list of strict weak orders each on A.

Input of the proposed method is a profile V. $N \in \mathbb{N}$ with $0 < N < \infty$ is the number of strict weak orders in $V := \{ \succ_1, ..., \succ_N \}$. These strict weak orders will sometimes be called “voters” or “ballots”.

“$a \succ_v b$” means “voter $v \in V$ strictly prefers alternative $a \in A$ to alternative b”. “$a \approx_v b$” means “voter $v \in V$ is indifferent between alternative a and alternative b”. “$a \succsim_v b$” means “$a \succ_v b$ or $a \approx_v b$”.

Output of the proposed method are (1) a strict partial order $\mathcal{O}$ on A and (2) a set $\varnothing \neq \mathcal{S} \subseteq A$ of winners.

A possible implementation of the Schulze method looks as follows:

> Each voter gets a complete list of all alternatives and ranks these alternatives in order of preference. The individual voter may give the same preference to more than one alternative and he may keep alternatives unranked. When a given voter does not rank all alternatives, then this means (1) that this voter strictly prefers all ranked alternatives to all not ranked alternatives and (2) that this voter is indifferent between all not ranked alternatives.

Suppose $N[e,f] := \| \{ v \in V \mid e \succ_v f \} \|$ is the number of voters who strictly prefer alternative e to alternative f. We presume that the strength of the link ef depends only on $N[e,f]$ and $N[f,e]$. Therefore, the strength of the link ef can be denoted $(N[e,f],N[f,e])$. We presume that a binary relation $\succ_D$ on $\mathbb{N}_0 \times \mathbb{N}_0$ is defined such that the link ef is stronger than the link gh if and only if $(N[e,f],N[f,e]) \succ_D (N[g,h],N[h,g])$. $N[e,f]$ is the *support* for the link ef; $N[f,e]$ is its *opposition*.

Example 1 (*margin*):

> When the strength of the link ef is measured by *margin*, then its strength is the difference $N[e,f] - N[f,e]$ between its support $N[e,f]$ and its opposition $N[f,e]$.
>
> $(N[e,f],N[f,e]) \succ_{margin} (N[g,h],N[h,g])$ if and only if $N[e,f] - N[f,e] > N[g,h] - N[h,g]$.

Example 2 (*ratio*):

> When the strength of the link ef is measured by *ratio*, then its strength is the ratio $N[e,f] \,/\, N[f,e]$ between its support $N[e,f]$ and its opposition $N[f,e]$.
>
> $(N[e,f],N[f,e]) \succ_{ratio} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:
>
> 1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
> 2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
> 3. $N[e,f] \cdot N[h,g] > N[f,e] \cdot N[g,h]$.
>
> 4*. $N[e,f] > N[g,h]$ and $N[f,e] \leq N[h,g]$.
>
> 5*. $N[e,f] \geq N[g,h]$ and $N[f,e] < N[h,g]$.

Example 3 (*winning votes*):

When the strength of the link ef is measured by *winning votes*, then its strength is measured primarily by its support $N[e,f]$.

$(N[e,f],N[f,e]) \succ_{win} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] > N[g,h]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.

5*. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[e,f] > N[g,h]$.

6*. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.

Example 4 (*losing votes*):

When the strength of the link ef is measured by *losing votes*, then its strength is measured primarily by its opposition $N[f,e]$.

$(N[e,f],N[f,e]) \succ_{los} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[f,e] < N[h,g]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.

5*. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] < N[h,g]$.

6*. $N[e,f] < N[f,e]$ and $N[g,h] < N[h,g]$ and $N[f,e] = N[h,g]$ and $N[e,f] > N[g,h]$.

Those conditions, that are marked with an asterisk (*), are actually superfluous for the definition of the Schulze method. But they make it easier to prove that $\succ_D$ satisfies property (2.1.1).

Conditions 4 and 5 in the definition of $\succ_{ratio}$ are needed e.g. to say that a pairwise victory of $(N[e,f],N[f,e]) = (5,0)$ is stronger than a pairwise victory of $(N[g,h],N[h,g]) = (1,0)$. This does not follow from conditions 1–3 in the definition of $\succ_{ratio}$. However, as it is not possible that there is a directed cycle that consists only of unanimous pairwise victories, unanimous pairwise victories cannot contradict each other. Therefore, the definition for the strength of an unanimous pairwise victory cannot have any impact on the result of an election. But conditions 4 and 5 in the definition of $\succ_{ratio}$ make it easier to prove that $\succ_{ratio}$ satisfies property (2.1.1).

Conditions 5 and 6 in the definitions for $\succ_{win}$ and $\succ_{los}$ are superfluous because, for each pair of alternatives $a,b \in A$, there is a path from alternative a to alternative b or a path from alternative b to alternative a that contains no pairwise defeats. Condition 2 in the definitions for $\succ_{win}$ and $\succ_{los}$ guarantees that a path that contains no pairwise defeats is always stronger than a path that contains a pairwise defeat. Therefore, the definition for the strength of a pairwise defeat cannot have any impact on the result of an election.

The most intuitive definitions for the strength of a link are its *margin* and its *ratio*. However, we only presume that $\succ_D$ is a strict weak order on $\mathbb{N}_0 \times \mathbb{N}_0$ with at least the following properties:

(2.1.1) $\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$:
$(\ (\ x_1 > y_1 \text{ and } x_2 \leq y_2\ ) \text{ or } (\ x_1 \geq y_1 \text{ and } x_2 < y_2\ )\ ) \Rightarrow (x_1,x_2) \succ_D (y_1,y_2)$.

(2.1.2) $\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$:
$(\ (\ x_1 > x_2 \text{ and } y_1 \leq y_2\ ) \text{ or } (\ x_1 \geq x_2 \text{ and } y_1 < y_2\ )\ ) \Rightarrow (x_1,x_2) \succ_D (y_1,y_2)$.

(2.1.3) $\forall\ (x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0\ \forall\ c_1,c_2 \in \mathbb{N}$:
$(c_1 \cdot x_1,c_1 \cdot x_2) \succ_D (c_1 \cdot y_1,c_1 \cdot y_2) \Rightarrow (c_2 \cdot x_1,c_2 \cdot x_2) \succ_D (c_2 \cdot y_1,c_2 \cdot y_2)$.

The presumption, that the strength of the link ef depends only on $N[e,f]$ and $N[f,e]$, guarantees (1) that the proposed method satisfies anonymity and neutrality, (2) that adding a ballot, on which all alternatives are ranked equally, cannot change the result of the elections, and (3) that the proposed method is a C2 *Condorcet social choice function* (CSCF) according to Fishburn’s (1977) terminology.

(2.1.1) says that, when the support of a link increases and its opposition doesn’t increase or when its opposition decreases and its support doesn’t decrease, then the strength of this link increases. So (2.1.1) says that the strength of a link responses to a change of its support or its opposition in the correct manner. (2.1.1) guarantees that the proposed method satisfies resolvability (section 4.2), Pareto (section 4.3), and monotonicity (section 4.5). When each voter $v \in V$ casts a linear order $\succ_v$ on A, then all definitions for $\succ_D$, that satisfy (2.1.1), are identical.

(2.1.2) says that every pairwise victory is stronger than every pairwise tie and that every pairwise tie is stronger than every pairwise defeat. (2.1.2) guarantees that the proposed method satisfies the Smith criterion (section 4.7).

Homogeneity means that the result depends only on the proportion of ballots of each type, not on their absolute numbers. (2.1.3) guarantees that the proposed method satisfies homogeneity.

Suppose $\emptyset \neq \mathcal{M} \subset \mathbb{N}_0 \times \mathbb{N}_0$ is finite and non-empty. Then “$\max_D\mathcal{M}$”, the *set of maximum elements* of $\mathcal{M}$, and “$\min_D\mathcal{M}$”, the *set of minimum elements* of $\mathcal{M}$, are defined as follows: $(\beta_1,\beta_2) \in \max_D\mathcal{M}$ if and only if (1) $(\beta_1,\beta_2) \in \mathcal{M}$ and (2) $(\beta_1,\beta_2) \succsim_D (\delta_1,\delta_2)\ \forall\ (\delta_1,\delta_2) \in \mathcal{M}$. $(\gamma_1,\gamma_2) \in \min_D\mathcal{M}$ if and only if (1) $(\gamma_1,\gamma_2) \in \mathcal{M}$ and (2) $(\gamma_1,\gamma_2) \precsim_D (\delta_1,\delta_2)\ \forall\ (\delta_1,\delta_2) \in \mathcal{M}$.

We write “$(\beta_1,\beta_2) := \max_D\mathcal{M}$” and “$(\gamma_1,\gamma_2) := \min_D\mathcal{M}$” for “$(\beta_1,\beta_2)$ is an arbitrarily chosen element of $\max_D\mathcal{M}$” and “(γ_1,γ_2) is an arbitrarily chosen element of $\min_D\mathcal{M}$”.

2.2. Basic Definitions

In this section, the Schulze method is defined. Concrete examples can be found in section 3.

Basic idea of the Schulze method is that the *strength* of the indirect comparison “alternative a vs. alternative b” is the *strength* of the *strongest path* $a \equiv c(1),\dots,c(n) \equiv b$ from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$ and that the *strength* of a path is the *strength* $(N[c(i),c(i+1)],N[c(i+1),c(i)])$ of its *weakest link* $c(i),c(i+1)$.

The Schulze method is defined as follows:

A *path* from alternative $x \in A$ to alternative $y \in A$ is a sequence of alternatives $c(1),\dots,c(n) \in A$ with the following properties:

1. $x \equiv c(1)$.
2. $y \equiv c(n)$.
3. $2 \leq n < \infty$.
4. For all $i = 1,\dots,(n-1)$: $c(i) \not\equiv c(i+1)$.

The *strength* of the path $c(1),\dots,c(n)$ is

$$\min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,\dots,(n-1) \}.$$

In other words: The strength of a path is the strength of its weakest link.

When a path $c(1),\dots,c(n)$ has the strength $z \in \mathbb{N}_0 \times \mathbb{N}_0$, then the *critical links* of this path are the links with $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \approx_D z$.

$$P_D[a,b] := \max_D \{ \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,\dots,(n-1) \} \mid c(1),\dots,c(n) \text{ is a path from alternative } a \text{ to alternative } b \}.$$

In other words: $P_D[a,b] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $a \in A$ to alternative $b \in A \setminus \{a\}$.

(2.2.1) The binary relation $\mathcal{O}$ on A is defined as follows:

$$ab \in \mathcal{O} :\Leftrightarrow P_D[a,b] \succ_D P_D[b,a].$$

(2.2.2) $\mathcal{S} := \{ a \in A \mid \forall\, b \in A \setminus \{a\}: ba \notin \mathcal{O} \}$ is the *set of winners*.

As the link ab is already a path from alternative a to alternative b of strength $(N[a,b],N[b,a])$, we get

(2.2.3) $\forall\, a,b \in A$: $P_D[a,b] \succsim_D (N[a,b],N[b,a])$.

With (2.2.1) and (2.2.3), we get

(2.2.4) $(N[a,b],N[b,a]) \succ_D P_D[b,a] \Rightarrow ab \in \mathcal{O}$.

Furthermore, we get

(2.2.5) $\forall\, a,b,c \in A$: $\min_D \{ P_D[a,b], P_D[b,c] \} \precsim_D P_D[a,c]$.

Otherwise, if $\min_D \{ P_D[a,b], P_D[b,c] \}$ was strictly larger than $P_D[a,c]$, then this would be a contradiction to the definition of $P_D[a,c]$ since there would be a path from alternative a to alternative c via alternative b with a strength of more than $P_D[a,c]$.

Furthermore, we get

(2.2.6) $\forall\ a,b \in A$: $P_D[a,b] \precsim_D \max_D$ { $(N[a,c],N[c,a])$ | $c \in A \setminus \{a\}$ }.

(2.2.7) $\forall\ a,b \in A$: $P_D[a,b] \precsim_D \max_D$ { $(N[c,b],N[b,c])$ | $c \in A \setminus \{b\}$ }.

The asymmetry of $\mathcal{O}$ follows directly from the asymmetry of $\succ_D$. The irreflexivity of $\mathcal{O}$ follows directly from the irreflexivity of $\succ_D$. Furthermore, in section 4.1, we will see that the binary relation $\mathcal{O}$ is transitive. This guarantees that there is always at least one winner.

Suppose $\emptyset \neq B \subsetneq A$. Then we get

(2.2.8) $\forall\ a \in B\ \forall\ b \notin B$: $P_D[a,b] \precsim_D \max_D$ { $(N[c,d],N[d,c])$ | $c \in B$ and $d \notin B$ }.

2.3. Implementation

The strength $P_D[i,j]$ of the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$ can be calculated with the Floyd (1962) algorithm. The runtime to calculate the strengths of all strongest paths is O(C^3), where C is the number of alternatives in A.

Input: $N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

Output: $P_D[i,j] \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

pred$[i,j] \in A \setminus \{j\}$ is the predecessor of alternative j in the strongest path from alternative $i \in A$ to alternative $j \in A \setminus \{i\}$.

$\mathcal{O}$ is the binary relation as defined in (2.2.1).

“*winner*[i] = *true*” if and only if $i \in \mathcal{S}$.

Stage 1 (initialization):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            P_D[i,j] : = (N[i,j],N[j,i])
            pred[i,j] : = i
        end
    end
end
```

Stage 2 (calculation of the strengths of the strongest paths):

```
for i : = 1 to C
begin
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            for k : = 1 to C
            begin
                if ( i ≠ k ) then
                begin
                    if ( j ≠ k ) then
                    begin
                        if ( P_D[j,k] ≺_D min_D { P_D[j,i], P_D[i,k] } ) then
                        begin
                            P_D[j,k] : = min_D { P_D[j,i], P_D[i,k] }
                            pred[j,k] : = pred[i,k]
                        end
                    end
                end
            end
        end
    end
end
```

Stage 3 (calculation of the binary relation $\mathcal{O}$ and the winners):

```
for i : = 1 to C
begin
    winner[i] : = true
    for j : = 1 to C
    begin
        if ( i ≠ j ) then
        begin
            if ( P_D[j,i] ≻_D P_D[i,j] ) then
            begin
                ji ∈ O
                winner[i] : = false
            end
            if ( P_D[j,i] ≾_D P_D[i,j] ) then
            begin
                ji ∉ O
            end
        end
    end
end
```

3. Examples

3.1. Example 1

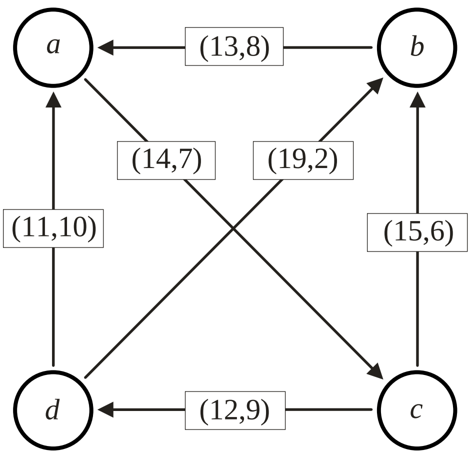
Example 1:

8 voters	$a \succ_v c \succ_v d \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v c$
4 voters	$c \succ_v d \succ_v b \succ_v a$
4 voters	$d \succ_v b \succ_v a \succ_v c$
3 voters	$d \succ_v c \succ_v b \succ_v a$

$N[i,j] \in \mathbb{N}_0$ is the number of voters who strictly prefer alternative $i \in A$ to alternative $j \in A \setminus \{i\}$. In example 1, the pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	8	14	10
$N[b,*]$	13	---	6	2
$N[c,*]$	7	15	---	12
$N[d,*]$	11	19	9	---

The following digraph illustrates the graph theoretic interpretation of pairwise elections. If $N[i,j] > N[j,i]$, then there is a link from vertex i to vertex j of strength $(N[i,j],N[j,i])$:



The above digraph can be used to determine the strengths of the strongest paths. In the following, "x, (Z_1,Z_2), y" means "$(N[x,y],N[y,x]) = (Z_1,Z_2)$".

$a \rightarrow b$: There are 2 paths from alternative a to alternative b.

Path 1: a, (14,7), c, (15,6), b
with a strength of $\min_D$ { (14,7), (15,6) } $\approx_D$ (14,7).

Path 2: a, (14,7), c, (12,9), d, (19,2), b
with a strength of $\min_D$ { (14,7), (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative a to alternative b is $\max_D$ { (14,7), (12,9) } $\approx_D$ (14,7).

$a \rightarrow c$: There is only one path from alternative a to alternative c.

Path 1: a, (14,7), c with a strength of (14,7).

$a \rightarrow d$: There is only one path from alternative a to alternative d.

Path 1: a, (14,7), c, (12,9), d
with a strength of $\min_D$ { (14,7), (12,9) } $\approx_D$ (12,9).

$b \rightarrow a$: There is only one path from alternative b to alternative a.

Path 1: b, (13,8), a with a strength of (13,8).

$b \rightarrow c$: There is only one path from alternative b to alternative c.

Path 1: b, (13,8), a, (14,7), c
with a strength of $\min_D$ { (13,8), (14,7) } $\approx_D$ (13,8).

$b \rightarrow d$: There is only one path from alternative b to alternative d.

Path 1: b, (13,8), a, (14,7), c, (12,9), d
with a strength of $\min_D$ { (13,8), (14,7), (12,9) } $\approx_D$ (12,9).

$c \rightarrow a$: There are 3 paths from alternative c to alternative a.

Path 1: c, (15,6), b, (13,8), a
with a strength of $\min_D$ { (15,6), (13,8) } $\approx_D$ (13,8).

Path 2: c, (12,9), d, (11,10), a
with a strength of $\min_D$ { (12,9), (11,10) } $\approx_D$ (11,10).

Path 3: c, (12,9), d, (19,2), b, (13,8), a
with a strength of $\min_D$ { (12,9), (19,2), (13,8) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative c to alternative a is $\max_D$ { (13,8), (11,10), (12,9) } $\approx_D$ (13,8).

$c \rightarrow b$: There are 2 paths from alternative c to alternative b.

Path 1: c, (15,6), b with a strength of (15,6).

Path 2: c, (12,9), d, (19,2), b
with a strength of $\min_D$ { (12,9), (19,2) } $\approx_D$ (12,9).

So the strength of the strongest path from alternative c to alternative b is $\max_D$ { (15,6), (12,9) } $\approx_D$ (15,6).

$c \rightarrow d$: There is only one path from alternative c to alternative d.

Path 1: c, (12,9), d with a strength of (12,9).

$d \rightarrow a$: There are 2 paths from alternative d to alternative a.

Path 1: d, (11,10), a with a strength of (11,10).

Path 2: d, (19,2), b, (13,8), a
with a strength of $\min_D$ { (19,2), (13,8) } $\approx_D$ (13,8).

So the strength of the strongest path from alternative d to alternative a is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

$d \rightarrow b$: There are 2 paths from alternative d to alternative b.

Path 1: d, (11,10), a, (14,7), c, (15,6), b
with a strength of $\min_D$ { (11,10), (14,7), (15,6) } $\approx_D$ (11,10).

Path 2: d, (19,2), b with a strength of (19,2).

So the strength of the strongest path from alternative d to alternative b is $\max_D$ { (11,10), (19,2) } $\approx_D$ (19,2).

$d \rightarrow c$: There are 2 paths from alternative d to alternative c.

Path 1: d, (11,10), a, (14,7), c
with a strength of $\min_D$ { (11,10), (14,7) } $\approx_D$ (11,10).

Path 2: d, (19,2), b, (13,8), a, (14,7), c
with a strength of $\min_D$ { (19,2), (13,8), (14,7) } $\approx_D$ (13,8).

So the strength of the strongest path from alternative d to alternative c is $\max_D$ { (11,10), (13,8) } $\approx_D$ (13,8).

The following table lists the strongest paths. The critical links of the strongest paths are underlined:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (14,7), *c*, (15,6), *b*	*a*, (14,7), *c*	*a*, (14,7), *c*, (12,9), *d*
from *b* ...	*b*, (13,8), *a*	---	*b*, (13,8), *a*, (14,7), *c*	*b*, (13,8), *a*, (14,7), *c*, (12,9), *d*
from *c* ...	*c*, (15,6), *b*, (13,8), *a*	*c*, (15,6), *b*	---	*c*, (12,9), *d*
from *d* ...	*d*, (19,2), *b*, (13,8), *a*	*d*, (19,2), *b*	*d*, (19,2), *b*, (13,8), *a*, (14,7), *c*	---

The strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(14,7)	(14,7)	(12,9)
$P_D[b,*]$	(13,8)	---	(13,8)	(12,9)
$P_D[c,*]$	(13,8)	(15,6)	---	(12,9)
$P_D[d,*]$	(13,8)	(19,2)	(13,8)	---

$xy \in \mathcal{O}$ if and only if $P_D[x,y] \succ_D P_D[y,x]$. So in example 1, we get $\mathcal{O} = \{ab, ac, cb, da, db, dc\}$.

$x \in \mathcal{S}$ if and only if $yx \notin \mathcal{O}$ for all $y \in A \setminus \{x\}$. So in example 1, we get $\mathcal{S} = \{d\}$.

3.2. Example 2

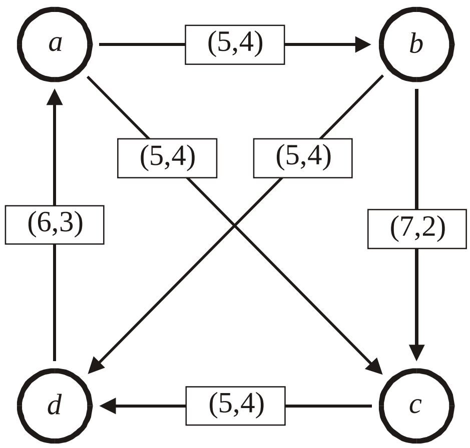
Example 2:

3 voters	$a \succ_v b \succ_v c \succ_v d$
2 voters	$c \succ_v b \succ_v d \succ_v a$
2 voters	$d \succ_v a \succ_v b \succ_v c$
2 voters	$d \succ_v b \succ_v c \succ_v a$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	5	5	3
$N[b,*]$	4	---	7	5
$N[c,*]$	4	2	---	5
$N[d,*]$	6	4	4	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to a	... to b	... to c	... to d
from a ...	---	a, $\underline{(5,4)}$, b	a, $\underline{(5,4)}$, c	a, $\underline{(5,4)}$, b, $\underline{(5,4)}$, d
from b ...	b, $\underline{(5,4)}$, d, (6,3), a	---	b, $\underline{(7,2)}$, c	b, $\underline{(5,4)}$, d
from c ...	c, $\underline{(5,4)}$, d, (6,3), a	c, $\underline{(5,4)}$, d, (6,3), a, $\underline{(5,4)}$, b	---	c, $\underline{(5,4)}$, d
from d ...	d, $\underline{(6,3)}$, a	d, (6,3), a, $\underline{(5,4)}$, b	d, (6,3), a, $\underline{(5,4)}$, c	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(5,4)	(5,4)	(5,4)
$P_D[b,*]$	(5,4)	---	(7,2)	(5,4)
$P_D[c,*]$	(5,4)	(5,4)	---	(5,4)
$P_D[d,*]$	(6,3)	(5,4)	(5,4)	---

We get $\mathcal{O} = \{bc, da\}$ and $\mathcal{S} = \{b, d\}$.

3.3. Example 3

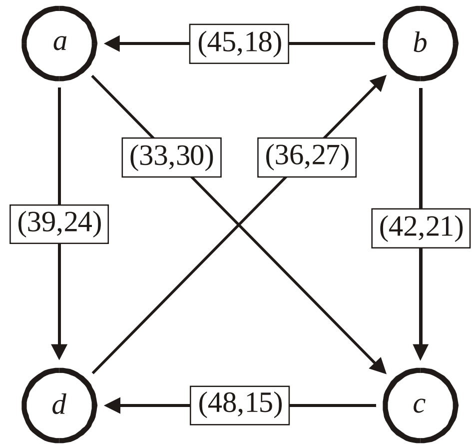
Example 3:

6 voters $a \succ_v b \succ_v c \succ_v d$
12 voters $a \succ_v c \succ_v d \succ_v b$
21 voters $b \succ_v c \succ_v a \succ_v d$
9 voters $c \succ_v d \succ_v b \succ_v a$
15 voters $d \succ_v b \succ_v a \succ_v c$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	18	33	39
$N[b,*]$	45	---	42	27
$N[c,*]$	30	21	---	48
$N[d,*]$	24	36	15	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (39,24), *d*, <u>(36,27)</u>, *b*	*a*, (39,24), *d*, <u>(36,27)</u>, *b*, (42,21), *c*	*a*, <u>(39,24)</u>, *d*
from *b* ...	*b*, <u>(45,18)</u>, *a*	---	*b*, <u>(42,21)</u>, *c*	*b*, <u>(42,21)</u>, *c*, (48,15), *d*
from *c* ...	*c*, (48,15), *d*, <u>(36,27)</u>, *b*, (45,18), *a*	*c*, (48,15), *d*, <u>(36,27)</u>, *b*	---	*c*, <u>(48,15)</u>, *d*
from *d* ...	*d*, <u>(36,27)</u>, *b*, (45,18), *a*	*d*, <u>(36,27)</u>, *b*	*d*, <u>(36,27)</u>, *b*, (42,21), *c*	---

Therefore, the strengths of the strongest paths are:

	$P_D[*,a]$	$P_D[*,b]$	$P_D[*,c]$	$P_D[*,d]$
$P_D[a,*]$	---	(36,27)	(36,27)	(39,24)
$P_D[b,*]$	(45,18)	---	(42,21)	(42,21)
$P_D[c,*]$	(36,27)	(36,27)	---	(48,15)
$P_D[d,*]$	(36,27)	(36,27)	(36,27)	---

We get $\mathcal{O} = \{ad, ba, bc, bd, cd\}$ and $\mathcal{S} = \{b\}$.

3.4. Example 4

3.4.1. Situation #1

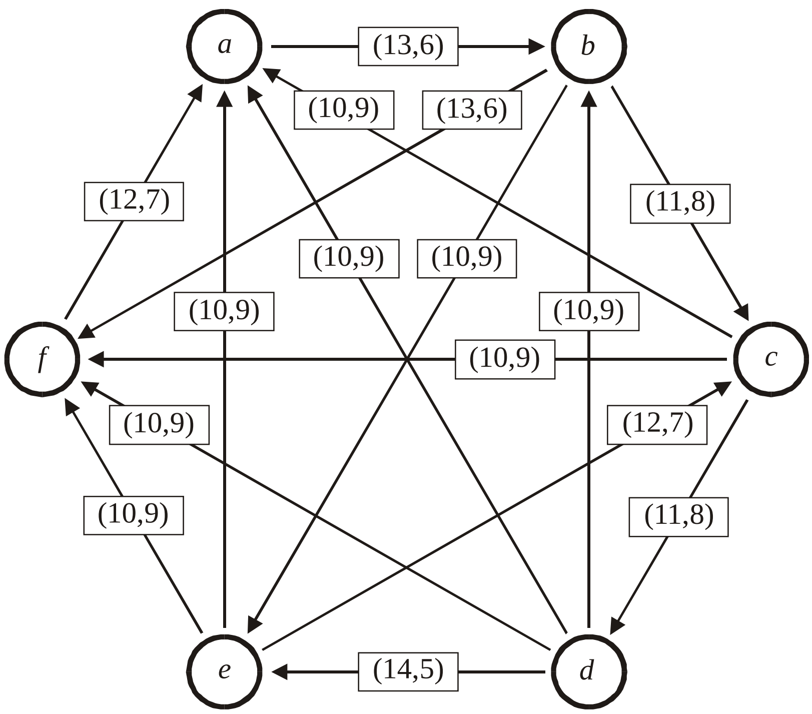
Example 4 (old):

3 voters	$a \succ_v d \succ_v e \succ_v b \succ_v c \succ_v f$
3 voters	$b \succ_v f \succ_v e \succ_v c \succ_v d \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v f \succ_v d \succ_v e$
1 voter	$d \succ_v b \succ_v c \succ_v e \succ_v f \succ_v a$
4 voters	$d \succ_v e \succ_v f \succ_v a \succ_v b \succ_v c$
2 voters	$e \succ_v c \succ_v b \succ_v d \succ_v f \succ_v a$
2 voters	$f \succ_v a \succ_v c \succ_v d \succ_v b \succ_v e$

The pairwise matrix N^{old} looks as follows:

	$N^{old}[*,a]$	$N^{old}[*,b]$	$N^{old}[*,c]$	$N^{old}[*,d]$	$N^{old}[*,e]$	$N^{old}[*,f]$
$N^{old}[a,*]$	---	13	9	9	9	7
$N^{old}[b,*]$	6	---	11	9	10	13
$N^{old}[c,*]$	10	8	---	11	7	10
$N^{old}[d,*]$	10	10	8	---	14	10
$N^{old}[e,*]$	10	9	12	5	---	10
$N^{old}[f,*]$	12	6	9	9	9	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*	... to *f*
from *a* ...	---	*a*, (13,6), *b*	*a*, (13,6), *b*, (11,8), *c*	*a*, (13,6), *b*, (11,8), *c*, (11,8), *d*	*a*, (13,6), *b*, (11,8), *c*, (11,8), *d*, (14,5), *e*	*a*, (13,6), *b*, (13,6), *f*
from *b* ...	*b*, (13,6), *f*, (12,7), *a*	---	*b*, (11,8), *c*	*b*, (11,8), *c*, (11,8), *d*	*b*, (11,8), *c*, (11,8), *d*, (14,5), *e*	*b*, (13,6), *f*
from *c* ...	*c*, (10,9), *a*	*c*, (10,9), *a*, (13,6), *b*	---	*c*, (11,8), *d*	*c*, (11,8), *d*, (14,5), *e*	*c*, (10,9), *f*
from *d* ...	*d*, (10,9), *a*	*d*, (10,9), *b*	*d*, (14,5), *e*, (12,7), *c*	---	*d*, (14,5), *e*	*d*, (10,9), *f*
from *e* ...	*e*, (10,9), *a*	*e*, (10,9), *a*, (13,6), *b*	*e*, (12,7), *c*	*e*, (12,7), *c*, (11,8), *d*	---	*e*, (10,9), *f*
from *f* ...	*f*, (12,7), *a*	*f*, (12,7), *a*, (13,6), *b*	*f*, (12,7), *a*, (13,6), *b*, (11,8), *c*	*f*, (12,7), *a*, (13,6), *b*, (11,8), *c*, (11,8), *d*	*f*, (12,7), *a*, (13,6), *b*, (11,8), *c*, (11,8), *d*, (14,5), *e*	---

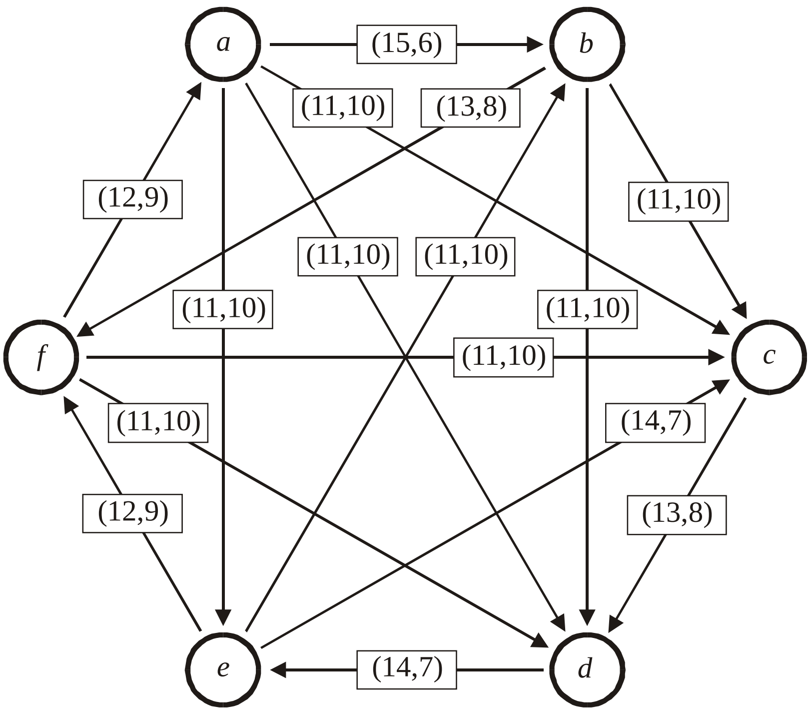
We get $\mathcal{O}^{\text{old}} = \{ab, ac, ad, ae, af, bc, bd, be, bf, dc, de, ec, fc, fd, fe\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

3.4.2. Situation #2

When 2 $a \succ_v e \succ_v f \succ_v c \succ_v b \succ_v d$ ballots are added, then the pairwise matrix N^{new} looks as follows:

	$N^{new}[*,a]$	$N^{new}[*,b]$	$N^{new}[*,c]$	$N^{new}[*,d]$	$N^{new}[*,e]$	$N^{new}[*,f]$
$N^{new}[a,*]$	---	15	11	11	11	9
$N^{new}[b,*]$	6	---	11	11	10	13
$N^{new}[c,*]$	10	10	---	13	7	10
$N^{new}[d,*]$	10	10	8	---	14	10
$N^{new}[e,*]$	10	11	14	7	---	12
$N^{new}[f,*]$	12	8	11	11	9	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*	... to *f*
from *a* ...	---	*a*, (15,6), *b*	*a*, (11,10), *c*	*a*, (11,10), *d*	*a*, (11,10), *e*	*a*, (15,6), *b*, (13,8), *f*
from *b* ...	*b*, (13,8), *f*, (12,9), *a*	---	*b*, (11,10), *c*	*b*, (11,10), *d*	*b*, (11,10), *d*, (14,7), *e*	*b*, (13,8), *f*
from *c* ...	*c*, (13,8), *d*, (14,7), *e*, (12,9), *f*, (12,9), *a*	*c*, (13,8), *d*, (14,7), *e*, (12,9), *f*, (12,9), *a*, (15,6), *b*	---	*c*, (13,8), *d*	*c*, (13,8), *d*, (14,7), *e*	*c*, (13,8), *d*, (14,7), *e*, (12,9), *f*
from *d* ...	*d*, (14,7), *e*, (12,9), *f*, (12,9), *a*	*d*, (14,7), *e*, (12,9), *f*, (12,9), *a*, (15,6), *b*	*d*, (14,7), *e*, (14,7), *c*	---	*d*, (14,7), *e*	*d*, (14,7), *e*, (12,9), *f*
from *e* ...	*e*, (12,9), *f*, (12,9), *a*	*e*, (12,9), *f*, (12,9), *a*, (15,6), *b*	*e*, (14,7), *c*	*e*, (14,7), *c*, (13,8), *d*	---	*e*, (12,9), *f*
from *f* ...	*f*, (12,9), *a*	*f*, (12,9), *a*, (15,6), *b*	*f*, (11,10), *c*	*f*, (11,10), *d*	*f*, (12,9), *a*, (11,10), *e*	---

We get $\mathcal{O}^{\text{new}}$ = {*ab*, *af*, *bf*, *ca*, *cb*, *cf*, *da*, *db*, *dc*, *de*, *df*, *ea*, *eb*, *ec*, *ef*} and $\mathcal{S}^{\text{new}}$ = {*d*}.

Thus the 2 $a \succ_v e \succ_v f \succ_v c \succ_v b \succ_v d$ voters change the winner from alternative *a* to alternative *d*.

3.5. Example 5

Suppose an alternative e is added with $N[d,e] > 0$ and $N[e,d] = 0$ for at least one already running alternative d. Then *independence from Pareto-dominated alternatives* (IPDA) says that we must get:

(3.5.1) $\forall\ x,y \in A \setminus \{e\}: xy \in \mathcal{O}^{\text{old}} \Leftrightarrow xy \in \mathcal{O}^{\text{new}}$.

(3.5.2) $\forall\ x \in A \setminus \{e\}: x \in \mathcal{S}^{\text{old}} \Leftrightarrow x \in \mathcal{S}^{\text{new}}$.

The following example demonstrates that the Schulze method, as defined in section 2.2, does not satisfy IPDA.

3.5.1. Situation #1

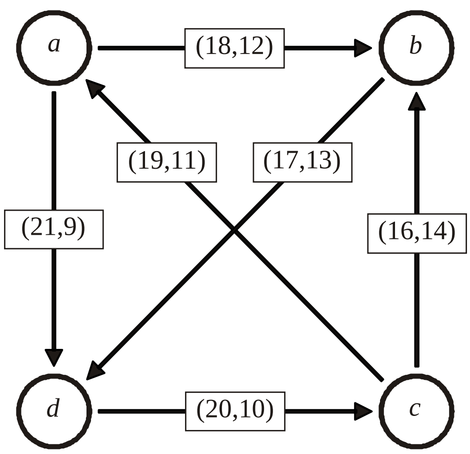
Example 5 (old):

3 voters	$a \succ_v b \succ_v d \succ_v c$
5 voters	$a \succ_v d \succ_v b \succ_v c$
1 voter	$a \succ_v d \succ_v c \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v c$
2 voters	$b \succ_v d \succ_v c \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v d$
6 voters	$c \succ_v b \succ_v a \succ_v d$
2 voters	$d \succ_v b \succ_v c \succ_v a$
5 voters	$d \succ_v c \succ_v a \succ_v b$

The pairwise matrix N^{old} looks as follows:

	$N^{\text{old}}[*,a]$	$N^{\text{old}}[*,b]$	$N^{\text{old}}[*,c]$	$N^{\text{old}}[*,d]$
$N^{\text{old}}[a,*]$	---	18	11	21
$N^{\text{old}}[b,*]$	12	---	14	17
$N^{\text{old}}[c,*]$	19	16	---	10
$N^{\text{old}}[d,*]$	9	13	20	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to a	... to b	... to c	... to d
from a ...	---	a, <u>(18,12)</u>, b	a, (21,9), d, <u>(20,10)</u>, c	a, <u>(21,9)</u>, d
from b ...	b, <u>(17,13)</u>, d, (20,10), c, (19,11), a	---	b, <u>(17,13)</u>, d, (20,10), c	b, <u>(17,13)</u>, d
from c ...	c, <u>(19,11)</u>, a	c, (19,11), a, <u>(18,12)</u>, b	---	c, <u>(19,11)</u>, a, (21,9), d
from d ...	d, (20,10), c, <u>(19,11)</u>, a	d, (20,10), c, (19,11), a, <u>(18,12)</u>, b	d, <u>(20,10)</u>, c	---

We get $\mathcal{O}^{\text{old}} = \{ab, ac, ad, cb, db, dc\}$ and $\mathcal{S}^{\text{old}} = \{a\}$.

3.5.2. Situation #2

Suppose alternative e is added as follows:

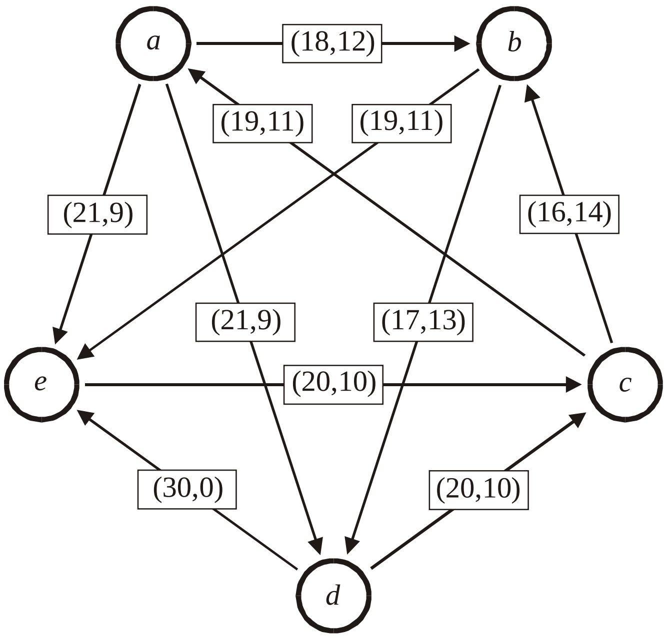
Example 5 (new):

3 voters	$a \succ_v b \succ_v d \succ_v e \succ_v c$
5 voters	$a \succ_v d \succ_v e \succ_v b \succ_v c$
1 voter	$a \succ_v d \succ_v e \succ_v c \succ_v b$
2 voters	$b \succ_v a \succ_v d \succ_v e \succ_v c$
2 voters	$b \succ_v d \succ_v e \succ_v c \succ_v a$
4 voters	$c \succ_v a \succ_v b \succ_v d \succ_v e$
6 voters	$c \succ_v b \succ_v a \succ_v d \succ_v e$
2 voters	$d \succ_v b \succ_v e \succ_v c \succ_v a$
5 voters	$d \succ_v e \succ_v c \succ_v a \succ_v b$

The pairwise matrix N^{new} looks as follows:

	$N^{\text{new}}[*,a]$	$N^{\text{new}}[*,b]$	$N^{\text{new}}[*,c]$	$N^{\text{new}}[*,d]$	$N^{\text{new}}[*,e]$
$N^{\text{new}}[a,*]$	---	18	11	21	21
$N^{\text{new}}[b,*]$	12	---	14	17	19
$N^{\text{new}}[c,*]$	19	16	---	10	10
$N^{\text{new}}[d,*]$	9	13	20	---	30
$N^{\text{new}}[e,*]$	9	11	20	0	---

The corresponding digraph looks as follows:



The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*	... to *e*
from *a* ...	---	*a*, <u>(18,12)</u>, *b*	*a*, (21,9), *d*, <u>(20,10)</u>, *c*	*a*, <u>(21,9)</u>, *d*	*a*, <u>(21,9)</u>, *e*
from *b* ...	*b*, <u>(19,11)</u>, *e*, (20,10), *c*, <u>(19,11)</u>, *a*	---	*b*, <u>(19,11)</u>, *e*, (20,10), *c*	*b*, <u>(19,11)</u>, *e*, (20,10), *c*, <u>(19,11)</u>, *a*, (21,9), *d*	*b*, <u>(19,11)</u>, *e*
from *c* ...	*c*, <u>(19,11)</u>, *a*	*c*, (19,11), *a*, <u>(18,12)</u>, *b*	---	*c*, <u>(19,11)</u>, *a*, (21,9), *d*	*c*, <u>(19,11)</u>, *a*, (21,9), *e*
from *d* ...	*d*, (20,10), *c*, <u>(19,11)</u>, *a*	*d*, (20,10), *c*, (19,11), *a*, <u>(18,12)</u>, *b*	*d*, <u>(20,10)</u>, *c*	---	*d*, <u>(30,0)</u>, *e*
from *e* ...	*e*, (20,10), *c*, <u>(19,11)</u>, *a*	*e*, (20,10), *c*, (19,11), *a*, <u>(18,12)</u>, *b*	*e*, <u>(20,10)</u>, *c*	*e*, (20,10), *c*, <u>(19,11)</u>, *a*, (21,9), *d*	---

We get $\mathcal{O}^{\text{new}} = \{ac, ad, ae, ba, bc, bd, be, dc, de, ec\}$ and $\mathcal{S}^{\text{new}} = \{b\}$.

3.6. Example 6

When each voter $v \in V$ casts a linear order $\succ_v$ on A, then all definitions for $\succ_D$, that satisfy presumption (2.1.1), are equivalent. However, when some voters cast non-linear orders, then there are many possible definitions for the strength of a link. The following example illustrates how the different definitions for the strength of a link can lead to different winners.

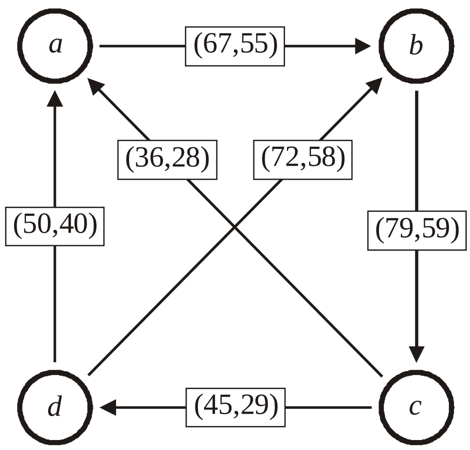
Example 6:

6	voters	$a \succ_v b \succ_v c \succ_v d$
8	voters	$a \approx_v b \succ_v c \approx_v d$
8	voters	$a \approx_v c \succ_v b \approx_v d$
18	voters	$a \approx_v c \succ_v d \succ_v b$
8	voters	$a \approx_v c \approx_v d \succ_v b$
40	voters	$b \succ_v a \approx_v c \approx_v d$
4	voters	$c \succ_v b \succ_v d \succ_v a$
9	voters	$c \succ_v d \succ_v a \succ_v b$
8	voters	$c \approx_v d \succ_v a \approx_v b$
14	voters	$d \succ_v a \succ_v b \succ_v c$
11	voters	$d \succ_v b \succ_v c \succ_v a$
4	voters	$d \succ_v c \succ_v a \succ_v b$

The pairwise matrix N looks as follows:

	$N[*,a]$	$N[*,b]$	$N[*,c]$	$N[*,d]$
$N[a,*]$	---	67	28	40
$N[b,*]$	55	---	79	58
$N[c,*]$	36	59	---	45
$N[d,*]$	50	72	29	---

The corresponding digraph looks as follows:



a) margin

We get: $(N[b,c],N[c,b]) \succ_{margin} (N[c,d],N[d,c]) \succ_{margin} (N[d,b],N[b,d]) \succ_{margin} (N[a,b],N[b,a]) \succ_{margin} (N[d,a],N[a,d]) \succ_{margin} (N[c,a],N[a,c])$.

The pairwise victories are:

bc with a margin of $N[b,c] - N[c,b] = 20$
cd with a margin of $N[c,d] - N[d,c] = 16$
db with a margin of $N[d,b] - N[b,d] = 14$
ab with a margin of $N[a,b] - N[b,a] = 12$
da with a margin of $N[d,a] - N[a,d] = 10$
ca with a margin of $N[c,a] - N[a,c] = 8$

The strongest paths are:

	... to a	... to b	... to c	... to d
from a ...	---	a, $\underline{(67,55)}$, b	a, $\underline{(67,55)}$, b, (79,59), c	a, $\underline{(67,55)}$, b, (79,59), c, (45,29), d
from b ...	b, (79,59), c, (45,29), d, $\underline{(50,40)}$, a	---	b, $\underline{(79,59)}$, c	b, (79,59), c, $\underline{(45,29)}$, d
from c ...	c, (45,29), d, $\underline{(50,40)}$, a	c, (45,29), d, $\underline{(72,58)}$, b	---	c, $\underline{(45,29)}$, d
from d ...	d, $\underline{(50,40)}$, a	d, $\underline{(72,58)}$, b	d, $\underline{(72,58)}$, b, (79,59), c	---

We get $\mathcal{O}_{margin} = \{ab, ac, ad, bc, bd, cd\}$ and $\mathcal{S}_{margin} = \{a\}$.

b) ratio

We get: $(N[c,d],N[d,c]) \succ_{ratio} (N[b,c],N[c,b]) \succ_{ratio} (N[c,a],N[a,c]) \succ_{ratio} (N[d,a],N[a,d]) \succ_{ratio} (N[d,b],N[b,d]) \succ_{ratio} (N[a,b],N[b,a])$.

The pairwise victories are:

cd with a ratio of $N[c,d] / N[d,c] = 1.552$
bc with a ratio of $N[b,c] / N[c,b] = 1.339$
ca with a ratio of $N[c,a] / N[a,c] = 1.286$
da with a ratio of $N[d,a] / N[a,d] = 1.250$
db with a ratio of $N[d,b] / N[b,d] = 1.241$
ab with a ratio of $N[a,b] / N[b,a] = 1.218$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (67,55), *b*	*a*, (67,55), *b*, (79,59), *c*	*a*, (67,55), *b*, (79,59), *c*, (45,29), *d*
from *b* ...	*b*, (79,59), *c*, (36,28), *a*	---	*b*, (79,59), *c*	*b*, (79,59), *c*, (45,29), *d*
from *c* ...	*c*, (36,28), *a*	*c*, (45,29), *d*, (72,58), *b*	---	*c*, (45,29), *d*
from *d* ...	*d*, (50,40), *a*	*d*, (72,58), *b*	*d*, (72,58), *b*, (79,59), *c*	---

We get $\mathcal{O}_{ratio}$ = {*ba*, *bc*, *bd*, *ca*, *cd*, *da*} and $\mathcal{S}_{ratio}$ = {*b*}.

c) winning votes

We get: $(N[b,c],N[c,b]) \succ_{win} (N[d,b],N[b,d]) \succ_{win} (N[a,b],N[b,a]) \succ_{win} (N[d,a],N[a,d]) \succ_{win} (N[c,d],N[d,c]) \succ_{win} (N[c,a],N[a,c])$.

The pairwise victories are:

bc with a support of $N[b,c]$ = 79
db with a support of $N[d,b]$ = 72
ab with a support of $N[a,b]$ = 67
da with a support of $N[d,a]$ = 50
cd with a support of $N[c,d]$ = 45
ca with a support of $N[c,a]$ = 36

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, (67,55), *b*	*a*, (67,55), *b*, (79,59), *c*	*a*, (67,55), *b*, (79,59), *c*, (45,29), *d*
from *b* ...	*b*, (79,59), *c*, (45,29), *d*, (50,40), *a*	---	*b*, (79,59), *c*	*b*, (79,59), *c*, (45,29), *d*
from *c* ...	*c*, (45,29), *d*, (50,40), *a*	*c*, (45,29), *d*, (72,58), *b*	---	*c*, (45,29), *d*
from *d* ...	*d*, (50,40), *a*	*d*, (72,58), *b*	*d*, (72,58), *b*, (79,59), *c*	---

We get $\mathcal{O}_{win}$ = {*ab*, *ac*, *bc*, *da*, *db*, *dc*} and $\mathcal{S}_{win}$ = {*d*}.

d) losing votes

We get: $(N[c,a],N[a,c]) \succ_{los} (N[c,d],N[d,c]) \succ_{los} (N[d,a],N[a,d]) \succ_{los} (N[a,b],N[b,a]) \succ_{los} (N[d,b],N[b,d]) \succ_{los} (N[b,c],N[c,b])$.

The pairwise victories are:

ca with an opposition of $N[a,c] = 28$
cd with an opposition of $N[d,c] = 29$
da with an opposition of $N[a,d] = 40$
ab with an opposition of $N[b,a] = 55$
db with an opposition of $N[b,d] = 58$
bc with an opposition of $N[c,b] = 59$

The strongest paths are:

	... to *a*	... to *b*	... to *c*	... to *d*
from *a* ...	---	*a*, $\underline{(67,55)}$, *b*	*a*, (67,55), *b*, $\underline{(79,59)}$, *c*	*a*, (67,55), *b*, $\underline{(79,59)}$, *c*, (45,29), *d*
from *b* ...	*b*, $\underline{(79,59)}$, *c*, (36,28), *a*	---	*b*, $\underline{(79,59)}$, *c*	*b*, $\underline{(79,59)}$, *c*, (45,29), *d*
from *c* ...	*c*, $\underline{(36,28)}$, *a*	*c*, (36,28), *a*, $\underline{(67,55)}$, *b*	---	*c*, $\underline{(45,29)}$, *d*
from *d* ...	*d*, $\underline{(50,40)}$, *a*	*d*, (50,40), *a* $\underline{(67,55)}$, *b*	*d*, (50,40), *a* (67,55), *b*, $\underline{(79,59)}$, *c*	---

We get $\mathcal{O}_{los} = \{ab, ca, cb, cd, da, db\}$ and $\mathcal{S}_{los} = \{c\}$.

4. Analysis of the Schulze Method

4.1. Transitivity

In this section, we will prove that the binary relation $\mathcal{O}$, as defined in (2.2.1), is *transitive*. This means: If $ab \in \mathcal{O}$ and $bc \in \mathcal{O}$, then $ac \in \mathcal{O}$. This guarantees that the set $\mathcal{S}$ of winners, as defined in (2.2.2), is non-empty. When we interpret the Schulze method as a method to find a set $\mathcal{S}$ of winners, rather than a method to generate a binary relation $\mathcal{O}$, then the proof of the transitivity of $\mathcal{O}$ is an essential part of the proof that the Schulze method is well defined.

Definition:

An election method satisfies *transitivity* if the following holds for all $a,b,c \in A$:

Suppose:

(4.1.1) $ab \in \mathcal{O}$.

(4.1.2) $bc \in \mathcal{O}$.

Then:

(4.1.3) $ac \in \mathcal{O}$.

Claim:

The binary relation $\mathcal{O}$, as defined in (2.2.1), is transitive.

Proof:

With (4.1.1), we get

(4.1.4) $P_D[a,b] \succ_D P_D[b,a]$.

With (4.1.2), we get

(4.1.5) $P_D[b,c] \succ_D P_D[c,b]$.

With (2.2.5), we get

(4.1.6) $\min_D \{ P_D[a,b], P_D[b,c] \} \precsim_D P_D[a,c]$.

(4.1.7) $\min_D \{ P_D[b,c], P_D[c,a] \} \precsim_D P_D[b,a]$.

(4.1.8) $\min_D \{ P_D[c,a], P_D[a,b] \} \precsim_D P_D[c,b]$.

Case 1: Suppose

(4.1.9a) $P_D[a,b] \succsim_D P_D[b,c]$.

Combining (4.1.5) and (4.1.9a) gives

(4.1.10a) $P_D[a,b] \succ_D P_D[c,b]$.

Combining (4.1.8) and (4.1.10a) gives

(4.1.11a) $P_D[c,a] \precsim_D P_D[c,b]$.

Combining (4.1.6) and (4.1.9a) gives

(4.1.12a) $P_D[b,c] \precsim_D P_D[a,c]$.

Combining (4.1.11a), (4.1.5), and (4.1.12a) gives

(4.1.13a) $P_D[c,a] \precsim_D P_D[c,b] \prec_D P_D[b,c] \precsim_D P_D[a,c]$.

With (4.1.13a), we get (4.1.3).

<u>Case 2:</u> Suppose

(4.1.9b) $P_D[a,b] \prec_D P_D[b,c]$.

Combining (4.1.4) and (4.1.9b) gives

(4.1.10b) $P_D[b,a] \prec_D P_D[b,c]$.

Combining (4.1.7) and (4.1.10b) gives

(4.1.11b) $P_D[c,a] \precsim_D P_D[b,a]$.

Combining (4.1.6) and (4.1.9b) gives

(4.1.12b) $P_D[a,b] \precsim_D P_D[a,c]$.

Combining (4.1.11b), (4.1.4), and (4.1.12b) gives

(4.1.13b) $P_D[c,a] \precsim_D P_D[b,a] \prec_D P_D[a,b] \precsim_D P_D[a,c]$.

With (4.1.13b), we get (4.1.3). □

The following corollary says that the set $\mathcal{S}$ of winners, as defined in (2.2.2), is non-empty:

<u>Corollary:</u>

For the Schulze method, as defined in section 2.2, we get

(4.1.14) $\forall\ b \notin \mathcal{S}\ \exists\ a \in \mathcal{S}: ab \in \mathcal{O}$.

<u>Proof of the corollary:</u>

As $b \notin \mathcal{S}$, there must be a $c(1) \in A$ with $c(1),b \in \mathcal{O}$.

If $c(1) \in \mathcal{S}$, then the corollary is proven. If $c(1) \notin \mathcal{S}$, then there must be a $c(2) \in A$ with $c(2),c(1) \in \mathcal{O}$. With the asymmetry and the transitivity of $\mathcal{O}$, we get $c(2),b \in \mathcal{O}$ and $c(2) \notin \{b, c(1)\}$.

We now proceed as follows: If $c(i) \in \mathcal{S}$, then the corollary is proven. If $c(i) \notin \mathcal{S}$, then there must be a $c(i+1) \in A$ with $c(i+1),c(i) \in \mathcal{O}$. With the asymmetry and the transitivity of $\mathcal{O}$, we get $c(i+1),b \in \mathcal{O}$ and $c(i+1) \notin \{b, c(1), ..., c(i)\}$.

We proceed until $c(i) \in \mathcal{S}$ for some $i \in \mathbb{N}$. Such an $i \in \mathbb{N}$ exists because A is finite. □

In example 2, we have $ba \notin \mathcal{O}$ and $ac \notin \mathcal{O}$ and $bc \in \mathcal{O}$. This shows that the Schulze relation, as defined in (2.2.1), is not necessarily negatively transitive.

4.2. Resolvability

Resolvability basically says that usually there is a unique winner $\mathcal{S} = \{a\}$. There are two different versions of the resolvability criterion. We will prove that the Schulze method, as defined in section 2.2, satisfies both.

4.2.1. Formulation #1

Definition:

An election method satisfies the first version of the *resolvability criterion* if (for every given number of alternatives) the proportion of profiles without a unique winner tends to zero as the number of voters in the profile tends to infinity.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the first version of the resolvability criterion.

Proof (overview):

Suppose $(x_1,x_2),(y_1,y_2) \in \mathbb{N}_0 \times \mathbb{N}_0$. Then, according to (2.1.1), there is a $v_1 \in \mathbb{N}_0$ such that for all $w_1 \in \mathbb{N}_0$:

1. $w_1 < v_1 \Rightarrow (x_1,x_2) \succ_D (w_1,y_2)$.

2. $w_1 > v_1 \Rightarrow (x_1,x_2) \prec_D (w_1,y_2)$.

When the number of voters tends to infinity (i.e. when x_1, x_2, y_1, and y_2 become large), then the proportion of profiles, where the condition "$y_1 = v_1$" happens to be satisfied, tends to zero. Therefore, when the number of voters tends to infinity, then the proportion of profiles, where two links ef and gh happen to have equivalent strengths $(N[e,f],N[f,e]) \approx_D (N[g,h],N[h,g])$, tends to zero.

Therefore, we will prove that, unless there are links ef and gh of equivalent strengths, there is always a unique winner. We will prove this by showing that, when we simultaneously presume (a) that there is more than one winner and (b) that there are no links ef and gh of equivalent strengths, then we necessarily get to a contradiction.

Proof (details):

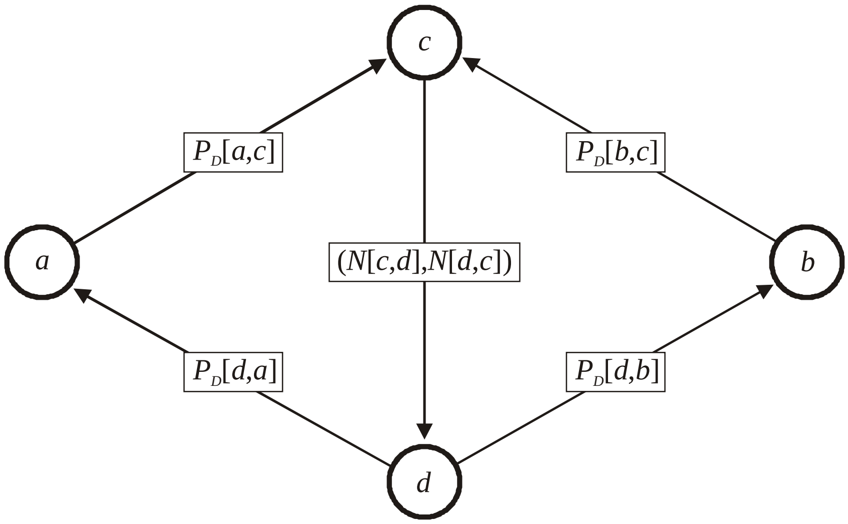
Suppose that there is more than one winner. Suppose alternative $a \in A$ and alternative $b \in A$ are winners. Then

(4.2.1.1) $\forall\ i \in A \setminus \{a\}$: $P_D[a,i] \succsim_D P_D[i,a]$.

(4.2.1.2) $\forall\ j \in A \setminus \{b\}$: $P_D[b,j] \succsim_D P_D[j,b]$.

(4.2.1.3) $P_D[a,b] \approx_D P_D[b,a]$.

Suppose there are no links *ef* and *gh* of equivalent strengths $(N[e,f],N[f,e]) \approx_D (N[g,h],N[h,g])$. Then $P_D[a,b] \approx_D P_D[b,a]$ means that the weakest link in the strongest path from alternative *a* to alternative *b* and the weakest link in the strongest path from alternative *b* to alternative *a* must be the same link, say *cd*. Therefore, the strongest paths have the following structure:



As *cd* is the weakest link in the strongest path from alternative *a* to alternative *b*, we get

(4.2.1.4) $P_D[a,d] \approx_D P_D[a,b]$.

(4.2.1.5) $P_D[d,b] \succ_D P_D[a,b]$.

As *cd* is the weakest link in the strongest path from alternative *b* to alternative *a*, we get

(4.2.1.6) $P_D[b,d] \approx_D P_D[b,a]$.

(4.2.1.7) $P_D[d,a] \succ_D P_D[b,a]$.

With (4.2.1.7), (4.2.1.3), and (4.2.1.4), we get

(4.2.1.8) $P_D[d,a] \succ_D P_D[b,a] \approx_D P_D[a,b] \approx_D P_D[a,d]$.

But (4.2.1.8) contradicts (4.2.1.1).

Similarly, with (4.2.1.5), (4.2.1.3), and (4.2.1.6), we get

(4.2.1.9) $P_D[d,b] \succ_D P_D[a,b] \approx_D P_D[b,a] \approx_D P_D[b,d]$.

But (4.2.1.9) contradicts (4.2.1.2). □

4.2.2. Formulation #2

The second version of the *resolvability criterion* says that, when there is more than one winner, then, for every alternative $a \in \mathcal{S}$, it is sufficient to add a single ballot w so that alternative a becomes the unique winner.

Definition:

An election method satisfies the second version of the *resolvability criterion* if the following holds:

$\forall\ a \in \mathcal{S}^{\text{old}}$: It is possible to construct a strict weak order w such that $\mathcal{S}^{\text{new}} = \{a\}$ for $V^{\text{new}} := V^{\text{old}} + \{w\}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the second version of the resolvability criterion.

Proof:

Suppose $a \in \mathcal{S}^{\text{old}}$. Then we get

(4.2.2.1) $\quad \forall\ b \in A \setminus \{a\}$: $P_D^{\text{old}}[a,b] \approx_D P_D^{\text{old}}[b,a]$.

Suppose the strict weak order w is chosen as follows:

(4.2.2.2) $\quad \forall\ f \in A \setminus \{a\}$: $pred^{\text{old}}[a,f] \succ_w f$.

(4.2.2.3) $\quad \forall\ e,f \in A \setminus \{a\}$: $(P_D^{\text{old}}[e,a] \succ_D P_D^{\text{old}}[f,a] \Rightarrow e \succ_w f)$.

With (4.2.2.2), we get

(4.2.2.4) $\quad \forall\ f \in A \setminus \{a\}$: $a \succ_w f$.

We will prove the following three claims:

Claim #1: It is not possible that (4.2.2.2) requires $e \succ_w f$ and that simultaneously (4.2.2.3) requires $f \succ_w e$.

Claim #2: $\forall\ g \in A \setminus \{a\}$: $P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]$.

Claim #3: $\forall\ g \in A \setminus \{a\}$: $P_D^{\text{new}}[g,a] \prec_D P_D^{\text{old}}[a,g]$.

With claim #2 and claim #3, we get

$P_D^{\text{new}}[a,g] \succ_D P_D^{\text{new}}[g,a]$ for all $g \in A \setminus \{a\}$

so that $ag \in \mathcal{O}^{\text{new}}$ for all $g \in A \setminus \{a\}$

so that $\mathcal{S}^{\text{new}} = \{a\}$.

Proof of claim #1:

Suppose $e,f \in A \setminus \{a\}$. With (2.2.3), we get

(4.2.2.5) $P_D^{\text{old}}[e,f] \approx_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e])$.

With (2.2.5), we get

(4.2.2.6) $\min_D \{ P_D^{\text{old}}[e,f], P_D^{\text{old}}[f,a] \} \precsim_D P_D^{\text{old}}[e,a]$.

With (4.2.2.1), we get

(4.2.2.7) $P_D^{\text{old}}[a,f] \approx_D P_D^{\text{old}}[f,a]$.

Suppose (4.2.2.2) requires $e \succ_w f$. Then $e = pred^{\text{old}}[a,f]$. Therefore, the link ef was in the strongest path from alternative a to alternative f. Thus, we get

(4.2.2.8) $P_D^{\text{old}}[a,f] \precsim_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e])$.

Suppose (4.2.2.3) requires $f \succ_w e$. Then

(4.2.2.9) $P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]$.

With (4.2.2.5), (4.2.2.8), (4.2.2.7), and (4.2.2.9), we get

(4.2.2.10) $P_D^{\text{old}}[e,f] \approx_D (N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \succsim_D P_D^{\text{old}}[a,f] \approx_D P_D^{\text{old}}[f,a] \succ_D P_D^{\text{old}}[e,a]$.

But (4.2.2.9) and (4.2.2.10) together contradict (4.2.2.6).

Proof of claim #2:

With (2.1.1) and (4.2.2.2), we get: The strength of each link of the strongest paths from alternative a to each other alternative $g \in A \setminus \{a\}$ is increased. Therefore

(4.2.2.11) $\forall\, g \in A \setminus \{a\}: P_D^{\text{new}}[a,g] \succ_D P_D^{\text{old}}[a,g]$.

Proof of claim #3:

Suppose $g \in A \setminus \{a\}$. Suppose

(4.2.2.12) $\quad \boldsymbol{\Upsilon}(g) := (\{a\} \cup \{ h \in A \setminus \{a\} \mid P_D^{\text{old}}[h,a] \succ_D P_D^{\text{old}}[a,g] \})$.

With (4.2.2.1) and (4.2.2.12), we get

(4.2.2.13) $\quad g \notin \boldsymbol{\Upsilon}(g)$ and $a \in \boldsymbol{\Upsilon}(g)$

and, therefore, $\varnothing \neq \boldsymbol{\Upsilon}(g) \subsetneq A$. Furthermore, we get

(4.2.2.14) $\quad \forall\, i \notin \boldsymbol{\Upsilon}(g)\ \forall\, j \in \boldsymbol{\Upsilon}(g)$: $(N^{\text{old}}[i,j],N^{\text{old}}[j,i]) \precsim_D P_D^{\text{old}}[a,g]$.

Otherwise, there was a path from alternative i to alternative a via alternative j with a strength of more than $P_D^{\text{old}}[a,g]$. But (as $i \notin \boldsymbol{\Upsilon}(g)$) this would contradict the definition of $\boldsymbol{\Upsilon}(g)$.

With (4.2.2.3), (4.2.2.4), and (4.2.2.12), we get

(4.2.2.15) $\quad \forall\, i \notin \boldsymbol{\Upsilon}(g)\ \forall\, j \in \boldsymbol{\Upsilon}(g)$: $j \succ_w i$.

With (2.1.1) and (4.2.2.15), we get

(4.2.2.16) $\quad \forall\, i \notin \boldsymbol{\Upsilon}(g)\ \forall\, j \in \boldsymbol{\Upsilon}(g)$: $(N^{\text{new}}[i,j],N^{\text{new}}[j,i]) \prec_D (N^{\text{old}}[i,j],N^{\text{old}}[j,i])$.

With (4.2.2.14) and (4.2.2.16), we get

(4.2.2.17) $\quad \forall\, i \notin \boldsymbol{\Upsilon}(g)\ \forall\, j \in \boldsymbol{\Upsilon}(g)$: $(N^{\text{new}}[i,j],N^{\text{new}}[j,i]) \prec_D P_D^{\text{old}}[a,g]$.

With (4.2.2.13) and (4.2.2.17), we get

(4.2.2.18) $\quad P_D^{\text{new}}[g,a] \prec_D P_D^{\text{old}}[a,g]$. □

In example 3, we have $\mathcal{O} = \{ad, ba, bc, bd, cd\}$ and $\mathcal{S} = \{b\}$. So we have $ac \notin \mathcal{O}$ and $ca \notin \mathcal{O}$, although there are no links of equivalent strengths. This non-linearity of $\mathcal{O}$ cannot be resolved by adding a single ballot w, since the difference between two different entries of the pairwise matrix N is always at least 3 votes. This demonstrates that the proofs of section 4.2 cannot be generalized to $\mathcal{O}$.

4.3. Pareto

The *Pareto criterion* says that the election method must respect unanimous opinions. There are two different versions of the Pareto criterion. The first version addresses situations with “ $a \succ_v b$ for all $v \in V$ ”, while the second version addresses situations with “ $a \succsim_v b$ for all $v \in V$ ” (for some pair of alternatives $a,b \in A$). The first version says that, when every voter strictly prefers alternative a to alternative b (i.e. $a \succ_v b$ for all $v \in V$), then alternative a must perform better than alternative b. The second version says that, when no voter strictly prefers alternative b to alternative a (i.e. $a \succsim_v b$ for all $v \in V$), then alternative b must not perform better than alternative a. We will prove that the Schulze method, as defined in section 2.2, satisfies both versions of the Pareto criterion.

4.3.1. Formulation #1

<u>Definition:</u>

An election method satisfies the first version of the *Pareto criterion* if the following holds:

Suppose:

(4.3.1.1) $\forall\, v \in V$: $a \succ_v b$.

Then:

(4.3.1.2) $ab \in \mathcal{O}$.

(4.3.1.3) $b \notin \mathcal{S}$.

<u>Claim:</u>

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the first version of the Pareto criterion.

<u>Proof:</u>

With (2.1.1) and (4.3.1.1), we get

(4.3.1.4) $\forall\, e,f \in A$: $(N[a,b],N[b,a]) \succsim_D (N[e,f],N[f,e])$.

(4.3.1.5) $[\ (N[a,b],N[b,a]) \approx_D (N[e,f],N[f,e])\ ] \Leftrightarrow [\ \forall\, v \in V\!: e \succ_v f\ ]$.

With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab.

However, as we presumed that the individual ballots $\succ_v$ are transitive and negatively transitive, it is not possible that there is a directed cycle of unanimous opinions. Therefore, it is not possible that the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab. Therefore, with (2.2.4), (4.3.1.4), and (4.3.1.5), we get (4.3.1.2). With (4.3.1.2), we get (4.3.1.3). □

4.3.2. Formulation #2

Definition:

An election method satisfies the second version of the *Pareto criterion* if the following holds:

Suppose:

(4.3.2.1) $\forall\, v \in V\text{: } a \succsim_v b$.

Then:

(4.3.2.2) $ba \notin \mathcal{O}$.

(4.3.2.3) $\forall\, f \in A \setminus \{a,b\}\text{: } bf \in \mathcal{O} \Rightarrow af \in \mathcal{O}$.

(4.3.2.4) $\forall\, f \in A \setminus \{a,b\}\text{: } fa \in \mathcal{O} \Rightarrow fb \in \mathcal{O}$.

(4.3.2.5) $b \in \mathcal{S} \Rightarrow a \in \mathcal{S}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies the second version of the Pareto criterion.

Proof:

With (4.3.2.1), we get

(4.3.2.6) $\forall\, e \in A \setminus \{a,b\}\text{: } N[a,e] \geq N[b,e]$.

With (4.3.2.1), we get

(4.3.2.7) $\forall\, e \in A \setminus \{a,b\}\text{: } N[e,b] \geq N[e,a]$.

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

(4.3.2.8) $\forall\, e \in A \setminus \{a,b\}\text{: } (N[a,e],N[e,a]) \succsim_D (N[b,e],N[e,b])$.

With (2.1.1), (4.3.2.6), and (4.3.2.7), we get

(4.3.2.9) $\forall\, e \in A \setminus \{a,b\}\text{: } (N[e,b],N[b,e]) \succsim_D (N[e,a],N[a,e])$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative b to alternative a. With (4.3.2.8) and (4.3.2.9), we get: $a,c(2),...,c(n-1),b$ is a path from alternative a to alternative b with at least the same strength. Therefore

(4.3.2.10) $P_D[a,b] \succsim_D P_D[b,a]$.

With (4.3.2.10), we get (4.3.2.2).

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative b to alternative $f \in A \setminus \{a,b\}$. With (4.3.2.8), we get: $a,c(m+1),...,c(n)$, where $c(m)$

is the last occurrence of an alternative of the set $\{a,b\}$, is a path from alternative a to alternative f with at least the same strength. Therefore

(4.3.2.11) $\forall\, f \in A \setminus \{a,b\}$: $P_D[a,f] \succsim_D P_D[b,f]$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative $f \in A \setminus \{a,b\}$ to alternative a. With (4.3.2.9), we get: $c(1),...,c(m–1),b$, where $c(m)$ is the first occurrence of an alternative of the set $\{a,b\}$, is a path from alternative f to alternative b with at least the same strength. Therefore

(4.3.2.12) $\forall\, f \in A \setminus \{a,b\}$: $P_D[f,b] \succsim_D P_D[f,a]$.

Part 1: Suppose $f \in A \setminus \{a,b\}$. Suppose

(4.3.2.13a) $bf \in \mathcal{O}$.

With (4.3.2.13a), we get

(4.3.2.14a) $P_D[b,f] \succ_D P_D[f,b]$.

With (4.3.2.11), (4.3.2.14a), and (4.3.2.12), we get

(4.3.2.15a) $P_D[a,f] \succsim_D P_D[b,f] \succ_D P_D[f,b] \succsim_D P_D[f,a]$.

With (4.3.2.15a), we get (4.3.2.3).

Part 2: Suppose $f \in A \setminus \{a,b\}$. Suppose

(4.3.2.13b) $fa \in \mathcal{O}$.

With (4.3.2.13b), we get

(4.3.2.14b) $P_D[f,a] \succ_D P_D[a,f]$.

With (4.3.2.12), (4.3.2.14b), and (4.3.2.11), we get

(4.3.2.15b) $P_D[f,b] \succsim_D P_D[f,a] \succ_D P_D[a,f] \succsim_D P_D[b,f]$.

With (4.3.2.15b), we get (4.3.2.4).

Part 3: Suppose

(4.3.2.13c) $b \in \mathcal{S}$.

With (4.3.2.13c), we get

(4.3.2.14c) $\forall\, f \in A \setminus \{b\}$: $fb \notin \mathcal{O}$.

With (4.3.2.4) and (4.3.2.14c), we get

(4.3.2.15c) $\forall\, f \in A \setminus \{a,b\}$: $fa \notin \mathcal{O}$.

With (4.3.2.2) and (4.3.2.15c), we get

(4.3.2.16c) $\forall\, f \in A \setminus \{a\}$: $fa \notin \mathcal{O}$.

With (4.3.2.16c), we get (4.3.2.5). □

4.4. Reversal Symmetry

Reversal symmetry as a criterion for single-winner election methods has been proposed by Saari (1994). This criterion says that, when $\succ_v$ is reversed for all $v \in V$, then also the result of the elections must be reversed; see (4.4.2). When alternative $a \in A$ was the unique winner in the original situation (i.e. $\mathcal{S}^{\text{old}} = \{a\}$), then alternative $a \in A$ should not be a winner in the reversed situation (i.e. $a \notin \mathcal{S}^{\text{new}}$); see (4.4.3). It should not be possible that the same alternatives are elected in the original situation and in the reversed situation, unless all alternatives are tied; see (4.4.4).

Basic idea of this criterion is that, when there is a vote on the best alternatives and then there is a vote on the worst alternatives and when in both cases the same alternatives are chosen, then this questions the logic of the underlying heuristic of the used election method.

<u>Definition:</u>

An election method satisfies *reversal symmetry* if the following holds:

Suppose:

(4.4.1) $\forall\, e,f \in A\ \forall\, v \in V\colon e \succ_v^{\text{old}} f \Leftrightarrow f \succ_v^{\text{new}} e.$

Then:

(4.4.2) $\forall\, a,b \in A\colon ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ba \in \mathcal{O}^{\text{new}}.$

(4.4.3) $(\ \exists\, i \in A\colon i \in \mathcal{S}^{\text{old}} \text{ and } i \notin \mathcal{S}^{\text{new}}\ ) \Leftrightarrow$
$(\ \exists\, j \in A\colon j \notin \mathcal{S}^{\text{old}} \text{ and } j \in \mathcal{S}^{\text{new}}\ ).$

(4.4.4) $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}} \Leftrightarrow \mathcal{S}^{\text{old}} = A.$

<u>Claim:</u>

The Schulze method, as defined in section 2.2, satisfies reversal symmetry.

Proof:

With (4.4.1), we get

(4.4.5) $\forall\ e,f \in A$: $N^{\text{old}}[e,f] = N^{\text{new}}[f,e]$.

With (4.4.5), we get

(4.4.6) $\forall\ e,f \in A$: $(N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \approx_D (N^{\text{new}}[f,e],N^{\text{new}}[e,f])$.

With (4.4.6), we get: When $c(1),...,c(n) \in A$ was a path from alternative $g \in A$ to alternative $h \in A \setminus \{g\}$, then $c(n),...,c(1)$ is a path from alternative h to alternative g with the same strength. Therefore

(4.4.7) $\forall\ g,h \in A$: $P_D^{\text{old}}[g,h] \approx_D P_D^{\text{new}}[h,g]$.

With (4.4.7), we get (4.4.2).

Part 1:

Suppose $\exists\ i \in A$: $i \in \mathcal{S}^{\text{old}}$ and $i \notin \mathcal{S}^{\text{new}}$. With $i \notin \mathcal{S}^{\text{new}}$ and (4.1.14), we get that there is a $j \in \mathcal{S}^{\text{new}}$ with $ji \in \mathcal{O}^{\text{new}}$. With (4.4.2), we get $ij \in \mathcal{O}^{\text{old}}$ and, therefore, $j \notin \mathcal{S}^{\text{old}}$. With $j \notin \mathcal{S}^{\text{old}}$ and $j \in \mathcal{S}^{\text{new}}$, we get the “$\Rightarrow$” direction of (4.4.3). The proof for the “$\Leftarrow$” direction of (4.4.3) is analogous.

Part 2:

Suppose $\mathcal{S}^{\text{old}} = A$. Then we get $\mathcal{O}^{\text{old}} = \emptyset$. Otherwise, if there was an $ij \in \mathcal{O}^{\text{old}}$, we would immediately get $j \notin \mathcal{S}^{\text{old}}$ and, therefore, $\mathcal{S}^{\text{old}} \neq A$. With $\mathcal{O}^{\text{old}} = \emptyset$ and (4.4.2), we get $\mathcal{O}^{\text{new}} = \emptyset$ and, therefore, $\mathcal{S}^{\text{new}} = A$. With $\mathcal{S}^{\text{old}} = A$ and $\mathcal{S}^{\text{new}} = A$, we get $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}}$.

Part 3:

Suppose $\mathcal{S}^{\text{old}} \neq A$. Then there is a $j \notin \mathcal{S}^{\text{old}}$. With (4.1.14), we get that there is an $i \in \mathcal{S}^{\text{old}}$ with $ij \in \mathcal{O}^{\text{old}}$. With (4.4.2), we get $ji \in \mathcal{O}^{\text{new}}$ and, therefore, $i \notin \mathcal{S}^{\text{new}}$. With $i \in \mathcal{S}^{\text{old}}$ and $i \notin \mathcal{S}^{\text{new}}$, we get $\mathcal{S}^{\text{old}} \neq \mathcal{S}^{\text{new}}$. With part 2 and part 3, we get (4.4.4). □

4.5. Monotonicity

Monotonicity says that, when some voters rank alternative $a \in A$ higher [see (4.5.1) and (4.5.2)] without changing the order in which they rank the other alternatives relatively to each other [see (4.5.3)], then this must not hurt alternative *a* [see (4.5.6)]. Monotonicity is also known as *mono-raise* and *non-negative responsiveness*.

Definition:

An election method satisfies *monotonicity* if the following holds:

Suppose $a \in A$. Suppose the ballots are modified in such a manner that the following three statements are satisfied:

(4.5.1) $\forall\, f \in A \setminus \{a\}\ \forall\, v \in V$: $a \succ_v^{\text{old}} f \Rightarrow a \succ_v^{\text{new}} f$.

(4.5.2) $\forall\, f \in A \setminus \{a\}\ \forall\, v \in V$: $a \succsim_v^{\text{old}} f \Rightarrow a \succsim_v^{\text{new}} f$.

(4.5.3) $\forall\, e,f \in A \setminus \{a\}\ \forall\, v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f$.

Then:

(4.5.4) $\forall\, b \in A \setminus \{a\}$: $ab \in \mathcal{O}^{\text{old}} \Rightarrow ab \in \mathcal{O}^{\text{new}}$.

(4.5.5) $\forall\, b \in A \setminus \{a\}$: $ba \notin \mathcal{O}^{\text{old}} \Rightarrow ba \notin \mathcal{O}^{\text{new}}$.

(4.5.6) $a \in \mathcal{S}^{\text{old}} \Rightarrow a \in \mathcal{S}^{\text{new}} \subseteq \mathcal{S}^{\text{old}}$.

Claim:

If $\succ_D$ satisfies (2.1.1), then the Schulze method, as defined in section 2.2, satisfies monotonicity.

Proof:

Part 1:

With (4.5.1), we get

(4.5.7) $\forall\, f \in A \setminus \{a\}$: $N^{\text{old}}[a,f] \leq N^{\text{new}}[a,f]$.

With (4.5.2), we get

(4.5.8) $\forall\, f \in A \setminus \{a\}$: $N^{\text{old}}[f,a] \geq N^{\text{new}}[f,a]$.

With (4.5.3), we get

(4.5.9) $\forall\, e,f \in A \setminus \{a\}$: $N^{\text{old}}[e,f] = N^{\text{new}}[e,f]$.

With (2.1.1), (4.5.7), and (4.5.8), we get

(4.5.10) $\forall\, f \in A \setminus \{a\}$: $(N^{\text{old}}[a,f],N^{\text{old}}[f,a]) \precsim_D (N^{\text{new}}[a,f],N^{\text{new}}[f,a])$.

With (2.1.1), (4.5.7), and (4.5.8), we get

(4.5.11) $\forall\, f \in A \setminus \{a\}$: $(N^{old}[f,a],N^{old}[a,f]) \gtrsim_D (N^{new}[f,a],N^{new}[a,f])$.

With (4.5.9), we get

(4.5.12) $\forall\, e,f \in A \setminus \{a\}$: $(N^{old}[e,f],N^{old}[f,e]) \approx_D (N^{new}[e,f],N^{new}[f,e])$.

Suppose $c(1),...,c(n) \in A$ was the strongest path from alternative a to alternative $b \in A \setminus \{a\}$. Then with (4.5.10) and (4.5.12), we get: $c(1),...,c(n)$ is a path from alternative a to alternative b with at least the same strength. Therefore

(4.5.13) $\forall\, b \in A \setminus \{a\}$: $P_D^{new}[a,b] \gtrsim_D P_D^{old}[a,b]$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative $b \in A \setminus \{a\}$ to alternative a. Then with (4.5.11) and (4.5.12), we get: $c(1),...,c(n)$ was a path from alternative b to alternative a with at least the same strength. Therefore

(4.5.14) $\forall\, b \in A \setminus \{a\}$: $P_D^{old}[b,a] \gtrsim_D P_D^{new}[b,a]$.

With (4.5.13) and (4.5.14), we get (4.5.4) and (4.5.5).

<u>Part 2:</u>

It remains to prove (4.5.6). Suppose $a \in \mathcal{S}^{old}$. Then “ $a \in \mathcal{S}^{new}$ ” follows directly from (4.5.5). To prove “ $\mathcal{S}^{new} \subseteq \mathcal{S}^{old}$ ”, we have to prove: $h \notin \mathcal{S}^{old} \Rightarrow h \notin \mathcal{S}^{new}$.

As $a \in \mathcal{S}^{old}$, we get

(4.5.15) $\forall\, b \in A \setminus \{a\}$: $P_D^{old}[a,b] \gtrsim_D P_D^{old}[b,a]$.

Suppose $h \notin \mathcal{S}^{old}$. Then there must have been an alternative $g \in A \setminus \{h\}$ with

(4.5.16) $P_D^{old}[g,h] \succ_D P_D^{old}[h,g]$.

With (4.5.10) – (4.5.12) and (4.5.16), we get: $P_D^{new}[g,h] \succ_D P_D^{new}[h,g]$, unless at least one of the following two cases occurred.

Case 1: xa was a weakest link in the strongest path from alternative g to alternative h.

Case 2: ay was the weakest link in the strongest path from alternative h to alternative g.

With (4.5.15), we get: $P_D^{old}[a,h] \gtrsim_D P_D^{old}[h,a]$. For $P_D^{old}[a,h] \succ_D P_D^{old}[h,a]$, we would, with (4.5.4), immediately get $P_D^{new}[a,h] \succ_D P_D^{new}[h,a]$, so that alternative h is still not a winner. Therefore, without loss of generality, we can presume $g \in A \setminus \{a,h\}$ and

(4.5.17) $P_D^{old}[a,h] \approx_D P_D^{old}[h,a]$.

With (4.5.15), we get

(4.5.18) $P^{\text{old}}_{D}[a,g] \succsim_D P^{\text{old}}_{D}[g,a]$.

With (2.2.5), we get

(4.5.19) $\min_D \{ P^{\text{old}}_{D}[g,h], P^{\text{old}}_{D}[h,a] \} \precsim_D P^{\text{old}}_{D}[g,a]$.

(4.5.20) $\min_D \{ P^{\text{old}}_{D}[h,a], P^{\text{old}}_{D}[a,g] \} \precsim_D P^{\text{old}}_{D}[h,g]$.

Case 1: Suppose *xa* was a weakest link in the strongest path from alternative *g* to alternative *h*. Then

(4.5.21a) $P^{\text{old}}_{D}[g,h] \approx_D P^{\text{old}}_{D}[g,a]$ and

(4.5.22a) $P^{\text{old}}_{D}[a,h] \succsim_D P^{\text{old}}_{D}[g,h]$.

Now (4.5.18), (4.5.21a), and (4.5.16) give

(4.5.23a) $P^{\text{old}}_{D}[a,g] \succsim_D P^{\text{old}}_{D}[g,a] \approx_D P^{\text{old}}_{D}[g,h] \succ_D P^{\text{old}}_{D}[h,g]$,

while (4.5.17), (4.5.22a), and (4.5.16) give

(4.5.24a) $P^{\text{old}}_{D}[h,a] \approx_D P^{\text{old}}_{D}[a,h] \succsim_D P^{\text{old}}_{D}[g,h] \succ_D P^{\text{old}}_{D}[h,g]$.

But (4.5.23a) and (4.5.24a) together contradict (4.5.20).

Case 2: Suppose *ay* was the weakest link in the strongest path from alternative *h* to alternative *g*. Then

(4.5.21b) $P^{\text{old}}_{D}[h,g] \approx_D P^{\text{old}}_{D}[a,g]$ and

(4.5.22b) $P^{\text{old}}_{D}[h,a] \succ_D P^{\text{old}}_{D}[h,g]$.

Now (4.5.22b), (4.5.21b), and (4.5.18) give

(4.5.23b) $P^{\text{old}}_{D}[h,a] \succ_D P^{\text{old}}_{D}[h,g] \approx_D P^{\text{old}}_{D}[a,g] \succsim_D P^{\text{old}}_{D}[g,a]$,

while (4.5.16), (4.5.21b), and (4.5.18) give

(4.5.24b) $P^{\text{old}}_{D}[g,h] \succ_D P^{\text{old}}_{D}[h,g] \approx_D P^{\text{old}}_{D}[a,g] \succsim_D P^{\text{old}}_{D}[g,a]$.

But (4.5.23b) and (4.5.24b) together contradict (4.5.19).

We have proven that neither case 1 nor case 2 is possible. Therefore

(4.5.25) $P^{\text{new}}_{D}[g,h] \succ_D P^{\text{new}}_{D}[h,g]$.

With (4.5.25), we get: $h \notin \mathcal{S}^{\text{new}}$. □

4.6. Independence of Clones

Independence of clones as a criterion for single-winner election methods has been proposed by Tideman (1987). This criterion says that running a large number of similar alternatives, so-called *clones*, must not have any impact on the result of the elections.

The precise definition for a *set of clones* stipulates that every voters ranks all the alternatives of this set in a consecutive manner; see (4.6.1) and (4.6.2). Replacing an alternative $d \in A^{\text{old}}$ by a set of clones K should not change the winner; see (4.6.7) and (4.6.8).

This criterion is very desirable especially for referendums because, while it might be difficult to find several candidates who are simultaneously sufficiently popular to campaign with them and sufficiently similar to misuse them for this strategy, it is usually very simple to formulate a large number of almost identical proposals. For example: In 1969, when the Canadian city that is now known as *Thunder Bay* was amalgamating, there was some controversy over what the name should be. In opinion polls, a majority of the voters preferred the name *The Lakehead* to the name *Thunder Bay*. But when the polls opened, there were three names on the referendum ballot: *Thunder Bay*, *Lakehead*, and *The Lakehead*. As the ballots were counted using *plurality voting*, it was not a surprise when *Thunder Bay* won. The votes were as follows: *Thunder Bay* 15870, *Lakehead* 15302, *The Lakehead* 8377.

Definition:

An election method is *independent of clones* if the following holds:

Suppose $d \in A^{\text{old}}$. Suppose $A^{\text{new}} := (A^{\text{old}} \cup K) \setminus \{d\}$.

Suppose alternative d is replaced by the set of alternatives K in such a manner that the following three statements are satisfied:

(4.6.1) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\ \forall\, v \in V$: $e \succ_v^{\text{old}} d \Leftrightarrow e \succ_v^{\text{new}} g$.

(4.6.2) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K\ \forall\, v \in V$: $d \succ_v^{\text{old}} f \Leftrightarrow g \succ_v^{\text{new}} f$.

(4.6.3) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}\ \forall\, v \in V$: $e \succ_v^{\text{old}} f \Leftrightarrow e \succ_v^{\text{new}} f$.

Then the following statements are satisfied:

(4.6.4) $\forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $ad \in \mathcal{O}^{\text{old}} \Leftrightarrow ag \in \mathcal{O}^{\text{new}}$.

(4.6.5) $\forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $db \in \mathcal{O}^{\text{old}} \Leftrightarrow gb \in \mathcal{O}^{\text{new}}$.

(4.6.6) $\forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $ab \in \mathcal{O}^{\text{old}} \Leftrightarrow ab \in \mathcal{O}^{\text{new}}$.

(4.6.7) $d \in \mathcal{S}^{\text{old}} \Leftrightarrow \mathcal{S}^{\text{new}} \cap K \neq \emptyset$.

(4.6.8) $\forall\, a \in A^{\text{old}} \setminus \{d\}$: $a \in \mathcal{S}^{\text{old}} \Leftrightarrow a \in \mathcal{S}^{\text{new}}$.

Claim:

The Schulze method, as defined in section 2.2, is independent of clones.

Proof:

With (4.6.1), we get

(4.6.9) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $N^{\text{old}}[e,d] = N^{\text{new}}[e,g]$.

With (4.6.2), we get

(4.6.10) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $N^{\text{old}}[d,f] = N^{\text{new}}[g,f]$.

With (4.6.3), we get

(4.6.11) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}$: $N^{\text{old}}[e,f] = N^{\text{new}}[e,f]$.

With (4.6.9) and (4.6.10), we get

(4.6.12) $\forall\, e \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $(N^{\text{old}}[e,d],N^{\text{old}}[d,e]) \approx_D (N^{\text{new}}[e,g],N^{\text{new}}[g,e])$.

With (4.6.9) and (4.6.10), we get

(4.6.13) $\forall\, f \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $(N^{\text{old}}[d,f],N^{\text{old}}[f,d]) \approx_D (N^{\text{new}}[g,f],N^{\text{new}}[f,g])$.

With (4.6.11), we get

(4.6.14) $\forall\, e,f \in A^{\text{old}} \setminus \{d\}$: $(N^{\text{old}}[e,f],N^{\text{old}}[f,e]) \approx_D (N^{\text{new}}[e,f],N^{\text{new}}[f,e])$.

Suppose $c(1),...,c(n) \in A^{\text{old}}$ was the strongest path from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative d. Then with (4.6.12) and (4.6.14), we get: $c(1),...,c(n-1),g$ is a path from alternative a to alternative $g \in K$ with the same strength. Therefore

(4.6.15) $\forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P^{\text{new}}_{D}[a,g] \succsim_D P^{\text{old}}_{D}[a,d]$.

Suppose $c(1),...,c(n) \in A^{\text{new}}$ is the strongest path from alternative $a \in A^{\text{new}} \setminus K$ to alternative $g \in K$. Then with (4.6.12) and (4.6.14), we get: $c(1),...,c(m-1),d$, where $c(m)$ is the first occurrence of an alternative of the set K, was a path from alternative a to alternative d with at least the same strength. Therefore

(4.6.16) $\forall\, a \in A^{\text{new}} \setminus K\ \forall\, g \in K$: $P^{\text{old}}_{D}[a,d] \succsim_D P^{\text{new}}_{D}[a,g]$.

Suppose $c(1),...,c(n) \in A^{\text{old}}$ was the strongest path from alternative d to alternative $b \in A^{\text{old}} \setminus \{d\}$. Then with (4.6.13) and (4.6.14), we get: $g,c(2),...,c(n)$ is a path from alternative $g \in K$ to alternative b with the same strength. Therefore

(4.6.17) $\forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P^{\text{new}}_{D}[g,b] \succsim_D P^{\text{old}}_{D}[d,b]$.

Suppose $c(1),...,c(n) \in A^{\text{new}}$ is the strongest path from alternative $g \in K$ to alternative $b \in A^{\text{new}} \setminus K$. Then with (4.6.13) and (4.6.14), we get: $d,c(m+1),...,c(n)$, where $c(m)$ is the last occurrence of an alternative of the set K, was a path from alternative d to alternative b with at least the same strength. Therefore

(4.6.18) $\forall\, b \in A^{\text{new}} \setminus K\ \forall\, g \in K$: $P^{\text{old}}_{D}[d,b] \succsim_D P^{\text{new}}_{D}[g,b]$.

(α) Suppose the strongest path $c(1),...,c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative $b \in A^{\text{old}} \setminus \{a,d\}$ did not contain alternative d. Then with (4.6.14), we get: $c(1),...,c(n)$ is still a path from alternative a to alternative b with the same strength. Therefore: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

(β) Suppose the strongest path $c(1),...,c(n) \in A^{\text{old}}$ from alternative $a \in A^{\text{old}} \setminus \{d\}$ to alternative $b \in A^{\text{old}} \setminus \{a,d\}$ contained alternative d. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),...,c(n)$, with alternative d replaced by an arbitrarily chosen alternative $g \in K$, is still a path from alternative a to alternative b with the same strength. Therefore: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

With (α) and (β), we get

(4.6.19) $\forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $P_D^{\text{new}}[a,b] \gtrsim_D P_D^{\text{old}}[a,b]$.

(γ) Suppose the strongest path $c(1),...,c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \setminus K$ to alternative $b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)$ does not contain alternatives of the set K. Then with (4.6.14), we get: $c(1),...,c(n)$ was a path from alternative a to alternative b with the same strength. Therefore: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{new}}[a,b]$.

(δ) Suppose the strongest path $c(1),...,c(n) \in A^{\text{new}}$ from alternative $a \in A^{\text{new}} \setminus K$ to alternative $b \in A^{\text{new}} \setminus (\, K \cup \{a\}\,)$ contains some alternatives of the set K. Then with (4.6.12), (4.6.13), and (4.6.14), we get: $c(1),...,c(s-1),d,c(t+1),...,c(n)$, where $c(s)$ is the first occurrence of an alternative of the set K and $c(t)$ is the last occurrence of an alternative of the set K, was a path from alternative a to alternative b with at least the same strength. Therefore: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{new}}[a,b]$.

With (γ) and (δ), we get

(4.6.20) $\forall\, a,b \in A^{\text{new}} \setminus K$: $P_D^{\text{old}}[a,b] \gtrsim_D P_D^{\text{new}}[a,b]$.

Combining (4.6.15) and (4.6.16) gives

(4.6.21) $\forall\, a \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P_D^{\text{old}}[a,d] \approx_D P_D^{\text{new}}[a,g]$.

Combining (4.6.17) and (4.6.18) gives

(4.6.22) $\forall\, b \in A^{\text{old}} \setminus \{d\}\ \forall\, g \in K$: $P_D^{\text{old}}[d,b] \approx_D P_D^{\text{new}}[g,b]$.

Combining (4.6.19) and (4.6.20) gives

(4.6.23) $\forall\, a,b \in A^{\text{old}} \setminus \{d\}$: $P_D^{\text{old}}[a,b] \approx_D P_D^{\text{new}}[a,b]$.

With (4.6.21) – (4.6.23), we get (4.6.4) – (4.6.6).

Part 1:

Suppose $d \in \mathcal{S}^{\text{old}}$. Then

(4.6.24) $\forall\, a \in A^{\text{old}} \setminus \{d\}$: $ad \notin \mathcal{O}^{\text{old}}$.

With (4.6.4) and (4.6.24), we get

(4.6.25) $\forall\, a \in A^{\text{new}} \setminus K\ \forall\, g \in K$: $ag \notin \mathcal{O}^{\text{new}}$.

Since the binary relation $\mathcal{O}^{\text{new}}$, as defined in (2.2.1), is asymmetric and transitive, there must be an alternative $k \in K$ with

(4.6.26) $\quad \forall\ l \in K \setminus \{k\}\colon lk \notin \mathcal{O}^{\text{new}}$.

With (4.6.25) and (4.6.26), we get $k \in \mathcal{S}^{\text{new}} \cap K$ and, therefore, $\mathcal{S}^{\text{new}} \cap K \neq \emptyset$.

Part 2:

Suppose $d \notin \mathcal{S}^{\text{old}}$. Then

(4.6.27) $\quad \exists\ a \in A^{\text{old}} \setminus \{d\}\colon ad \in \mathcal{O}^{\text{old}}$.

With (4.6.4) and (4.6.27), we get

(4.6.28) $\quad \exists\ a \in A^{\text{new}} \setminus K\ \forall\ g \in K\colon ag \in \mathcal{O}^{\text{new}}$.

With (4.6.28), we get: $\mathcal{S}^{\text{new}} \cap K = \emptyset$.

With part 1 and part 2, we get (4.6.7).

Part 3:

Suppose $a \in A^{\text{old}} \setminus \{d\}$ and $a \in \mathcal{S}^{\text{old}}$. Then

(4.6.29) $\quad da \notin \mathcal{O}^{\text{old}}$.

(4.6.30) $\quad \forall\ b \in A^{\text{old}} \setminus \{a,d\}\colon ba \notin \mathcal{O}^{\text{old}}$.

With (4.6.5) and (4.6.29), we get

(4.6.31) $\quad \forall\ g \in K\colon ga \notin \mathcal{O}^{\text{new}}$.

With (4.6.6) and (4.6.30), we get

(4.6.32) $\quad \forall\ b \in A^{\text{new}} \setminus (\ K \cup \{a\}\ )\colon ba \notin \mathcal{O}^{\text{new}}$.

With (4.6.31) and (4.6.32), we get: $a \in \mathcal{S}^{\text{new}}$.

Part 4:

Suppose $a \in A^{\text{old}} \setminus \{d\}$ and $a \notin \mathcal{S}^{\text{old}}$. Then at least one of the following two statements must have been valid:

(4.6.33a) $\quad da \in \mathcal{O}^{\text{old}}$.

(4.6.33b) $\quad \exists\ b \in A^{\text{old}} \setminus \{a,d\}\colon ba \in \mathcal{O}^{\text{old}}$.

With (4.6.5), (4.6.6), and (4.6.33), we get that at least one of the following two statements must be valid:

(4.6.34a) $\quad \forall\ g \in K\colon ga \in \mathcal{O}^{\text{new}}$.

(4.6.34b) $\quad \exists\ b \in A^{\text{new}} \setminus (\ K \cup \{a\}\ )\colon ba \in \mathcal{O}^{\text{new}}$.

With (4.6.34), we get: $a \notin \mathcal{S}^{\text{new}}$.

With part 3 and part 4, we get (4.6.8). □

4.7. Smith

The *Smith criterion* and *Smith-IIA* (where IIA means “independence of irrelevant alternatives”) say that *weak* alternatives should have no impact on the result of the elections.

Suppose:

(4.7.1) $\emptyset \neq B_1 \subsetneq A$, $\emptyset \neq B_2 \subsetneq A$, $B_1 \cup B_2 = A$, $B_1 \cap B_2 = \emptyset$.

(4.7.2) $\forall\, a \in B_1\ \forall\, b \in B_2$: $N[a,b] > N[b,a]$.

Then a *weak* alternative in the Smith paradigm is an alternative $b \in B_2$. Adding or removing a weak alternative $b \in B_2$ should have no impact on the set $\mathcal{S}$ of winners.

Definition:

An election method satisfies the *Smith criterion* if the following holds:

Suppose (4.7.1) and (4.7.2). Then:

(4.7.3) $\forall\, a \in B_1\ \forall\, b \in B_2$: $ab \in \mathcal{O}$.

(4.7.4) $\mathcal{S} \subseteq B_1$.

Remark:

If B_1 consists of only one alternative $a \in A$, then this is the so-called *Condorcet criterion*. If B_2 consists of only one alternative $b \in A$, then this is the so-called *Condorcet loser criterion*.

Claim:

If $\succ_D$ satisfies (2.1.2), then the Schulze method, as defined in section 2.2, satisfies the Smith criterion.

Proof:

The proof is trivial. Presumption (2.1.2) guarantees that any pairwise victory is stronger than any pairwise defeat. If $a \in B_1$ and $b \in B_2$, then already the link ab is a path from alternative a to alternative b that consists only of a pairwise victory. On the other side, (4.7.2) says that there cannot be a path from alternative b to alternative a that contains no pairwise defeat. So already the link ab is stronger than any path from alternative b to alternative a. □

Definition:

An election method satisfies *Smith-IIA* if the following holds:

Suppose (4.7.1) and (4.7.2). Then:

(4.7.5) If $d \in B_2$ is removed, then

(a) $\forall\ e,f \in B_1$: $ef \in \mathcal{O}^{\text{old}} \Leftrightarrow ef \in \mathcal{O}^{\text{new}}$.

(b) $\mathcal{S}^{\text{old}} = \mathcal{S}^{\text{new}}$.

(4.7.6) If $d \in B_1$ is removed, then
$\forall\ e,f \in B_2$: $ef \in \mathcal{O}^{\text{old}} \Leftrightarrow ef \in \mathcal{O}^{\text{new}}$.

Claim:

If $\succ_D$ satisfies (2.1.2), then the Schulze method, as defined in section 2.2, satisfies Smith-IIA.

Proof:

We will prove (4.7.5)(a). The proof for (4.7.6) is analogous.

(4.7.5)(b) follows directly from (4.7.4) and (4.7.5)(a).

Part 1: Suppose $e,f \in B_1$. Suppose $ef \in \mathcal{O}^{\text{old}}$. Then

(4.7.7) $P^{\text{old}}_D[e,f] \succ_D P^{\text{old}}_D[f,e]$.

With (2.2.3), we get

(4.7.8) $P^{\text{old}}_D[e,f] \succsim_D (N[e,f],N[f,e])$.

With (4.7.7) and (2.2.3), we get

(4.7.9) $P^{\text{old}}_D[e,f] \succ_D P^{\text{old}}_D[f,e] \succsim_D (N[f,e],N[e,f])$.

With (4.7.8) and (4.7.9), we get

(4.7.10) $P^{\text{old}}_D[e,f] \succsim_D \max_D \{\ (N[e,f],N[f,e]), (N[f,e],N[e,f])\ \}$.

With (4.7.2), we get: Any path from alternative $e \in B_1$ to alternative $f \in B_1$ that contained alternative $d \in B_2$ necessarily contained a pairwise defeat.

As it is not possible that the link ef is a pairwise defeat and that simultaneously the link fe is a pairwise defeat, $\max_D \{\ (N[e,f],N[f,e]), (N[f,e], N[e,f])\ \}$ is stronger than any pairwise defeat [because of (2.1.2)]. Therefore, with (4.7.2) and (4.7.10), we get: The strongest path from alternative $e \in B_1$ to alternative $f \in B_1$ did not contain alternative $d \in B_2$. Therefore

(4.7.11) $P^{\text{new}}_D[e,f] \approx_D P^{\text{old}}_D[e,f]$.

As the elimination of alternative $d \in B_2$ only removes paths, we get

(4.7.12) $P_D^{\text{new}}[f,e] \precsim_D P_D^{\text{old}}[f,e]$.

With (4.7.11), (4.7.7), and (4.7.12), we get

(4.7.13) $P_D^{\text{new}}[e,f] \approx_D P_D^{\text{old}}[e,f] \succ_D P_D^{\text{old}}[f,e] \succsim_D P_D^{\text{new}}[f,e]$.

With (4.7.13), we get: $ef \in \mathcal{O}^{\text{new}}$.

Part 2: The proof " $ef \notin \mathcal{O}^{\text{old}} \Rightarrow ef \notin \mathcal{O}^{\text{new}}$ " is analogous. □

The *majority criterion for solid coalitions* says that, when a majority of the voters strictly prefers every alternative of a given set of alternatives to every alternative outside this set of alternatives, then the winner must be chosen from this set. In short, an election method satisfies the *majority criterion for solid coalitions* if the following holds:

Suppose (4.7.1).
Suppose $\| \{ v \in V \mid \forall a \in B_1 \; \forall b \in B_2 : a \succ_v b \} \| > N/2$.
Then $\mathcal{S} \subseteq B_1$.

If B_1 consists of only one alternative $a \in A$, then this is the so-called *majority criterion*. If B_2 consists of only one alternative $b \in A$, then this is the so-called *majority loser criterion*.

Participation says that adding a list W of ballots, on which every alternative of a given set of alternatives is strictly preferred to every alternative outside this set, must not hurt the alternatives of this set. In short, an election method satisfies *participation* if the following holds:

Suppose (4.7.1).
Suppose $\forall a \in B_1 \; \forall b \in B_2 \; \forall w \in W : a \succ_w b$.
Suppose $V^{\text{new}} := V^{\text{old}} + W$.

Then (4.7.14) $\forall e \in B_1 \; \forall f \in B_2 : ef \in \mathcal{O}^{\text{old}} \Rightarrow ef \in \mathcal{O}^{\text{new}}$.
(4.7.15) $\forall e \in B_1 \; \forall f \in B_2 : fe \notin \mathcal{O}^{\text{old}} \Rightarrow fe \notin \mathcal{O}^{\text{new}}$.
(4.7.16) $\mathcal{S}^{\text{old}} \cap B_1 \neq \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_1 \neq \emptyset$.
(4.7.17) $\mathcal{S}^{\text{old}} \cap B_2 = \emptyset \Rightarrow \mathcal{S}^{\text{new}} \cap B_2 = \emptyset$.

The Smith criterion implies the majority criterion for solid coalitions, the Condorcet criterion, and the Condorcet loser criterion. The majority criterion for solid coalitions implies the majority criterion and the majority loser criterion. The Condorcet criterion implies the majority criterion. The Condorcet loser criterion implies the majority loser criterion. Unfortunately, the Condorcet criterion is incompatible with the participation criterion (Moulin, 1988). Example 4 shows a drastic violation of the participation criterion.

4.8. MinMax Set

For all $\emptyset \neq B \subsetneq A$, we define

(4.8.1) $\Gamma_D(B) := \max_D \{ (N[y,x],N[x,y]) \mid y \notin B, x \in B \}$.

Furthermore, we define

(4.8.2) $\beta_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \}$.

(4.8.3) $\mathfrak{B}_D := \cup \{ \emptyset \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$.

$\mathfrak{B}_D$ is the *MinMax set*. $\mathfrak{B}_D$ has the following properties:

1. $\mathfrak{B}_D \neq \emptyset$.

2. If $\mathfrak{B}_D$ consists of only one alternative $a \in A$, then alternative a is the unique Simpson-Kramer winner (i.e. that alternative $a \in A$ with minimum $\max_D \{ (N[b,a],N[a,b]) \mid b \in A \setminus \{a\} \}$).

3. If $d \in \mathfrak{B}_D$ is replaced by a set of alternatives K as described in (4.6.1) – (4.6.3), then $\mathfrak{B}_D^{\text{new}} = (\mathfrak{B}_D \cup K) \setminus \{d\}$.

4. If $d \notin \mathfrak{B}_D$ is replaced by a set of alternatives K as described in (4.6.1) – (4.6.3), then $\mathfrak{B}_D^{\text{new}} = \mathfrak{B}_D$.

So, in some sense, the MinMax set $\mathfrak{B}_D$ is a clone-proof generalization of the Simpson-Kramer winner.

When we want primarily that the used election method is independent of clones and secondarily that the strongest link *ef*, that is overruled when determining the winner, is minimized, then we have to demand that the winner is always chosen from the MinMax set $\mathfrak{B}_D$.

Claim:

The Schulze method, as defined in section 2.2, has the following properties:

(4.8.4) $\forall\, a \in \mathfrak{B}_D\ \forall\, b \notin \mathfrak{B}_D$: $ab \in \mathcal{O}$.

(4.8.5) $\mathcal{S} \subseteq \mathfrak{B}_D$.

Proof:

Suppose $a \in \mathfrak{B}_D$. Then we get

(4.8.6) $\exists\, \emptyset \neq B \subsetneq A$: $\Gamma_D(B) \approx_D \beta_D$ and $a \in B$.

Suppose $b \notin \mathfrak{B}_D$. Then we get

(4.8.7) $\gamma_D := \min_D \{ \Gamma_D(B) \mid \emptyset \neq B \subsetneq A \text{ and } b \in B \} \succ_D \beta_D$.

We will prove the following claims:

Claim #1: $P_D[b,a] \precsim_D \beta_D$.
Claim #2: $P_D[a,b] \succsim_D \gamma_D$.

With claim #1, claim #2, and (4.8.7), we get

(4.8.8) $P_D[a,b] \succsim_D \gamma_D \succ_D \beta_D \succsim_D P_D[b,a]$.

With (4.8.8), we get (4.8.4). With (4.8.4), we get (4.8.5).

Proof of claim #1:

With (4.8.6) and (4.8.7), we get

(4.8.9) $\exists\, \varnothing \neq B \subsetneq A$: $\Gamma_D(B) \approx_D \beta_D$ and $a \in B$ and $b \notin B$.

Suppose $c(1),...,c(n) \in A$ is the strongest path from alternative b to alternative a. Suppose $c(i)$ is the last alternative with $c(i) \notin B$. Then we get $(N[c(i),c(i+1)],N[c(i+1),c(i)]) \precsim_D \beta_D$. Therefore, we get

(4.8.10) $P_D[b,a] \precsim_D \beta_D$.

Proof of claim #2:

We can construct a path from alternative a to alternative b with a strength of at least γ_D as follows:

(1) We start with $E_1 := \{a\}$ and $i := 1$. Trivially, we get $b \notin E_1$ and $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_1 \setminus \{a\}$.

(2) At each stage, we consider the set $B_i := A \setminus E_i$.

With $b \in B_i$ and with (4.8.7), we get

(4.8.11) $\Gamma_D(B_i) \approx_D \max_D \{ (N[y,x],N[x,y]) \mid y \notin B_i, x \in B_i \} \succsim_D \gamma_D$.

We choose $f \in E_i$ and $g \in B_i$ with

(4.8.12) $(N[f,g],N[g,f]) \approx_D \max_D \{ (N[y,x],N[x,y]) \mid y \notin B_i, x \in B_i \} \succsim_D \gamma_D$.

We define $E_{i+1} := E_i \cup \{g\}$.

With $f \in E_i$, with $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_i \setminus \{a\}$, with $(N[f,g], N[g,f]) \succsim_D \gamma_D$, and with $E_{i+1} := E_i \cup \{g\}$, we get

(4.8.13) $P_D[a,h] \succsim_D \gamma_D$ for all $h \in E_{i+1} \setminus \{a\}$.

(3) We repeat stage 2 with $i \to i+1$, until $g \equiv b$.

Therefore, we get

(4.8.14) $P_D[a,b] \succsim_D \gamma_D$. □

Example 5 shows that IPDA and the desideratum, that the winner is always chosen from the MinMax set $\mathcal{B}_D$, are incompatible. In example 5(old), we get $\mathcal{B}_D^{\text{old}} = \{a, c, d\}$. In example 5(new), we get $\mathcal{B}_D^{\text{new}} = \{b\}$. Therefore, $\mathcal{B}_D^{\text{old}} \cap \mathcal{B}_D^{\text{new}} = \varnothing$. Thus, the desideratum, that the winner is always chosen from the MinMax set $\mathcal{B}_D$, implies that the winner is changed.

Actually, the Schulze method can be described completely with the desideratum to find a binary relation $\mathcal{O}$ on A that, primarily, is independent of clones (as defined in section 4.6) and that, secondarily, tries to rank the alternatives according to their worst defeats.

For all $a,b \in A$, we define

(4.8.15) $\gamma_D[a,b] := \min_D \{ \Gamma_D(B) \mid \varnothing \neq B \subsetneq A \text{ and } a \notin B \text{ and } b \in B \}$.

(4.8.16) $ab \in \mathcal{O} :\Leftrightarrow \gamma_D[a,b] \succ_D \gamma_D[b,a]$.

To prove that (4.8.16) is identical to (2.2.1), we have to prove $\gamma_D[a,b] = P_D[a,b]$. This proof is identical to the proof for (4.8.4).

4.9. Prudence

Prudence as a criterion for single-winner election methods has been popularized mainly by Arrow and Raynaud (1986). This criterion says that the strength λ_D of the strongest link ef, that is not supported by the binary relation $\mathcal{O}$, should be as small as possible. So $\lambda_D := \max_D \{ (N[e,f],N[f,e]) \mid ef \notin \mathcal{O} \}$ should be minimized.

When there is a directed cycle $c(1),...,c(n) \in A$ with $c(1) \equiv c(n)$, then it is obvious that the strongest link, that is not supported by the binary relation $\mathcal{O}$, is at least as strong as the weakest link $c(i),c(i+1)$ of this directed cycle. So we get:

(4.9.1) $\lambda_D \succsim_D \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \}$.

As we have to make this consideration for all directed cycles, the maximum, that we can ask for, is the following criterion.

Definition:

Suppose $\lambda_D \in \mathbb{N}_0 \times \mathbb{N}_0$ is the strength of the strongest directed cycle.

(4.9.2) $\lambda_D := \max_D \{ \min_D \{ (N[c(i),c(i+1)],N[c(i+1),c(i)]) \mid i = 1,...,(n-1) \} \mid c(1),...,c(n) \text{ is a path with } c(1) \equiv c(n) \}$.

Then an election method is *prudent* if the following holds:

(4.9.3) $\forall\, a,b \in A: (N[a,b],N[b,a]) \succ_D \lambda_D \Rightarrow ab \in \mathcal{O}$.

(4.9.4) $\forall\, a,b \in A: (N[a,b],N[b,a]) \succ_D \lambda_D \Rightarrow b \notin \mathcal{S}$.

Claim:

The Schulze method, as defined in section 2.2, is prudent.

Proof:

The proof is trivial. With (2.2.4), we get: $ab \in \mathcal{O}$, unless the link ab is in a directed cycle that consists of links of which each is at least as strong as the link ab. □

4.10. Schwartz

A *chain* from alternative $x \in A$ to alternative $y \in A$ is a sequence of alternatives $c(1),...,c(n) \in A$ with the following properties:

1. $x \equiv c(1)$.
2. $y \equiv c(n)$.
3. $2 \leq n < \infty$.
4. For all $i = 1,...,(n-1)$: $c(i) \not\equiv c(i+1)$.
5. For all $i = 1,...,(n-1)$: $N[c(i),c(i+1)] > N[c(i+1),c(i)]$.

Definition:

An election method satisfies the *Schwartz criterion* if the following holds:

Suppose there is a chain from alternative $a \in A$ to alternative $b \in A$ and no chain from alternative b to alternative a. Then:

(4.10.1) $ab \in \mathcal{O}$.

(4.10.2) $b \notin \mathcal{S}$.

Remark:

The Schwartz criterion has been proposed by Schwartz (1986). The Schwartz criterion implies the Smith criterion.

Claim:

If $\succ_D$ satisfies (2.1.2), then the Schulze method, as defined in section 2.2, satisfies the Schwartz criterion.

Proof:

The proof is trivial. □

5. Tie-Breaking

The Schulze relation O, as defined in (2.2.1), is only a strict partial order. However, sometimes, a linear order is needed. In this section, we will show how the Schulze relation O can be completed to a linear order without having to sacrifice any of the desired criteria.

Stage 1:

A *Tie-Breaking Ranking of the Candidates* (TBRC), a linear order $\succ_{TBRC}$ on A, is calculated as follows:

a) In the beginning: $a \approx_{TBRC} b \;\forall\; a,b \in A$.
b) Pick a random ballot $v \in V$ and use its rankings. (That means: $\forall\; a,b \in A$: If $a \approx_{TBRC} b$ and $a \succ_v b$, then replace “$a \approx_{TBRC} b$” by “$a \succ_{TBRC} b$”.)
c) Continue picking ballots randomly from those that have not yet been picked and use their rankings.
d) If you go through all ballots and there are still alternatives $a,b \in A$ with $b \in A \setminus \{a\}$ and $a \approx_{TBRC}$ b, then proceed as follows:

   d1) Pick a random alternative j and complete the TBRC in its favor. (That means: For all alternatives $k \in A \setminus \{j\}$ with $j \approx_{TBRC} k$: Replace “$j \approx_{TBRC} k$” by “$j \succ_{TBRC} k$”.)
   d2) Continue picking alternatives randomly from those that have not yet been picked and complete the TBRC in their favor.

When the bylaws require that the chairperson decides in the case of a tie, then the rules to create the TBRC have to be modified in such a manner that it is guaranteed that the ballot of the chairperson is always chosen first.

Stage 2:

A linear order $\succ_{second}$ of the $C\cdot(C–1)$ pairwise links is calculated.

<u>Variant 1:</u> $ij \succ_{second} mn$ if and only if at least one of the following conditions is satisfied:

1. $(N[i,j],N[j,i]) \succ_D (N[m,n],N[n,m])$.
2. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $i \succ_{TBRC} j$ and $n \succ_{TBRC} m$.
3. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $i \succ_{TBRC} j$ and $m \succ_{TBRC} n$ and $i \succ_{TBRC} m$.
4. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $j \succ_{TBRC} i$ and $n \succ_{TBRC} m$ and $i \succ_{TBRC} m$.
5. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $i \equiv m$ and $n \succ_{TBRC} j$.
6. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $j \equiv n$ and $i \succ_{TBRC} m$.

<u>Variant 2:</u> Alternatively, $ij \succ_{second} mn$ if and only if at least one of the following conditions is satisfied:

1. $(N[i,j],N[j,i]) \succ_D (N[m,n],N[n,m])$.
2. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $i \succ_{TBRC} j$ and $n \succ_{TBRC} m$.
3. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $i \succ_{TBRC} j$ and $m \succ_{TBRC} n$ and $n \succ_{TBRC} j$.
4. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $j \succ_{TBRC} i$ and $n \succ_{TBRC} m$ and $n \succ_{TBRC} j$.
5. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $i \equiv m$ and $n \succ_{TBRC} j$.
6. $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$ and $j \equiv n$ and $i \succ_{TBRC} m$.

The definition of $\succ_{second}$ is chosen in such a manner that e.g. when the TBRC $\succ_{TBRC}$ is *abcdefg* then links of otherwise equal strength, i.e. links with $(N[i,j],N[j,i]) \approx_D (N[m,n],N[n,m])$, are sorted *ag, af, ae, ad, ac, ab, bg, bf, be, bd, bc, cg, cf, ce, cd, dg, df, de, eg, ef, fg, ba, cb, ca, dc, db, da, ed, ec, eb, ea, fe, fd, fc, fb, fa, gf, ge, gd, gc, gb, ga* in variant 1 resp. *ag, bg, cg, dg, eg, fg, af, bf, cf, df, ef, ae, be, ce, de, ad, bd, cd, ac, bc, ab, gf, fe, ge, ed, fd, gd, dc, ec, fc, gc, cb, db, eb, fb, gb, ba, ca, da, ea, fa, ga* in variant 2, so that links that are in accordance with the TBRC are always stronger than links that are of otherwise equal strength and in contradiction with the TBRC.

Stage 3:

Suppose $\mathcal{O}_1$ and $\mathcal{O}_2$ are two linear orders. Then we write "$\mathcal{O}_1 \succ_{second} \mathcal{O}_2$" for "$\max_{second} \{ xy \in \mathcal{O}_1 \text{ and } xy \notin \mathcal{O}_2 \} \succ_{second} \max_{second} \{ xy \notin \mathcal{O}_1 \text{ and } xy \in \mathcal{O}_2 \}$".

Suppose $\mathcal{O}$ is the Schulze relation as defined in (2.2.1). Then the final Schulze ranking is that linear order $\mathcal{O}_{final}$ with

(1) $\mathcal{O} \subseteq \mathcal{O}_{final}$ and

(2) $\mathcal{O}_{final} \succ_{second} \mathcal{O}_{linear}$ for every other linear order $\mathcal{O}_{linear}$ with $\mathcal{O} \subseteq \mathcal{O}_{linear}$.

In example 3, we have $\mathcal{O} = \{ad, ba, bc, bd, cd\}$. The only linear orders, that contain $\mathcal{O}$, are $\mathcal{O}_1 = \{ac, ad, ba, bc, bd, cd\}$ and $\mathcal{O}_2 = \{ad, ba, bc, bd, ca, cd\}$. We get $\max_D \{ xy \in \mathcal{O}_1 \text{ and } xy \notin \mathcal{O}_2 \} = (N[a,c],N[c,a]) = (33,30)$ and $\max_D \{ xy \notin \mathcal{O}_1 \text{ and } xy \in \mathcal{O}_2 \} = (N[c,a],N[a,c]) = (30,33)$. Therefore, the final Schulze ranking is $\mathcal{O}_1 = \{ac, ad, ba, bc, bd, cd\}$.

6. Definition of the Strength of a Pairwise Link

There has been some debate about how to define $\succ_D$ when it is presumed that on the one side each voter has a sincere linear order of the alternatives, but on the other side some voters cast only a strict weak order because of strategic considerations. We got to the conclusion that the strength $(N[e,f], N[f,e])$ of the pairwise link $ef \in A \times A$ should be measured by *winning votes*, i.e. primarily by the support $N[e,f]$ of this link and secondarily by the opposition $N[f,e]$ to this link.

$(N[e,f],N[f,e]) \succ_{win} (N[g,h],N[h,g])$ if and only if at least one of the following conditions is satisfied:

1. $N[e,f] > N[f,e]$ and $N[g,h] \leq N[h,g]$.
2. $N[e,f] \geq N[f,e]$ and $N[g,h] < N[h,g]$.
3. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] > N[g,h]$.
4. $N[e,f] > N[f,e]$ and $N[g,h] > N[h,g]$ and $N[e,f] = N[g,h]$ and $N[f,e] < N[h,g]$.

Suppose $a,b \in A$. Suppose $R_1[a] := \| \{ v \in V \mid \forall c \in A \setminus \{a\}: a \succ_v c \} \|$ is the number of voters who strictly prefer alternative a to every other alternative. Suppose $R_2[b] := \| \{ v \in V \mid \exists c \in A \setminus \{b\}: b \succ_v c \} \|$ is the number of voters who strictly prefer alternative b to at least one other alternative. Suppose $R_1[a] > R_2[b]$. Then *Woodall’s plurality criterion* says: $b \notin \mathcal{S}$. Woodall (1997) writes: “If some candidate b has strictly fewer votes in total than some other candidate a has first-preference votes, then candidate b should not be elected.”

Claim:

If $\succ_{win}$ is being used, then the Schulze method satisfies Woodall’s plurality criterion.

Proof:

Suppose

(6.1) $R_1[a] > R_2[b]$.

With (6.1) and the definition for $\succ_{win}$, we get

(6.2) $(R_1[a],R_2[b]) \succ_{win} (R_2[b],0)$.

With the definitions for $R_1[a]$ and $R_2[b]$, we get

(6.3) $N[a,b] \geq R_1[a]$.

(6.4) $N[b,a] \leq R_2[b]$.

With (6.3), (6.4), and the definition for $\succ_{win}$, we get

(6.5) $(N[a,b],N[b,a]) \succsim_{win} (R_1[a],R_2[b])$.

With the definition for $R_2[b]$, we get

(6.6) $\forall c \in A \setminus \{b\}: N[b,c] \leq R_2[b]$.

With (6.6) and the definition for $\succ_{win}$, we get

(6.7) $\forall c \in A \setminus \{b\}: (N[b,c],N[c,b]) \precsim_{win} (R_2[b],0)$.

With (2.2.6) and (6.7), we get

(6.8) $P_{win}[b,a] \precsim_{win} (R_2[b],0)$.

With (2.2.3), (6.5), (6.2), and (6.8), we get

(6.9) $P_{win}[a,b] \succsim_{win} (N[a,b],N[b,a]) \succsim_{win} (R_1[a],R_2[b]) \succ_{win} (R_2[b],0) \succsim_{win} P_{win}[b,a]$

so that $ab \in \mathcal{O}$. □

7. Supermajority Requirements

When preferential ballots are being used in referendums, then sometimes alternatives have to fulfill some supermajority requirements to qualify. Typical supermajority requirements define some $M_1 \in \mathbb{N}$ or some $1 \leq M_2 \in \mathbb{R}$ and say that $N[a,b]$ must be strictly larger than max { $N[b,a]$, M_1 } or that $N[a,b]$ must be strictly larger than $M_2 \cdot N[b,a]$ to replace alternative $b \in A$ by alternative $a \in A$. Or they say that $N[a,b]$ must be strictly larger than $N[b,a]$ not only in the electorate as a whole, but also in a majority of its geographic parts or even in each of its geographic parts. It is also possible that in the same referendum the voters have to choose between alternatives that have to fulfill different supermajority requirements to qualify. In this section, we discuss a possible way to combine the Schulze method with supermajority requirements. Suppose $s \in A$ is the *status quo*.

These are the two tasks of supermajority requirements:

Task #1 (*protecting the status quo*):

> Supermajority requirements protect the status quo from accidental majorities. They make it more difficult to replace the status quo s by alternative $a \in A \setminus \{s\}$. Therefore, an important property of all supermajority requirements is that, when s had won in the absence of these requirements, then it also wins in the presence of these requirements.

Task #2 (*preventing the status quo from cycling*):

> Supermajority requirements prevent the status quo from cycling. Suppose $s(0)$ is the starting status quo. Suppose $s(k+1)$ is the new status quo when the method is applied to the same set of alternatives A, to the same set of ballots V, and to the status quo $s(k)$. Then we would expect that (for every possible set of alternatives A, for every possible set of ballots V, and for every possible starting status quo $s(0) \in A$) there is an $m < C$ such that $s(k) \equiv s(m)$ for all $k \geq m$.

We recommend the following method:

> The Schulze relation $\mathcal{O}$, as defined in (2.2.1), and the final Schulze ranking $\mathcal{O}_{final}$, as defined in section 5, are calculated.
>
> Alternative $a \in A \setminus \{s\}$ is *attainable* if and only if $N[a,s] > N[s,a]$ and (a) there is no supermajority requirement to replace the status quo s by alternative a or (b) alternative a has the supermajority required to replace the status quo s by alternative a.
>
> Alternative $a \in A$ is *eligible* if and only if ($a \equiv s$) or ((a is attainable) and ($as \in \mathcal{O}$)).
>
> A winner is an alternative $a \in A$ with (1) alternative a is eligible and (2) $ab \in \mathcal{O}_{final}$ for every other eligible alternative b.

The condition "$as \in \mathcal{O}$" in the definition of eligibility implies that alternative a can win only if it had disqualified the status quo s in the absence of supermajority requirements. This guarantees that, if s had won in the absence of supermajority requirements, then s also wins in the presence of these supermajority requirements.

In the above suggestion, the status quo s can only be replaced by an alternative a with $as \in \mathcal{O}$. As $\mathcal{O}$ is transitive, it is guaranteed that the status quo cannot be changed in a cyclic manner.

8. Electoral College

There has been some debate about how to combine the Schulze method with the Electoral College for the elections of the President of the USA. In our opinion, the Electoral College serves two important purposes:

> Purpose #1: The Electoral College gives more power to the smaller states.
>
> > The Senate, where each state has the same voting power regardless of its population, is more powerful than the House of Representatives, where each state has a voting power in proportion of its population. This is true especially for decisions that are close to the executive. For example, the President needs the consent of the Senate for treaties and for the appointment of officers and judges. Because of this reason, it is more important that the President has a reliable support in the Senate than that he has a reliable support in the House of Representatives.
>
> Purpose #2: The Electoral College makes it possible to count the ballots on the state levels and then to add the electoral votes up.
>
> > The Electoral College makes it possible that, to guarantee that all voters are treated in an equal manner, it is only necessary to guarantee that all voters *in the same state* are treated in an equal manner. However, if the ballots were added up on the national level, it would be necessary to guarantee that *all voters all over the USA* are treated in an equal manner. In the latter case, many provisions (e.g. the rules to gain suffrage and to be excluded from suffrage, the ballot access rules, the rules for postal voting, the opening hours of the polling places) would have to be harmonized all over the USA, leading to a very powerful central election authority.
> >
> > This property is desirable especially for the elections to the National Conventions for the nominations of the presidential candidates. Here, the election rules and the set of candidates differ significantly from state to state.

To combine the Schulze method with the Electoral College without losing any of its purposes, we recommend that, for each pair of candidates a and b separately, we should determine, how many electoral votes $N_{electors}[a,b]$ candidate a would get and how many electoral votes $N_{electors}[b,a]$ candidate b would get when only these two candidates were running. We then apply the Schulze method to the matrix $N_{electors}$.

So we recommend the following method:

Stage 1:

Suppose $A_X \subseteq A$ is the set of candidates who are running in state X.

For $a,b \in A_X$: $N_X[a,b] \in \mathbb{N}_0$ is the number of voters in state X who strictly prefer candidate a to candidate b.

Stage 2:

Suppose $y \in \mathbb{R}$ with $y > 0$. Then “smaller(y)” is the largest integer that is smaller than or equal to y. In other words: “smaller(y)” is that integer $z \in \mathbb{N}_0$ with $z \leq y < (z + 1)$.

Suppose $y \in \mathbb{R}$ with $y > 0$. Then “strictlysmaller(y)” is the largest integer that is strictly smaller than y. In other words: “strictlysmaller(y)” is that integer $z \in \mathbb{N}_0$ with $z < y \leq (z + 1)$.

Suppose $E_X \in \mathbb{N}$ is the number of electors of state X.

Suppose:

- $F_X[a,b] := E_X$, if $\{ a \in A_X$ and $b \notin A_X \}$ or $\{ a,b \in A_X$ and $N_X[a,b] > N_X[b,a] = 0 \}$.
- $F_X[a,b] := 0$, if $\{ a \notin A_X$ and $b \in A_X \}$ or $\{ a,b \in A_X$ and $N_X[b,a] > N_X[a,b] = 0 \}$.
- $F_X[a,b] := E_X / 2$, if $\{ a,b \notin A_X \}$ or $\{ a,b \in A_X$ and $N_X[a,b] = N_X[b,a] \}$.
- $F_X[a,b] := 0.01 \cdot \text{smaller}\left(\frac{N_X[a,b] \cdot (1 + 100 \cdot E_X)}{N_X[a,b] + N_X[b,a]} \right)$,
  if $a,b \in A_X$ and $N_X[a,b] > N_X[b,a] > 0$.
- $F_X[a,b] := 0.01 \cdot \text{strictlysmaller}\left(\frac{N_X[a,b] \cdot (1 + 100 \cdot E_X)}{N_X[a,b] + N_X[b,a]} \right)$,
  if $a,b \in A_X$ and $N_X[b,a] > N_X[a,b] > 0$.

$N_{electors}[a,b] := \sum_X F_X[a,b]$.

Stage 3:

The Schulze method, as defined in section 2.2, is applied to $N_{electors}$.

9. Comparison with other Methods

Table 9.2 compares the Schulze method with its main contenders. Extensive descriptions of the different methods can be found in publications by Fishburn (1977), Nurmi (1987), Kopfermann (1991), Levin and Nalebuff (1995), and Tideman (2006). As most of these methods only generate a set $\mathcal{S}$ of winners and don’t generate a binary relation $\mathcal{O}$, only that part of the different criteria is considered that refers to the set $\mathcal{S}$ of winners.

In terms of satisfied and violated criteria, that election method, that comes closest to the Schulze method, is Tideman’s ranked pairs method (Tideman, 1987). The only difference is that the ranked pairs method doesn’t choose from the MinMax set $\mathfrak{B}_D$.

The ranked pairs method works from the strongest to the weakest link. The link *xy* is locked if and only if it doesn’t create a directed cycle with already locked links. Otherwise, this link is locked in its opposite direction.

In example 1, the ranked pairs method locks *db*. Then it locks *cb*. Then it locks *ac*. Then it locks *ab*, since locking *ba* in its original direction would create a directed cycle with the already locked links *ac* and *cb*. Then it locks *cd*. Then it locks *ad*, since locking *da* in its original direction would create a directed cycle with the already locked links *ac* and *cd*.

The winner of the ranked pairs method is alternative $a \notin \mathfrak{B}_D = \{d\}$, because there is no locked link that ends in alternative *a*.

Although Tideman’s ranked pairs method is that election method that comes closest to the Schulze method in terms of satisfied and violated criteria, random simulations by Wright (2009) showed that that election method, that agrees the most frequently with the Schulze method, is the Simpson-Kramer method (table 9.1).

number of alternatives	A	B	C
3	100.0 %	100.0 %	100.0 %
4	99.7 %	98.5 %	98.2 %
5	99.2 %	96.0 %	95.3 %
6	99.1 %	93.0 %	92.3 %
7	98.9 %	90.0 %	89.1 %

Table 9.1: Simulations by Wright (2009)

A: Probability that the Schulze method conforms with the Simpson-Kramer method

B: Probability that the Schulze method conforms with the ranked pairs method

C: Probability that the ranked pairs method conforms with the Simpson-Kramer method

	resolvability	Pareto	reversal symmetry	monotonicity	independence of clones	Smith	Smith-IIA	Condorcet	Condorcet loser	majority for solid coalitions	majority	majority loser	participation	MinMax set	prudence	polynomial runtime
Baldwin	Y	Y	N	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
Black	Y	Y	Y	Y	N	N	N	Y	Y	N	Y	Y	N	N	N	Y
Borda	Y	Y	Y	Y	N	N	N	N	Y	N	N	Y	Y	N	N	Y
Bucklin	Y	Y	N	Y	N	N	N	N	N	Y	Y	Y	N	N	N	Y
Copeland	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	Y
Dodgson	Y	Y	N	N	N	N	N	Y	N	N	Y	N	N	N	N	N
instant runoff	Y	Y	N	N	Y	N	N	N	Y	Y	Y	Y	N	N	N	Y
Kemeny-Young	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Nanson	Y	Y	Y	N	N	Y	N	Y	Y	Y	Y	Y	N	N	N	Y
plurality	Y	Y	N	Y	N	N	N	N	N	N	Y	N	Y	N	N	Y
ranked pairs	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y
Schulze	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y
Simpson-Kramer	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	Y	Y
Slater	N	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	N	N	N	N
Young	Y	Y	N	Y	N	N	N	Y	N	N	Y	N	N	N	N	N

Table 9.2: Comparison of Election Methods

“Y” = compliance
“N” = violation

10. Discussion

Suppose $\Lambda_D(a) := \max_D \{ (N[b,a],N[a,b]) \mid b \in A \setminus \{a\} \}$ is the Simpson-Kramer score of alternative $a \in A$. Then the Simpson-Kramer method is defined as follows:

(10.1) $a \in \mathcal{S}_{SK} :\Leftrightarrow \Lambda_D(a) \precsim_D \Lambda_D(b)$ for all $b \in A \setminus \{a\}$.

Over a long period of time, this method was the most popular election method among Condorcet activists, because this method minimizes the number of overruled voters. However, a very serious problem of this method is that it is not independent of clones, because it can happen that, when alternative $a \in A$ is replaced by a set of clones K as described in (4.6.1) – (4.6.3), then the alternatives of the set K disqualify each other in such a manner that for some alternative $b \in A \setminus \{a\}$:

(10.2) $\Lambda_D^{\text{old}}(a) \prec_D \Lambda_D^{\text{old}}(b)$ and $\Lambda_D^{\text{new}}(b) \prec_D \Lambda_D^{\text{new}}(g)\ \forall\ g \in K$.

To make the Simpson-Kramer method clone-proof, the concept of Simpson-Kramer scores has to be generalized from individual alternatives $a \in A$ to sets of alternatives $\varnothing \neq B \subsetneq A$:

(10.3) $\Gamma_D(B) := \max_D \{ (N[b,a],N[a,b]) \mid b \notin B, a \in B \}$.

We get

(10.4) $\forall\ a \in A$: $\Lambda_D(a) \approx_D \Gamma_D(\{a\})$.

The Γ_D scores are clone-proof because, when alternative $a \in A$ is replaced by a set of clones K, then we get for all $\varnothing \neq B \subsetneq A$:

(10.5a) $a \in B \Rightarrow \Gamma_D^{\text{new}}((B \cup K) \setminus \{a\}) \approx_D \Gamma_D^{\text{old}}(B)$.

(10.5b) $a \notin B \Rightarrow \Gamma_D^{\text{new}}(B) \approx_D \Gamma_D^{\text{old}}(B)$.

Suppose $\beta_D := \min_D \{ \Gamma_D(B) \mid \varnothing \neq B \subsetneq A \}$ and $\mathfrak{B}_D := \cup \{ \varnothing \neq B \subsetneq A \mid \Gamma_D(B) \approx_D \beta_D \}$. Then when we want primarily that the used election method is clone-proof and secondarily that it minimizes the number of overruled voters, then the maximum, that we can ask for, is

(10.6) $\mathcal{S} \subseteq \mathfrak{B}_D$.

In this paper, we propose a new single-winner election method (*Schulze method*) that is clone-proof (section 4.6) and that always chooses from the MinMax set $\mathfrak{B}_D$ (section 4.8). The latter property is the most characteristic property of the Schulze method, since this is the first time that an election method with this property is proposed.

The Schulze method also satisfies many other criteria; some of them are also satisfied by the Simpson-Kramer method, like the Pareto criterion (section 4.3), resolvability (section 4.2), monotonicity (section 4.5), and prudence (section 4.9); some of them are violated by the Simpson-Kramer method, like the Smith criterion (section 4.7) and reversal symmetry (section 4.4). Because of this large number of satisfied criteria, we consider the

Schulze method to be a promising alternative to the Simpson-Kramer method for actual implementations, especially when manipulation through clones or weak alternatives is an issue.

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