Final Exam, Math 421
Differential Geometry: Curves and Surfaces in $\mathbf{R}^3$

Instructor: Hubert L. Bray

Monday, April 29, 2013

Your Name: Key

Honor Pledge Signature:

Instructions: This is a 3 hour, closed book exam. You may bring one $8\frac{1}{2}'' \times 11''$ piece of paper with anything you like written on it to use during the exam, but nothing else. No collaboration on this exam is allowed. All answers should be written in the space provided, but you may use the backs of pages if necessary.

Express your answers in essay form so that all of your ideas are clearly presented. Partial credit will be given for partial solutions which are understandable. If you want to make a guess, clearly say so. Partial credit will be maximized if you accurately describe what you know and what you are not sure about. Each problem is worth 12 points. Good luck on the exam!

Problem 1. Suppose we have a surface M parametrized by $x(u, v)$ with unit normal vector U. Let E, F, G and l, m, n be defined as usual in the book, but suppose that $F = 0$ everywhere.

(a) Prove that $\{\frac{x_u}{\sqrt{E}}, \frac{x_v}{\sqrt{G}}, U\}$ forms a orthonormal (length one, mutually perpendicular) basis of vectors at each point on the surface M.

$$\frac{X_u}{\sqrt{E}} \cdot \frac{X_u}{\sqrt{E}} = \frac{E}{E} = 1 \; ; \; \frac{X_v}{\sqrt{G}} \cdot \frac{X_v}{\sqrt{G}} = \frac{G}{G} = 1 \; ; \; U \cdot U = 1$$

$$\frac{X_u}{\sqrt{E}} \cdot \frac{X_v}{\sqrt{G}} = \frac{F}{\sqrt{EG}} = 0 \; ; \; U \cdot \left(\frac{X_u}{\sqrt{E}}\right) = 0 \; ; \; U \cdot \left(\frac{X_v}{\sqrt{G}}\right) = 0.$$

(b) Prove that

$$x_{uv} = \frac{E_v}{2E} x_u + \frac{G_u}{2G} x_v + mU.$$

Let $\boxed{X_{uv} = a \frac{X_u}{\sqrt{E}} + b \frac{X_v}{\sqrt{G}} + cU}$ in this basis.

Since this is an orthonormal basis,

$$a = X_{uv} \cdot \frac{X_u}{\sqrt{E}} = \frac{1}{2} (X_u \cdot X_u)_v / \sqrt{E} = \frac{E_v}{2\sqrt{E}}$$

$$b = X_{uv} \cdot \frac{X_v}{\sqrt{G}} = \frac{1}{2} (X_v \cdot X_v)_u / \sqrt{G} = \frac{G_u}{2\sqrt{G}}$$

$$c = X_{uv} \cdot U = - X_u \cdot U_v \quad \text{since } 0 = (X_u \cdot U)_v$$

$$= X_u \cdot \nabla_{X_v} U$$

$$= X_u \cdot S(X_v) = m$$

Plugging back into the boxed equation proves the result.

Problem 2. Suppose a unit speed curve $\alpha(s)$ has constant curvature $\kappa > 0$ and zero torsion τ.
(a) Show that

$$\gamma(s) = \alpha(s) + \frac{1}{\kappa} N$$

is a constant curve; that is, show that $\gamma(s) = p$ for some fixed point p.

$$\alpha'(s) = T \quad ; \quad N'(s) = -\kappa T + \tau B = -\kappa T.$$

(Frenet formula)

Hence,

$$\gamma'(s) = T(s) + \frac{1}{\kappa}(-\kappa T(s)) = T - T = 0,$$

so $\gamma(s) = p$

for some fixed point p.

(b) Using part (a), prove that the curve $\alpha(s)$ is part of a circle centered at the point p. What is the radius of the circle?

$$p = \alpha(s) + \frac{1}{\kappa} N.$$ Hence,

$$|\alpha(s) - p| = \left|-\frac{1}{\kappa} N\right| = \frac{1}{\kappa}$$ so that

$\alpha(s)$ is part of a circle centered at p of radius $R = \frac{1}{\kappa}$.

Problem 3.

(a) Given a surface M with unit normal vector field U, define the shape operator $S_p(v)$ at the point p on M, where v is a tangent vector to M at p.

$$S_p(v) = -\nabla_v U$$, where U is the unit normal on M.

(b) Prove that $S_p(v)$ is also a tangent vector to M at p. (Hence, $S_p : T_pM \to T_pM$.)

$$1 = U \cdot U$$

$$0 = v[U \cdot U] = 2U \cdot \nabla_v U \Rightarrow$$

$$S_p(v) = -\nabla_v U \perp U \Rightarrow$$

$$S_p(v) \in T_pM$$

(c) Prove that S_p is symmetric as follows: Given a coordinate chart $\vec{x}(u, v)$, show that

$$S_p(\vec{x}_u) \cdot \vec{x}_v = \vec{x}_u \cdot S_p(\vec{x}_v).$$

$$0 = U \cdot X_u$$

$$0 = (U \cdot X_u)_v = U_v \cdot X_u + U \cdot X_{uv} \rightarrow S(X_v) \cdot X_u = X_{uv} \cdot U$$

mixed partial derivatives are equal

$$0 = U \cdot X_v$$

$$0 = (U \cdot X_v)_u = U_u \cdot X_v + U \cdot X_{vu} \rightarrow S(X_u) \cdot X_v = X_{vu} \cdot U$$

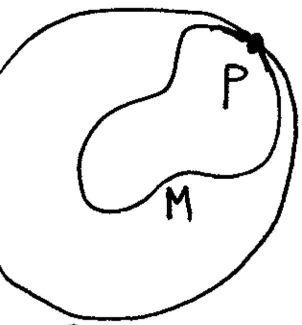
Thus,

$$S(X_u) \cdot X_v = S(X_v) \cdot X_u.$$

Problem 4.

(a) Prove that on every compact (closed and bounded) surface $M \subset \mathbf{R}^3$ there is at least one point of M with positive Gauss curvature K.

Consider the smallest sphere centered at the origin that bounds a region containing M. This sphere $S_r(0)$ will be tangent to M at some point p.



$S_r(0)$

By comparison, the principle curvatures of M will be larger than those of $S_r(0)$ which are both $1/r$.

Hence, $K(p) = k_1 k_2 \geq \frac{1}{r} \cdot \frac{1}{r} = 1/r^2 > 0$.

(b) Prove that there does not exist a compact minimal surface in $\mathbf{R}^3$.

A minimal surface has $0 = H = \frac{k_1 + k_2}{2} \Rightarrow k_2 = -k_1$.

Hence, $K = k_1 k_2 =$

$= k_1(-k_1)$

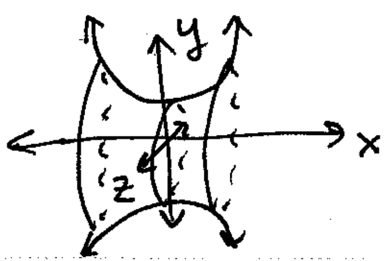
$= -(k_1)^2 \leq 0$ at every point,

which is impossible by part (a). Hence, there does not exist a <u>compact</u> minimal surface in $\mathbb{R}^3$.

(c) Give two examples of a minimal surface in $\mathbf{R}^3$.

(1) The xy plane: $(z=0)$

(2) The catenoid: $\sqrt{y^2+z^2} = \cosh(x)$



Problem 5. Define a geodesic of a surface M to be any curve $\alpha(t)$ on M such that $\alpha''(t)$ is perpendicular to M.

(a) Prove that a geodesic has constant speed.

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\alpha'(t)\cdot\alpha'(t)) = 2\underbrace{\alpha'(t)}_{\text{tangent to } M}\cdot\underbrace{\alpha''(t)}_{\text{perpendicular to } M} = 0.$$

Thus, v^2 and hence v are constant.

(b) Give the definition of geodesic curvature for a general curve on a surface.

$$\kappa_g = \alpha''(s)\cdot\underbrace{J(T)}_{\substack{90^\circ \text{ counterclockwise} \\ \text{rotation of } T=\alpha'(s)}}$$ when $\alpha(s)$ is parametrized by arc length.

(c) Prove that a geodesic has zero geodesic curvature.

$$\kappa_g = \underbrace{\alpha''(s)}_{\text{perp to } M}\cdot\underbrace{J(T)}_{\text{tangent to } M} = 0$$

(d) Suppose α is a geodesic on the standard unit sphere. Prove that it is contained in a plane.

WLOG: unit speed

geodesic: $\alpha''(s) = T'(s) = \kappa N$ is perp. to M

Thus, $N = \pm U$.

But the unit sphere has $S_p(v) = -\nabla_v U = V$, so

$$N'(s) = \pm U'(s) = \pm\nabla_{\alpha'(s)}U = \mp S_p(\alpha'(s)) = \mp\alpha'(s) = \mp T$$

By the Frenet formulas,

$N'(s) = -\kappa T + \tau B$, so $\tau = 0$ (and $\kappa = \pm 1$).

But $\tau = 0$ implies that α is contained in a plane.

Problem 6. The Gauss curvature of a surface of revolution of the curve $\alpha(u) = (g(u), h(u))$, $h(u) > 0$, is given by

$$K = \frac{g'(g''h' - h''g')}{h(g'^2 + h'^2)^2}.$$

(a) If we parametrize the curve by arc length so that the velocity of the curve is one, then we may let

$$\alpha'(u) = (g'(u), h'(u)) = (\cos(\theta(u)), \sin(\theta(u))),$$

where θ is the angle that the velocity of the curve makes with the x axis. Compute the Gauss curvature K in terms of $h(u)$ and $\theta(u)$.

$$g'(u) = \cos\theta(u) \rightarrow g''(u) = -\sin\theta \cdot \theta'(u)$$
$$h'(u) = \sin\theta(u) \longrightarrow h''(u) = \cos\theta \cdot \theta'(u)$$

$$K = \frac{\cos\theta\left(-\sin^2\theta - \cos^2\theta\right)\theta'(u)}{h\left(\cos^2\theta + \sin^2\theta\right)} = -\frac{\cos\theta \cdot \theta'(u)}{h(u)} = -\frac{h''(u)}{h(u)}.$$

either is fine

(b) Prove that the only flat ($K = 0$) surfaces of revolution are planes, cones, and cylinders. (Hint: The solution to part (a) makes this easy.)

$K = 0 \rightarrow h'' = 0 \rightarrow$ constant $= h' = \sin\theta(u) \longrightarrow$

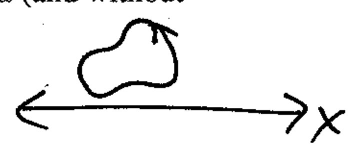
$\theta(u)$ is constant.

$\theta = 0 \rightarrow \boxed{\text{Cylinder}}$

$\theta = \pi/2 \rightarrow \boxed{\text{plane}}$

All other $\theta \longrightarrow \boxed{\text{Cone}}$

(c) Let M be a toroidal (like a donut) surface of revolution formed by a unit speed curve $\alpha(u)$, $a \le u \le b$, which begins and ends at the same point ($\alpha(a) = \alpha(b)$) with the same velocity vector ($\alpha'(a) = \alpha'(b)$). Using part (a) and the area form formula $dA = 2\pi h(u)du$ (and without using the Gauss-Bonnet theorem), prove that



$$\int_M K\, dA = c.$$

for some constant c, and compute the value of c.

$$\int_M K dA = \int_a^b -\frac{h''(u)}{h(u)} 2\pi h(u) du = -2\pi h'(u)\Big]_a^b$$
$$= 2\pi\left(h'(a) - h'(b)\right) = 0.$$

Hence, $c = 0$, which makes sense since M is a torus with Euler characteristic zero.

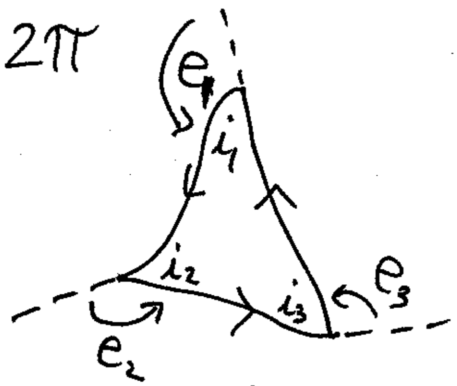
Problem 7.

(a) State the "Gauss Bonnet Theorem for a Disk."

$$\iint_D K\,dA + \int_{\partial D} \kappa_g\,ds = 2\pi$$

(b) State the "Gauss Bonnet Theorem for a Disk with Corners" both in terms of exterior angles and interior angles.

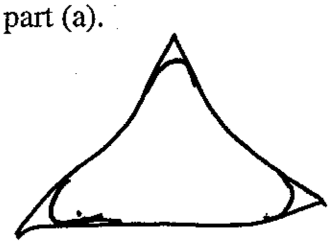
$$\iint_D K\,dA + \int_{\partial D - \{\text{corners}\}} \kappa_g\,ds + \sum_{k \in \{\text{corners}\}} (\pi - i_k) = 2\pi$$



$$\iint_D K\,dA + \int_{\partial D - \{\text{corners}\}} \kappa_g\,ds + \sum_{k \in \{\text{corners}\}} e_k = 2\pi$$

$$e_k = \pi - i_k$$

(c) Explain, in general terms, how the statement in part (b) can be derived from the statement in part (a).



Round out the corners and then take the limit.

$\int \kappa_g\,ds \approx \int \theta'(s)\,ds \approx \Delta\theta = \text{exterior angle}$

and is exact at each corner in the limit as the radius of curvature of each rounding goes to zero.

(d) Using the statement in part (b), prove the Gauss Bonnet Theorem for a general compact surface with boundary.

Triangulate M:

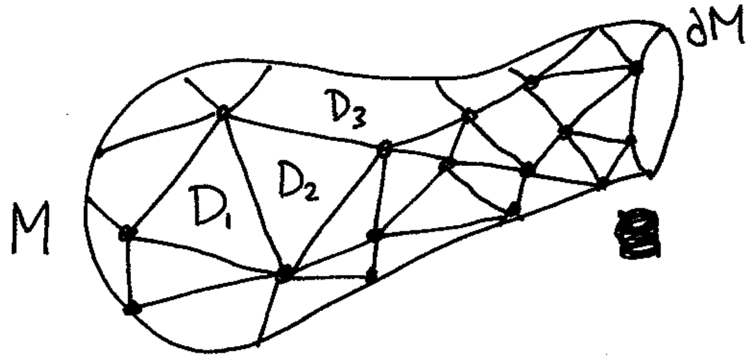
F - # of faces

$E_{interior}$ - # of interior edges

$E_{boundary}$ - # of edges on ∂M

$V_{interior}$ - # of interior vertices

$V_{boundary}$ - # of vertices on ∂M.



Sum Gauss-Bonnet for a Disk over each face:

$$\iint_{D_i} K\,dA + \int_{\partial D_i - \{\text{corners}\}} \kappa_g\,ds + \sum_{\text{edges}} \pi - \sum_{\text{corners}} i_k = 2\pi$$

Cancels except on ∂M

$$\iint_M K\,dA + \int_{\partial M} \kappa_g\,ds + 2\pi E_{interior} - 2\pi V_{interior} + \pi E_{boundary} - \pi V_{boundary} = 2\pi F$$

But ∂M has the same number of boundary edges and vertices, so $E_{boundary} = V_{boundary}$, so we may add in: $\pi E_{boundary} - \pi V_{boundary}$ to the left hand side to get

$$\iint_M K\,dA + \int_{\partial M} \kappa_g\,ds = 2\pi(F - E + V).$$

$$= 2\pi\chi(M)$$

Problem 8. Suppose α is a geodesic on M and is also contained in a plane P. Prove that α is also a line of curvature of M. (Recall that a line of curvature is any curve whose tangent direction T is an eigenvector of the shape operator at every point.)

WLOG let $\alpha(s)$ be a unit speed geodesic. Then

$\alpha''(s) = T'(s) = \kappa N$ is perpendicular to M $\Rightarrow$

$N \parallel U \rightarrow N = \pm U.$

Since α is contained in a plane, $\tau = 0$, so the Frenet formula

$N'(s) = -\kappa T + \tau B$ implies

$\pm U'(s) = -\kappa T$

$\pm \nabla_T U = -\kappa T$

$S_{\alpha(s)}(T) = \pm \kappa T \Rightarrow$

T is an eigenvector of the shape operator everywhere on $\alpha(s)$ $\Rightarrow$

$\alpha(s)$ is a line of curvature of M

Problem 9. Suppose that M is a connected surface without boundary which has Gauss curvature larger than the Gauss curvature of the round sphere S of radius one in $\mathbb{R}^3$.

(a) What is the Gauss curvature of the round sphere S of radius one in $\mathbb{R}^3$?

$k_1 = k_2 = 1$

$$\Rightarrow K = k_1 k_2 = 1 \text{ at each point.}$$

(b) Prove that M has positive Euler characteristic.

By Gauss-Bonnet, $\iint_M K \, dA = \chi(M)$.

$$K > 1 \Rightarrow \chi(M) > 0.$$

(c) Using the fact that connected surfaces all have Euler characteristic less than or equal to two, prove that the area of M is less than the area of S.

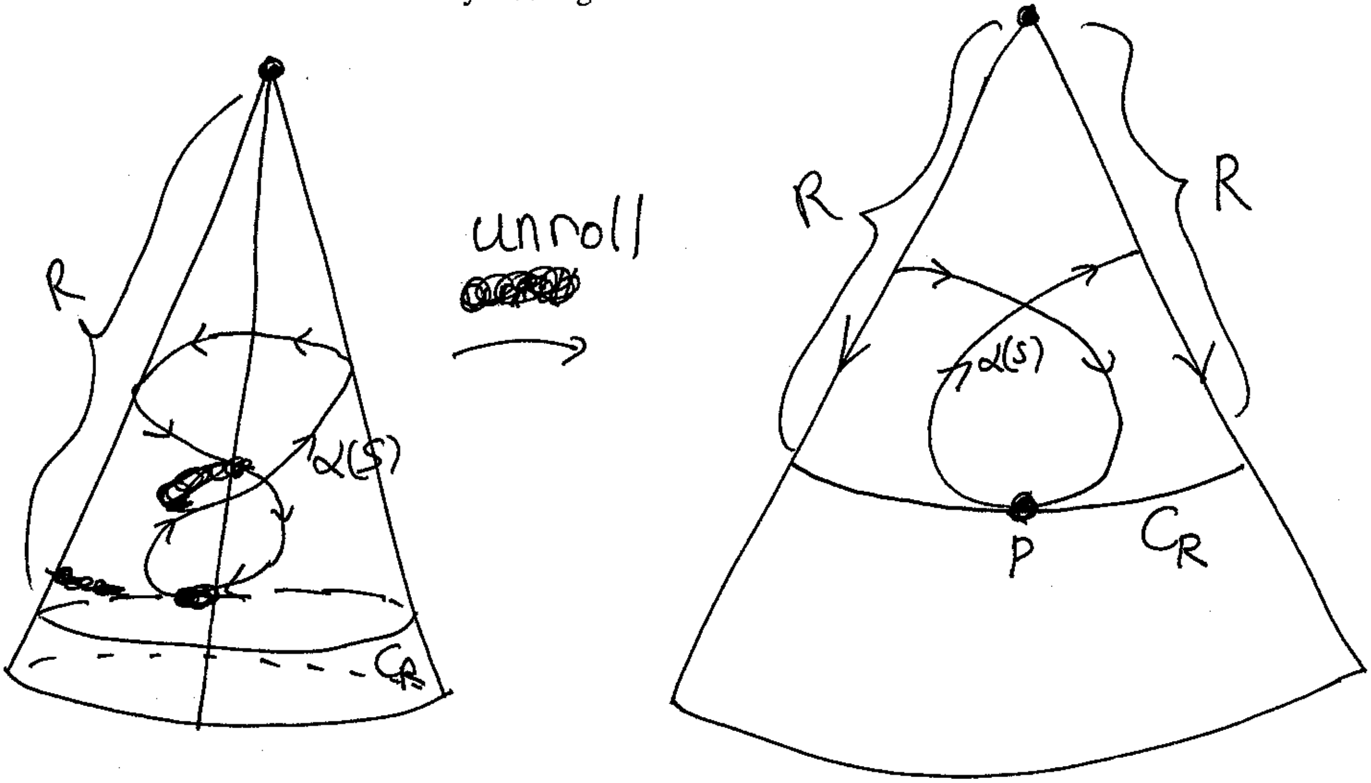
$$4\pi = 2\pi \cdot 2 \geq 2\pi \cdot \chi(M) = \iint_M K \, dA > \iint_M 1 \, dA = \text{area}(M)$$

Hence,

$$\text{area}(M) < 4\pi = \text{area}(S).$$

Bonus Problem: A closed geodesic is one which makes a complete loop back to where it started and then continues in the same direction so that $\alpha(s) = \alpha(s+L)$, for some L. For example, great circles are closed geodesics on the round sphere.

Prove that there are not any closed geodesics on a circular cone.



Suppose a closed geodesic $\alpha(s)$ did exist. Choose the smallest circle C_R which bounds it which will be tangent at some point p. Note that since C_R is part of a circle of radius R in the plane when the cone is unrolled, it has $\kappa_g^{C_R} = \frac{1}{R}$.

By comparison, $\kappa_g^{\alpha}(p) > \kappa_g^{C_R} = \frac{1}{R}$.

Hence, the geodesic curvature $\kappa_g^{\alpha}(p)$ of $\alpha(s)$ at p must be positive and hence can not be zero. Contradiction. Thus, a closed geodesic does not exist on any cone.