Midterm, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

Instructor: Hubert L. Bray

February 27, 2014

1	12	
2	12	
3	15	
4	12	
5	12	
6	12	
	75	

Your Name: Key

Honor Pledge Signature:

Instructions: This is a 75 minute, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points, except Problem 3 which is worth 15 points. Good luck on the exam!

Problem 1. The Circular Cylinder

Consider the surface parametrized by

$$\vec{x}(u,v) = (\cos u, \sin u, v).$$

(a) Compute E, F, and G for this surface.

$$\vec{X}_u = (-\sin u, \cos u, 0)$$
$$\vec{X}_v = (0, 0, 1)$$

$$E = \vec{X}_u \cdot \vec{X}_u = 1$$
$$F = \vec{X}_u \cdot \vec{X}_v = 0$$
$$G = \vec{X}_v \cdot \vec{X}_v = 1$$

(b) Compute l, m, and n for this surface.

$$\vec{U} = \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|} = (\cos u, \sin u, 0)$$

$$l = \vec{U} \cdot \vec{X}_{uu} = (\cos u, \sin u, 0) \cdot (-\cos u, -\sin u, 0) = -1$$
$$m = \vec{U} \cdot \vec{X}_{uv} = 0$$
$$n = \vec{U} \cdot \vec{X}_{vv} = 0$$

$$\longrightarrow S(X_u) = -X_u \qquad \longrightarrow \qquad k_1 = 0$$
$$S(X_v) = 0 \qquad\qquad\qquad k_2 = -1$$

(c) Compute the mean curvature H and the Gauss curvature K for this surface.

$$H = \frac{k_1 + k_2}{2} = -\frac{1}{2}$$

$$K = k_1 k_2 = 0$$

Problem 2. Umbilic Surfaces

Suppose M is parametrized by $\vec{x}(u,v)$ with unit normal vector field U and consists entirely of umbilic points. That is, suppose

$$S_p(\vec{v}) = k(p)\,\vec{v},$$

where $k(p)$ is a real-valued function for $p \in M$.

(a) Prove that $k(p)$ equals a constant (call it k_0) on M.

$$U_{uv} = \nabla_{X_v}(\nabla_{X_u} U) = \nabla_{X_v}(-k X_u) = -k X_{uv} - k_v X_u$$

$$\| $$

$$U_{vu} = \nabla_{X_u}(\nabla_{X_v} U) = \nabla_{X_u}(-k X_v) = -k X_{uv} - k_u X_v$$

$\therefore\ k_u \vec{X}_v = k_v \vec{X}_u \Rightarrow 0 = k_u = k_v$ since $\vec{X}_u, \vec{X}_v$ are linearly independent.

$\therefore\ k(p) =$ a constant on M.

(b) If $k_0 = 0$, show that M is contained in a plane. (From scratch - do not quote the theorem that zero shape operator implies that M is in a plane - prove this.)

Let α be a curve from any point p to any point $q = \alpha(1)$.
$\overset{\|}{\alpha(0)}$

$$\frac{d}{dt}\left\{\left[\alpha(t) - \alpha(0)\right] \cdot \vec{U}(\alpha(t))\right\}$$

$$= \alpha'(t) \cdot \vec{U}(\alpha(t)) + \nabla_{\alpha'(t)} \vec{U} = 0 + 0 = 0.$$

But $\{\ \} = 0$ when $t = 0$. Hence at $t = 1$ it is also zero $\Rightarrow$

$$(q - p) \cdot \vec{U}(q) = 0$$

$\Rightarrow$ p is on the plane through $\vec{q}$ with unit normal $\vec{U}(q)$.

(c) If $k_0 \neq 0$, then show that

$$\vec{y}(u,v) = \vec{x}(u,v) + \frac{1}{k_0}U$$

is a constant. That is, show that $\vec{y}(u,v) = p$ for some fixed point p.

$$\vec{y}_u = \left(\vec{X} + \frac{1}{k_0}\vec{U}\right)_u = \vec{X}_u + \frac{1}{k_0}\nabla_{X_u}\vec{U} = \vec{X}_u + \frac{1}{k_0}(-k_0\vec{X}_u) = 0$$

$$\vec{y}_v = \left(\vec{X} + \frac{1}{k_0}\vec{U}\right)_v = \vec{X}_v + \frac{1}{k_0}(-S(X_v)) = \vec{X}_v + \frac{1}{k_0}(-k_0\vec{X}_v) = 0$$

$\therefore\ \vec{y}(u,v) =$ constant $= \vec{p}$

(d) Using part (c), prove that the surface is part of a sphere centered at the point p. What is the radius of the sphere?

$$\vec{p} = \vec{X} + \frac{1}{k_0}\vec{U}$$

$$|\vec{X} - \vec{p}| = \left|-\frac{1}{k_0}\vec{U}\right| = \frac{1}{k_0}$$

$\therefore\ \vec{X}(u,v)$ is on the sphere of $\boxed{\text{radius } \frac{1}{k_0}}$ centered at $\vec{p}$.

Problem 3. (A Surface in 4-Dimensional Euclidean Space)

Consider the surface in 4-dimensional Euclidean space parametrized by

$$\vec{x}(u,v) = (\underbrace{3\cos(u), 3\sin(u)}_{\text{circle}}, \underbrace{4\cos(v), 4\sin(v)}_{\text{circle}}).$$

(a) If we allow $-\infty \leq u, v \leq \infty$, this parametrization wraps around the surface infinity many times. Topologically, what is this surface? (Possible answers: a plane, a disk, a sphere, a torus (surface of a donut), the surface of a donut with two holes, etc.)

circle of circles =  = surface of a donut = torus

(b) Compute $\vec{x}_u$ and $\vec{x}_v$.

$$\vec{X}_u = (-3\sin u, 3\cos u, 0, 0)$$
$$\vec{X}_v = (0, 0, -4\sin v, 4\cos v)$$

(c) Since this surface is not embedded in 3-dimensions, our usual unit normal vector $\vec{U}$ is not well defined, meaning that our usual shape operator is not well defined. Hence, we cannot compute l, m, or n. However, we can compute E, F, and G. What are they?

$$E = \vec{X}_u \cdot \vec{X}_u = 3^2 = 9$$
$$F = \vec{X}_u \cdot \vec{X}_v = 0$$
$$G = \vec{X}_v \cdot \vec{X}_v = 4^2 = 16$$

(d) Since the shape operator is not well-defined, the principal curvatures, principal curvature directions, and mean curvature of this surface are not well defined in the usual sense. However, we do have a formula for the Gauss curvature of this surfaces in terms of E, F, G when $F = 0$, which is

$$K = -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right]$$

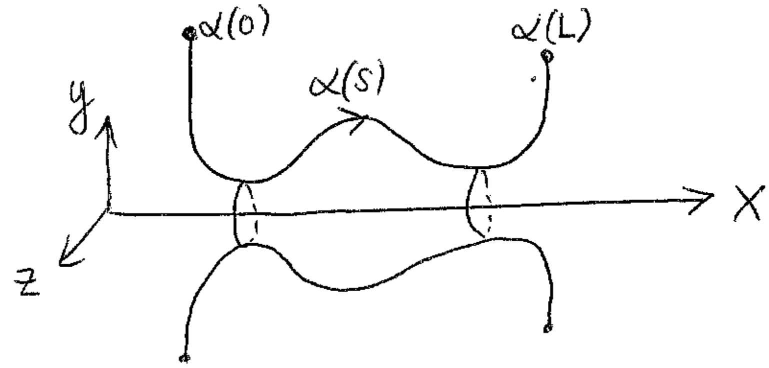
Compute K for this surface.

$$K = -\frac{1}{2\sqrt{12^2}}\left[\frac{\partial}{\partial v}(0) + \frac{\partial}{\partial u}(0)\right] = 0$$

(e) Prove that this surface with this metric (meaning this E, F, G) cannot be embedded as a smooth compact surface in 3 dimensional Euclidean space.

Theorem from book: Every compact surface (in 3 dim'l Euclidean space) without boundary has at least one point with $K > 0$. Hence, since the metric on this surface has $K = 0$ everywhere, this surface with this metric cannot be embedded as a smooth compact surface in 3 dim'l Euclidean space.

Problem 4. (The Gauss Bonnet Theorem for a "Catenoidal" Surface of Revolution)



Let $\alpha(s) = (x(s), y(s))$, $0 \leq s \leq L$, be a *unit speed* smooth [redacted] curve of length L in the xy plane which does not intersect itself or the x-axis. Suppose also that the curve starts out going straight down and ends by going straight up, as drawn in the figure, meaning that

$$y'(0) = -1 \quad \text{and} \quad y'(L) = 1.$$

Let M be the "catenoidal" surface of revolution created by rotating the curve α around the x-axis. Show that

$$\int_M K dA = c$$

for some constant c, and compute the constant c. The only formula for the Gauss curvature of M you may assume without proof is

$$K = \frac{x'(s)}{y(s)} \left(x''(s)y'(s) - y''(s)x'(s)\right).$$

Hints: You will need to use $dA = 2\pi y ds$ and the fact that $x'(s)^2 + y'(s)^2 = 1$.

$$\int_M K\,dA = \int_0^L \left(-\frac{y''}{y}\right) \cdot 2\pi y\,ds$$

$$= \int_0^L -2\pi y''(s)\,ds = -2\pi y'(s)\Big]_0^L$$

$$= -2\pi(1-(-1)) = -4\pi.$$

$\frac{d}{ds}$

$$2x'x'' + 2y'y'' = 0$$

$$x'x'' = -y'y''$$

$$K = \frac{1}{y}\left[-y'y''y' - x'y''x'\right]$$

$$= -\frac{y''}{y}(y'^2 + x'^2)$$

$$= -\frac{y''}{y}$$

Problem 5. (The Willmore Functional of a Surface)

The Willmore functional of a surface S in 3 dimensional Euclidean space is defined to be

$$W(S) = \iint_S H^2 \, dA,$$

where H is the mean curvature of the surface. After the midterm, we will learn the Gauss-Bonnet theorem, which says that

$$\iint_S K \, dA = 4\pi,$$

when S is a topological sphere (meaning not necessarily round, like a deflated beach ball).

(a) What are H and $W(S)$ when S is a sphere of radius R?

$$k_1 = k_2 = \frac{1}{R} \rightarrow H = \frac{1}{R} \rightarrow$$

$$W = \iint H^2 dA = \frac{1}{R^2} \cdot \text{area}(S) = \frac{1}{R^2} \cdot 4\pi R^2 = 4\pi.$$

(b) Using the Gauss-Bonnet theorem above, prove that when S is a topological sphere,

$$W(S) \geq 4\pi.$$

(**Hint:** Simplify $(a+b)^2 - (a-b)^2$ and then relate your identity to the definitions of K and H.)

$$\left(\frac{k_1+k_2}{2}\right)^2 - \left(\frac{k_1-k_2}{2}\right)^2 = k_1 k_2$$

$$H^2 = K + \left(\frac{k_1-k_2}{2}\right)^2$$

$$\therefore \iint_S H^2 = \iint_S K + \iint_S \left(\frac{k_1-k_2}{2}\right)^2 \geq 4\pi + 0 = 4\pi$$

(c) Use a theorem that we learned in class to prove that if S is a topological sphere with $W(S) = 4\pi$, then S must be a round sphere.

From above, $\iint_S H^2 = 4\pi \Rightarrow \iint_S \left(\frac{k_1-k_2}{2}\right)^2 = 0 \Rightarrow k_1 = k_2$ everywhere.

$\Rightarrow$ umbilic at every point $\Rightarrow$ sphere of radius $1/k_0$ by Problem 2d.

Problem 6. (Curves which Wrap around a Cylinder)

Consider the curve $\alpha(\theta)$ on the circular cylinder $x^2 + y^2 = 1$, where

$$\alpha(\theta) = (\cos(\theta), \sin(\theta), h(\theta))$$

for some function h which has period 2π, meaning that $h(\theta + 2\pi) = h(\theta)$. Note that this curve goes around the cylinder once when $0 \le \theta \le 2\pi$.

(a) Compute $\alpha'(\theta)$, $\alpha''(\theta)$, and $\alpha'''(\theta)$.

$$\alpha'(\theta) = (-\sin\theta, \cos\theta, h'(\theta))$$
$$\alpha''(\theta) = (-\cos\theta, -\sin\theta, h''(\theta))$$
$$\alpha'''(\theta) = (\sin\theta, -\cos\theta, h'''(\theta))$$

(b) Compute the torsion τ of this curve in terms of h. You may use the formula

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}.$$

$$\alpha'(\theta) \times \alpha''(\theta) = (h''\cos\theta + h'\sin\theta,\ h''\sin\theta - h'\cos\theta,\ 1)$$
$$|\alpha'(\theta) \times \alpha''(\theta)|^2 = 1 + h'(\theta)^2 + h''(\theta)^2$$

$$\tau = \frac{h'(\theta) + h'''(\theta)}{1 + h'(\theta)^2 + h''(\theta)^2}$$

(since h has period 2π)

(c) Prove that

$$0 = \int_0^{2\pi} \tau(\theta)\left(1 + h'(\theta)^2 + h''(\theta)^2\right)\, d\theta$$

$$\int_0^{2\pi} \tau\left(1 + h'(\theta)^2 + h''(\theta)^2\right) = \int_0^{2\pi} \left(h'(\theta) + h'''(\theta)\right) d\theta = h(\theta) + h''(\theta)\Big]_0^{2\pi} = 0$$

(d) Whether you got part (c) or not, use the identity in part (c) to prove that if α has $\tau \ge 0$, then the curve α must be contained in a plane.

If $\tau > 0$ anywhere, the above integral would be > 0. Hence, $\tau = 0$ everywhere $\Rightarrow \alpha(\theta)$ is contained in a plane.