

Calcul02.pdf

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Abstract. In this paper we illustrate the calculation of the strengths of the vote managements.

Introduction

We use instance A53 with $N = 460$ voters, $C = 10$ candidates, and $M = 4$ seats to illustrate the calculation of the strengths of the vote managements. To be more concrete, we are looking for that situation where the candidates $\{b, c, e, j\}$ are running a vote management strategy against candidate a . In file *a53_stv.dat*, we can see that the strength of this vote management is

$$r (V(\overline{a, \{b, c, e, j\}})) = 77.389937$$

This entry can be found here:

11	A	B	C	E	J	77.389937	99.563758	107.281879	101.107383	69.463087
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When the used multi-winner election method satisfies Droop proportionality, then vote management of the candidates b_1, \dots, b_M against candidate a is possible at least when the electorate can be divided into $M + 1$ disjoint sets B_1, \dots, B_M, B_{M+1} such that

- (1) $\forall k \in \{1, \dots, M\} : |B_k| > N / (M + 1)$.
- (2) $\forall k \in \{1, \dots, M\}$: Every voter in B_k strictly prefers candidate b_k to candidate a .

The candidates b_1, \dots, b_M can then ask each voter of set B_k to give his first preference to candidate b_k , so that all the candidates of the set $\{b_1, \dots, b_M\}$ are necessarily elected according to Droop proportionality.

Suppose r is the maximum value such that the electorate can be divided into $M + 1$ disjoint sets B_1, \dots, B_M, B_{M+1} such that

- (1) $\forall k \in \{1, \dots, M\} : |B_k| \geq r$.
- (2) $\forall k \in \{1, \dots, M\}$: Every voter in B_k strictly prefers candidate b_k to candidate a .

Then, r is a measure for the probability of success of this vote management. Therefore, it makes sense to define the *strength* of this vote management by r .

Suppose $t_{xy} \in \mathbb{R}$ is the share of voter x for candidate b_y . Then, to calculate r , we have to solve the following linear program LP:

Find the maximum $r \in \mathbb{R}$ such that there is a $t \in \mathbb{R}^{N \times M}$ such that

$$(C1) \quad \forall x \in \{1, \dots, N\} \quad \forall y \in \{1, \dots, M\}: t_{xy} \geq 0.$$

$$(C2) \quad \forall x \in \{1, \dots, N\}: \sum_{y=1}^M t_{xy} \leq 1.$$

$$(C3) \quad \forall x \in \{1, \dots, N\} \quad \forall y \in \{1, \dots, M\}: a \succ_x b_y \Rightarrow t_{xy} = 0.$$

$$(C4) \quad \forall y \in \{1, \dots, M\}: \sum_{x=1}^N t_{xy} \geq r.$$

We define the following linear program LP*:

Find the maximum $r^* \in \mathbb{R}$ such that there is a $t^* \in \mathbb{R}^{N \times M}$ such that

$$(C1) \quad \forall x \in \{1, \dots, N\} \quad \forall y \in \{1, \dots, M\}: t^*_{xy} \geq 0.$$

$$(C2) \quad \forall x \in \{1, \dots, N\}: \sum_{y=1}^M t^*_{xy} \leq 1.$$

$$(C3) \quad \forall x \in \{1, \dots, N\} \quad \forall y \in \{1, \dots, M\}: a \succ_x b_y \Rightarrow t^*_{xy} = 0.$$

$$(C4^*) \quad \sum_{x=1}^N \sum_{y=1}^M t^*_{xy} \geq M \cdot r^*.$$

As condition (C4) implies condition (C4*), we get: $r \leq r^*$.

Suppose t^* is a solution of LP*. Then we define:

$$s^* := \min \left\{ \sum_{x=1}^N t^*_{xy} \mid 1 \leq y \leq M \right\}.$$

When we replace t by t^* and r by s^* , then the conditions (C1) -- (C4) of LP are satisfied. Therefore, as r is (by definition) the maximum value such that the conditions (C1) -- (C4) can be satisfied, we get: $s^* \leq r$.

Thus, we get: $s^* \leq r \leq r^*$.

Compared to LP, LP* has the advantage that it describes a trivial max-flow problem. A max-flow problem can be solved significantly more easily than a general linear program. Therefore, we solve the linear program LP by solving a series of max-flow problems as follows:

Suppose \mathfrak{W} is the number of voters who strictly prefer candidate a to every candidate of the set $\{b_1, \dots, b_M\}$. Then we know that r cannot be larger than $(N - \mathfrak{W}) / M$.

Therefore, we start with

$$r^{(0)} := (N - \mathfrak{W}) / M.$$

$$s^{(0)} := 0.$$

For $z = 1, 2, 3, \dots$, we solve the following linear programs $\text{LP}^{(z)}$:

Find the maximum $r^{(z)} \in \mathbb{R}$ such that there is a $t^{(z)} \in \mathbb{R}^{N \times M}$ such that

$$(C1) \quad \forall x \in \{1, \dots, N\} \quad \forall y \in \{1, \dots, M\}: t_{xy}^{(z)} \geq 0.$$

$$(C2) \quad \forall x \in \{1, \dots, N\}: \sum_{y=1}^M t_{xy}^{(z)} \leq 1.$$

$$(C3) \quad \forall x \in \{1, \dots, N\} \quad \forall y \in \{1, \dots, M\}: a \succ_x b_y \Rightarrow t_{xy}^{(z)} = 0.$$

$$(C4^*) \quad \sum_{x=1}^N \sum_{y=1}^M t_{xy}^{(z)} \geq M \cdot r^{(z)}.$$

$$(C5) \quad \forall y \in \{1, \dots, M\}: \sum_{x=1}^N t_{xy}^{(z)} \leq r^{(z-1)}.$$

Furthermore, we define for $z = 1, 2, 3, \dots$:

$$s^{(z)} := \min \left\{ \sum_{x=1}^N t_{xy}^{(z)} \mid 1 \leq y \leq M \right\}.$$

The idea behind condition (C5) is: As condition (C4) implies condition (C4*), we have already found out in $\text{LP}^{(z-1)}$ that r cannot be larger than $r^{(z-1)}$. Therefore, if there is a $1 \leq y \leq M$ with $\sum_{x=1}^N t_{xy}^{(z)} > r^{(z-1)}$, then this surplus is wasted and cannot contribute to any further increase of $s^{(z)}$.

When we solve $\text{LP}^{(1)}, \text{LP}^{(2)}, \text{LP}^{(3)}, \dots$, then we get a decreasing sequence $r^{(0)}, r^{(1)}, r^{(2)}, r^{(3)}, \dots$ and an increasing sequence $s^{(0)}, s^{(1)}, s^{(2)}, s^{(3)}, \dots$. These two sequences converge to the same limit. This limit is the solution of LP.

Now, we use this algorithm to calculate the strength of the vote management of the candidates b, c, e, j against candidate a in instance A53. After proportional completion, the voter profile looks as follows:

		b	c	e	j
voter01	36.597383	1	1	1	1
voter02	5.481150	1	1	1	3
voter03	13.279131	1	1	3	1
voter04	4.859413	1	1	3	3
voter05	35.425375	1	3	1	1
voter06	5.490934	1	3	1	3
voter07	22.855333	1	3	3	1
voter08	19.835570	1	3	3	3
voter09	22.928716	3	1	1	1
voter10	5.538309	3	1	1	3
voter11	13.130227	3	1	3	1
voter12	6.056291	3	1	3	3
voter13	23.992772	3	3	1	1
voter14	16.699207	3	3	1	3
voter15	98.165759	3	3	3	1
voter16	129.664430	3	3	3	3
	460.000000				

The corresponding max-flow problem has the following form:

Each voting pattern, where voters strictly prefer at least one candidate of the set $\{b, c, e, j\}$ to candidate a , is represented by a vertex. Each candidate of the set $\{b, c, e, j\}$ is represented by a vertex. Furthermore, there is a vertex “source” and a vertex “drain”.

From the vertex “source” we draw a link to each vertex that represents a voting pattern. The maximum capacity of this link is the number of voters with this voting pattern.

From each vertex, that represents a voting pattern, we draw a link to each vertex that represents a candidate who is strictly preferred to candidate a by voters with this voting pattern. The maximum capacity of this link is the number of voters with this voting pattern.

From each vertex, that represents a candidate, we draw a link to the vertex “drain”. The maximum capacity of this link is $r^{(z-1)}$.

The task is: Maximize the total flow from the vertex “source” to the vertex “drain”.

In our case, we get a digraph with 21 vertices and 51 links.

Furthermore, we get:

$$r^{(0)} := (N - \mathbf{qW}) / M = (460 - 129.664430) / 4 = 82.583893$$

Our digraph has the following form:

link	start	end	capacity
1	source	voter01	36.597383
2	source	voter02	5.481150
3	source	voter03	13.279131
4	source	voter04	4.859413
5	source	voter05	35.425375
6	source	voter06	5.490934
7	source	voter07	22.855333
8	source	voter08	19.835570
9	source	voter09	22.928716
10	source	voter10	5.538309
11	source	voter11	13.130227
12	source	voter12	6.056291
13	source	voter13	23.992772
14	source	voter14	16.699207
15	source	voter15	98.165759
16	voter01	candidate b	36.597383
17	voter01	candidate c	36.597383
18	voter01	candidate e	36.597383
19	voter01	candidate j	36.597383
20	voter02	candidate b	5.481150
21	voter02	candidate c	5.481150
22	voter02	candidate e	5.481150
23	voter03	candidate b	13.279131
24	voter03	candidate c	13.279131
25	voter03	candidate j	13.279131
26	voter04	candidate b	4.859413
27	voter04	candidate c	4.859413
28	voter05	candidate b	35.425375
29	voter05	candidate e	35.425375
30	voter05	candidate j	35.425375
31	voter06	candidate b	5.490934
32	voter06	candidate e	5.490934
33	voter07	candidate b	22.855333
34	voter07	candidate j	22.855333
35	voter08	candidate b	19.835570
36	voter09	candidate c	22.928716
37	voter09	candidate e	22.928716
38	voter09	candidate j	22.928716
39	voter10	candidate c	5.538309
40	voter10	candidate e	5.538309
41	voter11	candidate c	13.130227
42	voter11	candidate j	13.130227
43	voter12	candidate c	6.056291
44	voter13	candidate e	23.992772
45	voter13	candidate j	23.992772
46	voter14	candidate e	16.699207
47	voter15	candidate j	98.165759
48	candidate b	drain	$r^{(z-1)}$
49	candidate c	drain	$r^{(z-1)}$
50	candidate e	drain	$r^{(z-1)}$
51	candidate j	drain	$r^{(z-1)}$

The following 13 pages document the solutions for LP⁽¹⁾ to LP⁽¹³⁾.

We get:

$r^{(0)} = 82.583893;$	$s^{(0)} = 0.000000$
$r^{(1)} = 78.688426;$	$s^{(1)} = 71.469640$
$r^{(2)} = 77.714559;$	$s^{(2)} = 75.365107$
$r^{(3)} = 77.471093;$	$s^{(3)} = 76.740693$
$r^{(4)} = 77.410226;$	$s^{(4)} = 77.227626$
$r^{(5)} = 77.395009;$	$s^{(5)} = 77.349359$
$r^{(6)} = 77.391205;$	$s^{(6)} = 77.379793$
$r^{(7)} = 77.390254;$	$s^{(7)} = 77.387401$
$r^{(8)} = 77.390016;$	$s^{(8)} = 77.389303$
$r^{(9)} = 77.389957;$	$s^{(9)} = 77.389779$
$r^{(10)} = 77.389942;$	$s^{(10)} = 77.389897$
$r^{(11)} = 77.389938;$	$s^{(11)} = 77.389927$
$r^{(12)} = 77.389937;$	$s^{(12)} = 77.389935$

We get:

$$r = \lim_{z \rightarrow \infty} r^{(z)} = \lim_{z \rightarrow \infty} s^{(z)} = 77.389937$$

Stage z = 1:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	82.583893
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	34.239372
18	voter01	candidate e	36.597383	2.358011
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	35.425375
29	voter05	candidate e	35.425375	0.000000
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	82.583893
48	candidate b	drain	82.583893	78.116279
49	candidate c	drain	82.583893	82.583893
50	candidate e	drain	82.583893	71.469640
51	candidate j	drain	82.583893	82.583893

Stage $z = 2$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	78.688426
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	30.343905
18	voter01	candidate e	36.597383	6.253478
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	35.425375
29	voter05	candidate e	35.425375	0.000000
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	78.688426
48	candidate b	drain	78.688426	78.116279
49	candidate c	drain	78.688426	78.688426
50	candidate e	drain	78.688426	75.365107
51	candidate j	drain	78.688426	78.688426

Stage z = 3:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.714559
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.370038
18	voter01	candidate e	36.597383	7.227344
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	35.023656
29	voter05	candidate e	35.425375	0.401719
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.714559
48	candidate b	drain	77.714559	77.714559
49	candidate c	drain	77.714559	77.714559
50	candidate e	drain	77.714559	76.740693
51	candidate j	drain	77.714559	77.714559

Stage $z = 4$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.471093
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.126572
18	voter01	candidate e	36.597383	7.470811
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.780190
29	voter05	candidate e	35.425375	0.645186
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.471093
48	candidate b	drain	77.471093	77.471093
49	candidate c	drain	77.471093	77.471093
50	candidate e	drain	77.471093	77.227626
51	candidate j	drain	77.471093	77.471093

Stage z = 5:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.410226
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.065705
18	voter01	candidate e	36.597383	7.531678
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.719323
29	voter05	candidate e	35.425375	0.706053
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.410226
48	candidate b	drain	77.410226	77.410226
49	candidate c	drain	77.410226	77.410226
50	candidate e	drain	77.410226	77.349359
51	candidate j	drain	77.410226	77.410226

Stage z = 6:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.395009
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.050488
18	voter01	candidate e	36.597383	7.546894
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.704106
29	voter05	candidate e	35.425375	0.721269
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.395009
48	candidate b	drain	77.395009	77.395009
49	candidate c	drain	77.395009	77.395009
50	candidate e	drain	77.395009	77.379793
51	candidate j	drain	77.395009	77.395009

Stage $z = 7$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.391205
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.046684
18	voter01	candidate e	36.597383	7.550699
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.700302
29	voter05	candidate e	35.425375	0.725073
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.391205
48	candidate b	drain	77.391205	77.391205
49	candidate c	drain	77.391205	77.391205
50	candidate e	drain	77.391205	77.387401
51	candidate j	drain	77.391205	77.391205

Stage $z = 8$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.390254
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.045733
18	voter01	candidate e	36.597383	7.551650
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.699351
29	voter05	candidate e	35.425375	0.726024
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.390254
48	candidate b	drain	77.390254	77.390254
49	candidate c	drain	77.390254	77.390254
50	candidate e	drain	77.390254	77.389303
51	candidate j	drain	77.390254	77.390254

Stage z = 9:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.390016
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.045495
18	voter01	candidate e	36.597383	7.551887
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.699113
29	voter05	candidate e	35.425375	0.726262
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.390016
48	candidate b	drain	77.390016	77.390016
49	candidate c	drain	77.390016	77.390016
50	candidate e	drain	77.390016	77.389779
51	candidate j	drain	77.390016	77.390016

Stage $z = 10$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389957
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.045436
18	voter01	candidate e	36.597383	7.551947
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.699054
29	voter05	candidate e	35.425375	0.726322
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.389957
48	candidate b	drain	77.389957	77.389957
49	candidate c	drain	77.389957	77.389957
50	candidate e	drain	77.389957	77.389897
51	candidate j	drain	77.389957	77.389957

Stage $z = 11$:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389942
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.045421
18	voter01	candidate e	36.597383	7.551962
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.699039
29	voter05	candidate e	35.425375	0.726336
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.389942
48	candidate b	drain	77.389942	77.389942
49	candidate c	drain	77.389942	77.389942
50	candidate e	drain	77.389942	77.389927
51	candidate j	drain	77.389942	77.389942

Stage z = 12:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389938
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.045417
18	voter01	candidate e	36.597383	7.551965
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.699035
29	voter05	candidate e	35.425375	0.726340
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.389938
48	candidate b	drain	77.389938	77.389938
49	candidate c	drain	77.389938	77.389938
50	candidate e	drain	77.389938	77.389935
51	candidate j	drain	77.389938	77.389938

Stage z = 13:

link	start	end	capacity	flow
1	source	voter01	36.597383	36.597383
2	source	voter02	5.481150	5.481150
3	source	voter03	13.279131	13.279131
4	source	voter04	4.859413	4.859413
5	source	voter05	35.425375	35.425375
6	source	voter06	5.490934	5.490934
7	source	voter07	22.855333	22.855333
8	source	voter08	19.835570	19.835570
9	source	voter09	22.928716	22.928716
10	source	voter10	5.538309	5.538309
11	source	voter11	13.130227	13.130227
12	source	voter12	6.056291	6.056291
13	source	voter13	23.992772	23.992772
14	source	voter14	16.699207	16.699207
15	source	voter15	98.165759	77.389937
16	voter01	candidate b	36.597383	0.000000
17	voter01	candidate c	36.597383	29.045416
18	voter01	candidate e	36.597383	7.551966
19	voter01	candidate j	36.597383	0.000000
20	voter02	candidate b	5.481150	0.000000
21	voter02	candidate c	5.481150	5.481150
22	voter02	candidate e	5.481150	0.000000
23	voter03	candidate b	13.279131	0.000000
24	voter03	candidate c	13.279131	13.279131
25	voter03	candidate j	13.279131	0.000000
26	voter04	candidate b	4.859413	0.000000
27	voter04	candidate c	4.859413	4.859413
28	voter05	candidate b	35.425375	34.699034
29	voter05	candidate e	35.425375	0.726341
30	voter05	candidate j	35.425375	0.000000
31	voter06	candidate b	5.490934	0.000000
32	voter06	candidate e	5.490934	5.490934
33	voter07	candidate b	22.855333	22.855333
34	voter07	candidate j	22.855333	0.000000
35	voter08	candidate b	19.835570	19.835570
36	voter09	candidate c	22.928716	0.000000
37	voter09	candidate e	22.928716	22.928716
38	voter09	candidate j	22.928716	0.000000
39	voter10	candidate c	5.538309	5.538309
40	voter10	candidate e	5.538309	0.000000
41	voter11	candidate c	13.130227	13.130227
42	voter11	candidate j	13.130227	0.000000
43	voter12	candidate c	6.056291	6.056291
44	voter13	candidate e	23.992772	23.992772
45	voter13	candidate j	23.992772	0.000000
46	voter14	candidate e	16.699207	16.699207
47	voter15	candidate j	98.165759	77.389937
48	candidate b	drain	77.389937	77.389937
49	candidate c	drain	77.389937	77.389937
50	candidate e	drain	77.389937	77.389936
51	candidate j	drain	77.389937	77.389937