Directed Graphs
Math 218

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A *directed graph* is a collection of
Directed Graphs

Definition

A directed graph is a collection of nodes.
Directed Graphs

Definition

A directed graph is a collection of nodes.
Directed Graphs

Definition

A *directed graph* is a collection of *nodes* and *arrows*.
A directed graph is a collection of nodes and arrows.
Other terms for ”directed graph”

“digraph”  “directed network”  “quiver” =
Directed Graphs
Basic Terminology

Other terms for ”directed graph”
“digraph”  “directed network”  “quiver” =

Other terms for “node”
“vertex”  “object”  “point”
Directed Graphs

Basic Terminology

Other terms for "directed graph"

"digraph"  "directed network"  "quiver" =

Other terms for "node"

"vertex"  "object"  "point"

Other terms for "arrow"

"directed edge"  "directed arc"  "directed line"
Directed Graphs

Basic Terminology

digraph on six vertices and seven directed edges
Directed Graphs

Basic Terminology

quiver on four nodes and six arrows
Directed Graphs

Applications

Digraphs are often used to describe *relationships* between *objects*.
Directed Graphs

Applications

In finance, digraphs can be used to model *transactional data.*
In finance, digraphs can be used to model *transactional data*.

*nodes* financial institutions *arrows* transactions
Directed Graphs

Applications

In finance, digraphs can be used to model transactional data.

nodes financial institutions

arrows transactions
Directed Graphs

Applications

In finance, digraphs can be used to model *transactional data*.

- **Nodes**: financial institutions
- **Arrows**: transactions
Directed Graphs

Applications

The internet can be modeled by a digraph organizing *hyperlink data*.
Directed Graphs

Applications

The internet can be modeled by a digraph organizing *hyperlink data*.

**nodes** webpages  **arrows** hyperlinks
Directed Graphs

Applications

The internet can be modeled by a digraph organizing hyperlink data.

nodes webpages arrows hyperlinks
Directed Graphs

Applications

The internet can be modeled by a digraph organizing hyperlink data.

nodes webpages arrows hyperlinks

Diagram showing directed edges between webpages (labels: $l_1, l_2, \ldots, l_9$).
Family trees are digraphs organizing *parental relationships*.
Family trees are digraphs organizing *parental relationships*.

**nodes** people

**arrows** parental relationships
Directed Graphs

Applications

Family trees are digraphs organizing *parental relationships*.

- **nodes** people
- **arrows** parental relationships
Family trees are digraphs organizing *parental relationships*.

**nodes** people

**arrows** parental relationships
Directed Graphs

Applications

Facebook can be modeled by a digraph organizing *friendships*.

- **nodes** user accounts
- **arrows** friendships

https://medium.com/@johnrobb/facebook-the-complete-social-graph-b58157ee6594
Directed Graphs

Applications

Uber can be modeled by a digraph organizing *routes*.

![Nodes locations arrows routes](https://dabrownstein.files.wordpress.com/2014/06/rides-into-neighborhoods-bradley-voytek.jpg)
Directed Graphs
Applications

Logical arguments can be modeled by digraphs organizing *logical implications*. 
Directed Graphs

Applications

Logical arguments can be modeled by digraphs organizing *logical implications*.

- **nodes** statements
- **arrows** logical implications
Directed Graphs

Applications

Logical arguments can be modeled by digraphs organizing *logical implications*.

- **Nodes**: statements
- **Arrows**: logical implications

**Axiom 1**

**Lemma 1**

**Lemma 2**

**Lemma 3**

**Theorem 1**

**Theorem 2**
Logical arguments can be modeled by digraphs organizing *logical implications*.

**nodes** statements

**arrows** logical implications

- Axiom 1
- Axiom 2
- Lemma 1
- Lemma 2
- Lemma 3
- Theorem 1
- Theorem 2
A *neural network* is a type of digraph used to solve artificial intelligence problems.

**Nodes** “artificial neurons”  **Arrows** “connections”
Directed Graphs

Applications

Transportation

- **nodes**: street intersections
- **arrows**: one-way streets
Directed Graphs

Applications

**Transportation**

- nodes: street intersections
- arrows: one-way streets

**Communication**

- nodes: cell phone numbers
- arrows: text messages
Directed Graphs

Applications

Transportation

(nodes) street intersections (arrows) one-way streets

Communication

(nodes) cell phone numbers (arrows) text messages

Food Chain

(nodes) species (arrows) predator-prey relationships
Directed Graphs

Applications

Transportation

- **nodes**: street intersections
- **arrows**: one-way streets

Communication

- **nodes**: cell phone numbers
- **arrows**: text messages

Food Chain

- **nodes**: species
- **arrows**: predator-prey relationships

Game Theory

- **nodes**: board positions
- **arrows**: legal moves
Directed Graphs

Invariants

Interesting properties of digraphs are often referred to as invariants.
Directed Graphs

Invariants

**Connectedness**

How can we determine if a digraph is connected?
Connectedness

How can we determine if a digraph is connected?
Connectedness

How can we determine if a digraph is connected?
Connected Components

How can we determine the number of *connected components* of a digraph?
Directed Graphs

Invariants

**Connected Components**

How can we determine the number of *connected components* of a digraph?

![Diagram showing two connected components](image)

*one* connected component
Connected Components

How can we determine the number of connected components of a digraph?

one connected component

two connected components
Directed Graphs

Trees

How can we determine if a digraph is a tree (connected and no cycles)?
Trees
How can we determine if a digraph is a tree (connected and no cycles)?

directed graph representation
Directed Graphs

Invariants

Trees
How can we determine if a digraph is a tree (connected and no cycles)?

- Tree:
  - Connected
  - No cycles

- Not a tree:
  - Connected
  - Contains a cycle

```
  +---+  +---+
  |   |  |   |
  +---+  +---+  +---+
      |  |     |  |
      +---+     +---+

  tree

  +---+  +---+
  |   |  |   |
  +---+  +---+  +---+
      |  |     |  |
      +---+     +---+

  not a tree
```
Directed Graphs

Invariants

Girth

What is the length of a shortest cycle?
Directed Graphs

Invariants

Girth

What is the length of a shortest cycle?

girth = 5
Directed Graphs

Invariants

**Girth**

What is the length of a shortest cycle?

\[ \text{girth} = 5 \]
Directed Graphs

Invariants

How can we use mathematics to solve problems involving digraphs?
Incidence Vectors

Definition

Each arrow in a digraph is associated to an *incidence vector*.
Incidence Vectors

Definition

Each arrow in a digraph is associated to an *incidence vector*.
Incidence Vectors

Definition

Each arrow in a digraph is associated to an *incidence vector*.

$$\vec{a}_1 = \begin{bmatrix} a_1 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
Each arrow in a digraph is associated to an *incidence vector*.

The incidence vector associates *nodes* to *scalars* ("scalar" means "number").
Each arrow in a digraph is associated to an *incidence vector*.

\[ \vec{a}_1 = \begin{bmatrix} a_1 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \]

The incidence vector associates *nodes* to *scalars* ("scalar" means "number").

"source" node $\rightarrow -1$  "target" node $\rightarrow 1$  other nodes $\rightarrow 0$
Incidence Vectors

Definition

The incidence vectors are

\[ \vec{a}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \]
\[ \vec{a}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \]
\[ \vec{a}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \]
Incidence Vectors

Definition

The incidence vectors are

\[
\vec{a}_1 = \begin{bmatrix} v_1 & -1 \\ v_2 & 1 \\ v_3 & 0 \\ v_4 & 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}
\]
Incidence Vectors

Definition

The incidence vectors are

\[
\vec{a}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\]
Incidence Vectors

Definition

The incidence vectors are

\[ \vec{a}_1 = \begin{bmatrix} v_1 & -1 \\ v_2 & 1 \\ v_3 & 0 \\ v_4 & 0 \end{bmatrix} \]

\[ \vec{a}_2 = \begin{bmatrix} v_1 & 0 \\ v_2 & -1 \\ v_3 & 1 \\ v_4 & 0 \end{bmatrix} \]

\[ \vec{a}_3 = \begin{bmatrix} v_1 & 0 \\ v_2 & 1 \\ v_3 & 0 \\ v_4 & -1 \end{bmatrix} \]
Incidence Vectors
General Vectors

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many scalars are stored inside a given incidence vector?
Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many scalars are stored inside a given incidence vector?

Answer
Each incidence vector is of the form

\[
\begin{bmatrix}
    v_1 \\
v_2 \\
    \vdots \\
v_m
\end{bmatrix}
\]

There are \( m \) scalars stored inside \( \vec{a} \).

To indicate this, we write \( \vec{a} \in \mathbb{R}^m \).
Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many scalars are stored inside a given incidence vector?

Answer
Each incidence vector is of the form

\[
\vec{a} = \begin{bmatrix}
    v_1 & \ast \\
    v_2 & \ast \\
    \vdots & \vdots \\
    v_m & \ast
\end{bmatrix}
\]

There are \( m \) scalars stored inside \( \vec{a} \). To indicate this, we write \( \vec{a} \in \mathbb{R}^m \).
Incidence Vectors

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many scalars are stored inside a given incidence vector?

Answer
Each incidence vector is of the form

\[
\vec{a} = \begin{bmatrix}
v_1 & \ast \\
v_2 & \ast \\
\vdots & \vdots \\
v_m & \ast \\
\end{bmatrix}
\]

There are \( m \) scalars stored inside \( \vec{a} \).
Incidence Vectors

General Vectors

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many scalars are stored inside a given incidence vector?

Answer
Each incidence vector is of the form

\[
\overrightarrow{a} = \begin{bmatrix}
    v_1 & * \\
    v_2 & * \\
    \vdots & \vdots \\
    v_m & *
\end{bmatrix}
\]

There are \( m \) scalars stored inside \( \overrightarrow{a} \). To indicate this, we write \( \overrightarrow{a} \in \mathbb{R}^m \).
Definition

A vector in $\mathbb{R}^m$ is a list of $m$ scalars.
Incidence Vectors

General Vectors

Set-Membership Notation

We use the symbol $\in$ to indicate membership.
Set-Membership Notation

We use the symbol $\in$ to indicate membership.

$\vec{v} \in \mathbb{R}^m$ means $\vec{v}$ is a vector in $\mathbb{R}^m$.
Incidence Vectors
General Vectors

Set-Membership Notation
We use the symbol $\in$ to indicate membership.

$\vec{v} \in \mathbb{R}^m$ means $\vec{v}$ is a vector in $\mathbb{R}^m$

$\vec{v} \notin \mathbb{R}^m$ means $\vec{v}$ is not a vector in $\mathbb{R}^m$
Incidence Vectors

General Vectors

Set-Membership Notation

We use the symbol $\in$ to indicate membership.

- $\vec{v} \in \mathbb{R}^m$ means $\vec{v}$ is a vector in $\mathbb{R}^m$
- $\vec{v} \not\in \mathbb{R}^m$ means $\vec{v}$ is not a vector in $\mathbb{R}^m$

Horizontal and Vertical Notation

Two ways to indicate that $\vec{v} \in \mathbb{R}^m$.

- $\vec{v} = \langle x_1, x_2, \ldots, x_m \rangle$
  Horizontal Notation
- $\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$
  Vertical Notation
Incidence Vectors

General Vectors

Set-Membership Notation
We use the symbol $\in$ to indicate membership.

$\vec{v} \in \mathbb{R}^m$ means $\vec{v}$ is a vector in $\mathbb{R}^m$

$\vec{v} \notin \mathbb{R}^m$ means $\vec{v}$ is not a vector in $\mathbb{R}^m$

Horizontal and Vertical Notation
Two ways to indicate that $\vec{v} \in \mathbb{R}^m$.

$$\vec{v} = \langle x_1, x_2, \ldots, x_m \rangle$$

Horizontal Notation

$$\vec{v} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Vertical Notation

The scalars $x_1, x_2, \ldots, x_m$ are called the coordinates of $\vec{v}$. 
Example
Consider the vector $\vec{v} = \langle 3, -2, 4, 9 \rangle$. 
Example
Consider the vector \( \vec{v} = \langle 3, -2, 4, 9 \rangle \). Then \( \vec{v} \in \mathbb{R}^4 \).
Example
Consider the vector \( \vec{v} = \langle 3, -2, 4, 9 \rangle \). Then \( \vec{v} \in \mathbb{R}^4 \).
Example
Consider the vector \( \vec{v} = \langle 3, -2, 4, 9 \rangle \). Then \( \vec{v} \in \mathbb{R}^4 \). In vertical and horizontal notation, we have

\[
\vec{v} = \langle 3, -2, 4, 9 \rangle = \begin{bmatrix}
3 \\
-2 \\
4 \\
9
\end{bmatrix}
\]
Example

Consider the vector $\mathbf{v} = \langle 3, -2, 4, 9 \rangle$. Then $\mathbf{v} \in \mathbb{R}^4$. In vertical and horizontal notation, we have

$$\mathbf{v} = \langle 3, -2, 4, 9 \rangle = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 9 \end{bmatrix}$$

The coordinates of $\mathbf{v}$ are

first coordinate is 3
Incidence Vectors
General Vectors

Example
Consider the vector \( \mathbf{v} = \langle 3, -2, 4, 9 \rangle \). Then \( \mathbf{v} \in \mathbb{R}^4 \). In vertical and horizontal notation, we have

\[
\mathbf{v} = \langle 3, -2, 4, 9 \rangle = \begin{bmatrix}
3 \\
-2 \\
4 \\
9
\end{bmatrix}
\]

The coordinates of \( \mathbf{v} \) are

first coordinate is 3
second coordinate is \(-2\)
Example
Consider the vector $\vec{v} = \langle 3, -2, 4, 9 \rangle$. Then $\vec{v} \in \mathbb{R}^4$. In vertical and horizontal notation, we have

$$\vec{v} = \langle 3, -2, 4, 9 \rangle = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 9 \end{bmatrix}$$

The coordinates of $\vec{v}$ are

- first coordinate is 3
- second coordinate is $-2$
- third coordinate is 4
- fourth coordinate is 9
Incidence Vectors
General Vectors

Example
Consider the vector \( \vec{v} = \langle 3, -2, 4, 9 \rangle \). Then \( \vec{v} \in \mathbb{R}^4 \). In vertical and horizontal notation, we have

\[
\vec{v} = \langle 3, -2, 4, 9 \rangle = \begin{bmatrix}
3 \\
-2 \\
4 \\
9
\end{bmatrix}
\]

The coordinates of \( \vec{v} \) are

first coordinate is 3  
second coordinate is \(-2\)  
third coordinate is 4  
fourth coordinate is 9
Incidence Vectors

General Vectors

Example
Consider the vectors

\[ \vec{v} = \langle 1, 1 \rangle \quad \vec{w} = \langle -6, -2, 0 \rangle \quad \vec{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 0 \end{bmatrix} \]
Incidence Vectors

General Vectors

**Example**

Consider the vectors

\[ \vec{v} = \langle 1, 1 \rangle \]
\[ \vec{w} = \langle -6, -2, 0 \rangle \]
\[ \vec{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \]
\[ \vec{y} = \begin{bmatrix} 1/2 \\ 2 \\ 0 \end{bmatrix} \]

These vectors satisfy
Incidence Vectors

Example
Consider the vectors

\[ \vec{v} = \langle 1, 1 \rangle \quad \vec{w} = \langle -6, -2, 0 \rangle \quad \vec{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} 1/2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \]

These vectors satisfy

\[ \vec{v} \in \mathbb{R}^2 \quad \vec{w} \in \mathbb{R}^3 \quad \vec{x} \in \mathbb{R}^3 \quad \vec{y} \in \mathbb{R}^4 \]
Example

Consider the vectors

\[ \mathbf{v} = \langle 1, 1 \rangle \]
\[ \mathbf{w} = \langle -6, -2, 0 \rangle \]
\[ \mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \]
\[ \mathbf{y} = \begin{bmatrix} 1/2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \]

These vectors satisfy

\[ \mathbf{v} \in \mathbb{R}^2 \quad \mathbf{w} \in \mathbb{R}^3 \quad \mathbf{x} \in \mathbb{R}^3 \quad \mathbf{y} \in \mathbb{R}^4 \]
\[ \mathbf{v} \notin \mathbb{R}^3 \quad \mathbf{w} \notin \mathbb{R}^2 \quad \mathbf{x} \notin \mathbb{R}^5 \quad \mathbf{y} \notin \mathbb{R}^2 \]
The incidence vector associated to the arrow $a_4$ is

$$
\vec{a}_4 = \begin{bmatrix}
-1 \\
0 \\
1 \\
0
\end{bmatrix} = \langle -1, 0, 1, 0 \rangle \in \mathbb{R}^4
$$
Our Convention

We will typically use the “arrow” or “harpoon” notation $\vec{v}$ to refer to vectors.
**Our Convention**
We will typically use the “arrow” or “harpoon” notation $\vec{v}$ to refer to vectors.

**Other Common Notation**
You might also encounter

$$
\begin{align*}
\mathbf{v} & \quad \vec{v} & \quad \mathbf{V} & \quad \mathbf{V}
\end{align*}
$$

Strang uses bold-font notation $\mathbf{v} \in \mathbb{R}^m$. 
Incidence Matrices

Definition

Question
How can we use a computer to analyze a digraph?
Incidence Matrices

Definition

**Question**
How can we use a computer to analyze a digraph?

**Answer**
Feed the computer all of the incidence vectors!
Incidence Matrices

Definition

Every digraph is associated to an *incidence matrix*.

\[
A = \begin{bmatrix}
\vec{a}_1 & \vec{a}_2 & \vec{a}_3
\end{bmatrix}
\]

Here, \(A\) has three rows and three columns.
Incidence Matrices

Definition

Every digraph is associated to an *incidence matrix*.

The incidence matrix is an array of numbers obtained by inserting each incidence vector into a column.

\[
A = \begin{bmatrix}
\vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
-1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1 \\
\end{bmatrix}
\]
Incidence Matrices

Definition

Every digraph is associated to an *incidence matrix*.  

\[
A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}
\]

The incidence matrix is an array of numbers obtained by inserting each incidence vector into a column.

\[
A = \begin{bmatrix}
v_1 & a_1 & a_2 & a_3 \\ 
v_2 & -1 & -1 & 0 \\ 
v_3 & 1 & 0 & -1 \\ 
0 & 1 & 1 \\ 
\end{bmatrix}
\]

Here, \( A \) has *three rows* and *three columns*. 
Incidence Matrices

Definition

The incidence matrix is

\[ A = \begin{bmatrix}
  a_1 & a_2 & a_3 \\
  v_1 & v_2 & v_3 & v_4
\end{bmatrix} \]

Here, \( A \) has four rows and three columns.
The incidence matrix is

\[
A = \begin{bmatrix}
  a_1 & a_2 & a_3 \\
  v_1 & \begin{bmatrix} -1 & 1 \end{bmatrix} \\
  v_2 & 1 \\
  v_3 & 0 \\
  v_4 & 0 
\end{bmatrix}
\]
Incidence Matrices

Definition

The incidence matrix is

\[
A = \begin{bmatrix}
-1 & 0 & a_3 \\
1 & -1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Here, \(A\) has four rows and three columns.
The incidence matrix is

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
1 & -1 & 1 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]
Incidence Matrices

Definition

The incidence matrix is

\[
A = \begin{bmatrix}
    -1 & 0 & 0 \\
    1 & -1 & 1 \\
    0 & 1 & 0 \\
    0 & 0 & -1
\end{bmatrix}
\]

Here, \( A \) has four rows and three columns.
Incidence Matrices

Definition

Every digraph is defined by its incidence matrix.
Incidence Matrices

Definition

Every digraph is defined by its incidence matrix.

\[
A = \begin{bmatrix}
-1 & 0 & -1 & 1 \\
1 & -1 & 0 & -1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]
Every digraph is defined by its incidence matrix.

\[ A = \begin{bmatrix}
  v_1 & a_1 & a_2 & a_3 & a_4 \\
  v_2 & -1 & 0 & -1 & 1 \\
  v_3 & 1 & -1 & 0 & -1 \\
  v_4 & 0 & 1 & 1 & 0
\end{bmatrix} \]
Incidence Matrices

Definition

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many rows and columns does its incidence matrix have?
Incidence Matrices

Definition

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many rows and columns does its incidence matrix have?

Answer
The incidence matrix is of the form

\[
A = \begin{bmatrix}
     \ast & \ast & \cdots & \ast \\
v_1 & a_1 & a_2 & \cdots & a_n \\
v_2 & \ast & \ast & \cdots & \ast \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_m & \ast & \ast & \cdots & \ast
\end{bmatrix}
\]
Incidence Matrices

Definition

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many rows and columns does its incidence matrix have?

Answer
The incidence matrix is of the form

\[
A = \begin{bmatrix}
*a* & * & \cdots & * \\
* & * & \cdots & * \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & *
\end{bmatrix}
\]

The incidence matrix has \( m \) rows and \( n \) columns.
Incidence Matrices

Definition

Question
Suppose our digraph has \( m \) nodes and \( n \) arrows. How many rows and columns does its incidence matrix have?

Answer
The incidence matrix is of the form

\[
A = \begin{bmatrix}
\ast & \ast & \ldots & \ast \\
\ast & \ast & \ldots & \ast \\
\vdots & \vdots & \ddots & \vdots \\
\ast & \ast & \ldots & \ast \\
\end{bmatrix}
\]

The incidence matrix has \( m \) rows and \( n \) columns.

\text{nodes} \leftrightarrow \text{rows} \quad \text{arrows} \leftrightarrow \text{columns}
Incidence Matrices

General Matrices

**Definition**

A $m \times n$ matrix is an array of numbers with $m$ rows and $n$ columns.
Incidence Matrices

General Matrices

Definition

A \( m \times n \) matrix is an array of numbers with \( m \) rows and \( n \) columns.

Note

We often represent matrices as

\[
A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\]
Incidence Matrices

General Matrices

Definition
A $m \times n$ matrix is an array of numbers with $m$ rows and $n$ columns.

Note
We often represent matrices as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We use the notation $a_{ij}$ to refer to the element of $A$ in the $i$th row and $j$th column.
Incidence Matrices

General Matrices

Example

Consider the matrix $A$ given by

$$A = \begin{bmatrix}
-5 & 2 & 9 & 9 & -1 \\
-8 & -10 & -6 & -6 & 6 \\
7 & -3 & 9 & -1 & -7
\end{bmatrix}$$
Example

Consider the matrix $A$ given by

$$A = \begin{bmatrix}
-5 & 2 & 9 & 9 & -1 \\
-8 & -10 & -6 & -6 & 6 \\
7 & -3 & 9 & -1 & -7
\end{bmatrix}$$

Then $A$ is a $3 \times 5$ matrix with

$$a_{11} = -5 \quad a_{12} = 2 \quad a_{13} = 9 \quad a_{14} = 9 \quad a_{15} = -1$$
$$a_{21} = -8 \quad a_{22} = -10 \quad a_{23} = -6 \quad a_{24} = -6 \quad a_{25} = 6$$
$$a_{31} = 7 \quad a_{32} = -3 \quad a_{33} = 9 \quad a_{34} = -1 \quad a_{35} = -7$$
Incidence Matrices

General Matrices

**Convention**

We typically use capital letters to denote matrices.
Incidence Matrices

General Matrices

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Definition
The collection of $m \times n$ matrices is denoted by $M_{m \times n}(\mathbb{R})$ or $\mathbb{R}^{m \times n}$. 
Incidence Matrices

General Matrices

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We typically use capital letters to denote matrices.

Definition
The collection of $m \times n$ matrices is denoted by $M_{m \times n}(\mathbb{R})$ or $\mathbb{R}^{m \times n}$.

Example
The incidence matrix $A$ of a digraph on 4 nodes and 10 arrows satisfies $A \in \mathbb{R}^{4 \times 10}$.
Example

Consider the matrices $A$, $B$, and $C$ given by

\[
A = \begin{bmatrix}
1 & 7 & -7 & 0 & 1 \\
2 & -1 & 0 & 1 & 2 \\
-2 & 2 & 0 & 25 & 2
\end{bmatrix} \quad B = \begin{bmatrix}
-1 & -1 \\
0 & 1 \\
1 & -1/2 \\
1 & 2
\end{bmatrix} \quad C = \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix}
\]
Example

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\end{bmatrix}$$

These matrices satisfy
Incidence Matrices

General Matrices

Example

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0 \\
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0
\end{bmatrix}
\]

These matrices satisfy

$A \in \mathbb{R}^{3 \times 5}$ \quad $A \not\in \mathbb{R}^{5 \times 3}$
### Example

Consider the matrices $A$, $B$, and $C$ given by

\[
A = \begin{bmatrix}
1 & 7 & -7 & 0 & 1 \\
2 & -1 & 0 & 1 & 2 \\
-2 & 2 & 0 & 25 & 2
\end{bmatrix} \\
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-1 & -1 \\
0 & 1 \\
1 & -1/2 \\
1 & 2
\end{bmatrix} \\
C = \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix}
\]

These matrices satisfy

\[
A \in \mathbb{R}^{3 \times 5} \\
B \in M_{4 \times 2}(\mathbb{R}) \\
A \notin \mathbb{R}^{5 \times 3} \\
B \notin M_{2 \times 4}(\mathbb{R})
\]
Example

Consider the matrices $A$, $B$, and $C$ given by

$A = \begin{bmatrix}
1 & 7 & -7 & 0 & 1 \\
2 & -1 & 0 & 1 & 2 \\
-2 & 2 & 0 & 25 & 2
\end{bmatrix}$

$B = \begin{bmatrix}
-1 & -1 \\
0 & 1 \\
1 & -1/2 \\
1 & 2
\end{bmatrix}$

$C = \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix}$

These matrices satisfy

$A \in \mathbb{R}^{3 \times 5}$

$B \in M_{4 \times 2}(\mathbb{R})$

$C \in \mathbb{R}^{3 \times 1}$

$A \not\in \mathbb{R}^{5 \times 3}$

$B \not\in M_{2 \times 4}(\mathbb{R})$

$C \not\in \mathbb{R}^{1 \times 3}$
Observation
Each row and each column of a matrix $A$ is a vector.
Incidence Matrices

General Matrices

Observation
Each row and each column of a matrix $A$ is a vector.

Row and Column Extraction
Let $A$ be a $m \times n$ matrix. We write

$$A = \begin{bmatrix}
\vec{r}_1 \\
\vec{r}_2 \\
\vdots \\
\vec{r}_m
\end{bmatrix} = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$$

to define the rows of $A$ as the vectors $\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m\}$ and the columns of $A$ as the vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$. 
Incidence Matrices
General Matrices

Example

By writing

\[
A = \begin{bmatrix}
4 & 1 & 0 & 0 \\
2 & 0 & 2 & -1 \\
-2 & 0 & -1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
\vec{r}_1 \\
\vec{r}_2 \\
\vec{r}_3 \\
\end{bmatrix} = \begin{bmatrix}
\vec{a}_1 \\
\vec{a}_2 \\
\vec{a}_3 \\
\vec{a}_4 \\
\end{bmatrix}
\]

we define the rows and columns of \( A \) as

\[
\vec{r}_1 = \langle 4, 1, 0, 0 \rangle \\
\vec{r}_2 = \langle 2, 0, 2, -1 \rangle \\
\vec{r}_3 = \langle -2, 0, -1, 1 \rangle
\]

\[
\vec{a}_1 = \langle 4, 2, -2 \rangle \\
\vec{a}_2 = \langle 1, 0, 0 \rangle \\
\vec{a}_3 = \langle 0, 2, -1 \rangle \\
\vec{a}_4 = \langle 0, -1, 1 \rangle
\]
The choice to organize “node data” in rows and “arrow data” in columns is, of course, arbitrary.
The choice to organize “node data” in rows and “arrow data” in columns is, of course, arbitrary.

\[
A = \begin{bmatrix}
-1 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 1 \\
\end{bmatrix}
\]
Transposition

Incidence Matrices

The choice to organize “node data” in rows and “arrow data” in columns is, of course, arbitrary.

\[ \begin{pmatrix} v_1 & \vdots & v_3 \end{pmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \]

The same information is stored in the transpose of the incidence matrix.

\[ A^T = \begin{bmatrix} a_1 & v_1 & v_2 & v_3 \\ a_2 & -1 & 1 & 0 \\ a_3 & -1 & 0 & 1 \end{bmatrix} \]

\[ A = \begin{bmatrix} v_1 & v_2 & v_3 \\ a_1 & -1 & -1 & 0 \\ a_2 & 1 & 0 & -1 \\ a_3 & 0 & 1 & 1 \end{bmatrix} \]
The incidence matrix $A$ and its transpose $A^T$ are given by
The incidence matrix $A$ and its transpose $A^T$ are given by

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ v_1 & -1 & 0 & 0 \\ v_2 & 1 & -1 & 1 \\ v_3 & 0 & 1 & 0 \\ v_4 & 0 & 0 & -1 \end{bmatrix} \quad A^T = \begin{bmatrix} a_1 & v_1 & v_2 & v_3 & v_4 \\ a_2 & -1 & 1 & 0 & 0 \\ a_3 & 0 & -1 & 1 & 0 \end{bmatrix}$$
The incidence matrix $A$ and its transpose $A^T$ are given by

$$A = \begin{bmatrix} v_1 & a_1 & 0 & 0 \\ v_2 & 1 & -1 & 1 \\ v_3 & 0 & 1 & 0 \\ v_4 & 0 & 0 & -1 \end{bmatrix} \quad \quad A^T = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ a_1 & -1 & 1 & 0 & 0 \\ a_2 & 0 & -1 & 1 & 0 \\ a_3 & 0 & 1 & 0 & -1 \end{bmatrix}$$

Note that $A$ is $4 \times 3$ while $A^T$ is $3 \times 4$. 
Transposition
Incidence Matrices

Since the incidence matrix $A$ and its transpose $A^T$ contain the same information, some mathematicians call $A^T$ the incidence matrix!
Transposition

Definition

The *transpose* of a $m \times n$ matrix $A$ is the $n \times m$ matrix $A^\top$ formed by interchanging the rows and columns of $A$. 

Definition
The transpose of a $m \times n$ matrix $A$ is the $n \times m$ matrix $A^\top$ formed by interchanging the rows and columns of $A$.

Cool animation: https://en.wikipedia.org/wiki/Transpose
Transposition
General Matrices

Example
Consider the matrices $A$ and $B$ given by

\[
A = \begin{bmatrix}
6 & -10 & 1 \\
2 & 1 & -2 \\
\end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix}
1 & -11 & -1 \\
1 & 2 & 0 \\
-4 & 0 & 1 \\
-1 & 1 & 1 \\
\end{bmatrix}_{4 \times 3}
\]
Transposition
General Matrices

Example
Consider the matrices $A$ and $B$ given by

$$A = \begin{bmatrix} 6 & -10 & 1 \\ 2 & 1 & -2 \end{bmatrix}_{2 \times 3} \quad B = \begin{bmatrix} 1 & -11 & -1 \\ 1 & 2 & 0 \\ -4 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}_{4 \times 3}$$

The transposes of these matrices are given by

$$A^T = \begin{bmatrix} 6 & -10 \\ -10 & 1 \\ 1 & -2 \end{bmatrix}_{3 \times 2} \quad B^T = \begin{bmatrix} 1 & 1 & -4 & -1 \\ -11 & 2 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{bmatrix}_{3 \times 4}$$
Transposition
The Involution Property

Question
What happens if we transpose twice?

Example
For
\[
A = \begin{bmatrix}
1 & -2 & 3 \\
-1 & 1 & 0
\end{bmatrix}
\]
we have
\[
A^\top = \begin{bmatrix}
1 & -1 \\
-2 & 1
\end{bmatrix}
\]
\[
(A^\top)^\top = \begin{bmatrix}
1 & -2 & 3 \\
-1 & 1 & 0
\end{bmatrix}
\]
So \((A^\top)^\top = A\).
Transposition
The Involution Property

**Question**
What happens if we transpose twice?

**Example**
For \( A = \begin{bmatrix} 1 & -28 & 3 \\ 1 & -1 & 1 \end{bmatrix} \) we have

\[
A^\top = \begin{bmatrix} 1 & 1 \\ -28 & -1 \\ 3 & 1 \end{bmatrix} \quad \quad (A^\top)^\top = \begin{bmatrix} 1 & -28 & 3 \\ 1 & -1 & 1 \end{bmatrix}
\]

So \((A^\top)^\top = A\).
Transposition
The Involution Property

**Question**
What happens if we transpose twice?

**Example**
For $A = \begin{bmatrix} 1 & -28 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ we have

$$A^\top = \begin{bmatrix} 1 & -28 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$(A^\top)^\top = \begin{bmatrix} 1 & -28 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

So $(A^\top)^\top = A$.

**Answer**
Transposing twice results in the original matrix.
Involution Property of Transposition

The transpose operation is an *involution*, meaning \((A^T)^T = A\).
Net Flow

Definition

Question
What is the “net flow” through each node?

For each node, take the difference of the number of "target arrows" and the number of "source arrows."

- Through $v_1$: $a_4 - a_1 - a_2 = -1$
- Through $v_2$: $a_1 + a_5 - a_3 - a_4 = 0$
- Through $v_3$: $a_3 + a_5 - a_5 = 1$
Net Flow

Definition

Question
What is the “net flow” through each node?

Answer
For each node, take the difference of the number of “target arrows” and the number of “source arrows.”

\[
\begin{align*}
\text{through } v_1 & \quad 1 - 1 - 1 = -1 \\
\end{align*}
\]
Net Flow

Definition

Question
What is the “net flow” through each node?

Answer
For each node, take the difference of the number of “target arrows” and the number of “source arrows.”

Through $v_1$:
$$\frac{1}{a_4} - \frac{1}{a_1} - \frac{1}{a_2} = -1$$

Through $v_2$:
$$\frac{1}{a_1} + \frac{1}{a_5} - \frac{1}{a_3} - \frac{1}{a_4} = 0$$
**Net Flow**

**Definition**

What is the “net flow” through each node?

**Question**

**Answer**

For each node, take the difference of the number of “target arrows” and the number of “source arrows.”

- **through \( v_1 \)**
  \[
  1 - 1 - 1 = -1
  \]

- **through \( v_2 \)**
  \[
  1 + 1 - 1 - 1 = 0
  \]

- **through \( v_3 \)**
  \[
  1 + 1 - 1 = 1
  \]
Net Flow
Via Incidence Vectors

Note
The net flows can be computed by *summing* the incidence vectors.
Net Flow
Via Incidence Vectors

Note
The net flows can be computed by *summing* the incidence vectors.

\[
\begin{bmatrix}
  v_1 \\
  v_2 \\
  v_3
\end{bmatrix} = \begin{bmatrix}
  -1 & 1 & 0 \\
  1 & 0 & 1 \\
  0 & 1 & 0
\end{bmatrix} + \begin{bmatrix}
  -1 & 0 & -1 \\
  0 & 1 & 0 \\
  1 & -1 & -1
\end{bmatrix} + \begin{bmatrix}
  0 & -1 & 1 \\
  1 & 0 & 0 \\
  0 & 1 & 1
\end{bmatrix} + \begin{bmatrix}
  1 & -1 & 0 \\
  0 & 1 & 1 \\
  0 & 0 & 0
\end{bmatrix} + \begin{bmatrix}
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  1 & 0 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  -1 - 1 + 0 + 1 + 0 \\
  1 + 0 - 1 - 1 + 1 \\
  0 + 1 + 1 + 0 - 1
\end{bmatrix} = \begin{bmatrix}
  -1 \\
  0 \\
  1
\end{bmatrix}
\]
**Net Flow**

**Summing General Vectors**

**Definition**

Let \( \mathbf{v}, \mathbf{w} \in \mathbb{R}^n \) be vectors. The *sum* of \( \mathbf{v} \) and \( \mathbf{w} \) is

\[
\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}
\]
Net Flow
Summing General Vectors

Definition
Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ be vectors. The sum of $\mathbf{v}$ and $\mathbf{w}$ is

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

Example
$$\begin{bmatrix} -1 \\ -8 \\ 9 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 + 5 \\ -8 + (-1) \\ 9 + (-1) \\ 1 + (-3) \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ 8 \\ -2 \end{bmatrix}$$
Note
If $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^n$, then $\vec{v} + \vec{w}$ is a vector in $\mathbb{R}^n$. 
Net Flow
Summing General Vectors

Note
If \( \vec{v} \) and \( \vec{w} \) are vectors in \( \mathbb{R}^n \), then \( \vec{v} + \vec{w} \) is a vector in \( \mathbb{R}^n \).

Note
The sum \( \vec{v} + \vec{w} \) is only defined if \( \vec{v} \) and \( \vec{w} \) have the same dimension.
Net Flow
Summing General Vectors

Note
If \( \vec{v} \) and \( \vec{w} \) are vectors in \( \mathbb{R}^n \), then \( \vec{v} + \vec{w} \) is a vector in \( \mathbb{R}^n \).

Note
The sum \( \vec{v} + \vec{w} \) is only defined if \( \vec{v} \) and \( \vec{w} \) have the same dimension.

Example
The sum
\[
\begin{bmatrix}
3 \\
-1 \\
2
\end{bmatrix}
+ \begin{bmatrix}
1 \\
1 \\
-1
\end{bmatrix}
\]
is nonsensical!
Weighted Digraphs

Definition

Arrows in digraphs are often associated to scalars. These scalars are called *weights* and these digraphs are called *weighted digraphs*.
Financial transactions are measured in dollars.
Weighted Digraphs

Definition

When modeling road networks, one might use speed limits as weights.

\[ v_1 \xrightarrow{a_1} 45 \text{ mph} \xrightarrow{a_2} 30 \text{ mph} \xrightarrow{a_4} 40 \text{ mph} \xleftarrow{a_5} 50 \text{ mph} \xleftarrow{a_3} 25 \text{ mph} \xrightarrow{a_2} 30 \text{ mph} \]

\[ v_2 \]

\[ v_3 \]

\[ v_4 \]
The weighted incidence vectors of a weighted digraph are obtained by “scaling” the original incidence vectors by the corresponding weights.
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\[
\begin{align*}
4 \cdot \vec{a}_1 &= \begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix} & 11 \cdot \vec{a}_2 &= \begin{bmatrix} 0 \\ -11 \\ 11 \end{bmatrix} & -2 \cdot \vec{a}_3 &= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}
\end{align*}
\]
Weighted Digraphs
Scaling General Vectors

Definition (Algebraic)

Let $\vec{v} \in \mathbb{R}^n$ be a vector and let $c \in \mathbb{R}$ be a scalar. The scalar product of $c$ and $\vec{v}$ is

$$c \cdot \vec{v} = c \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_n \end{bmatrix}$$

Example $(−4) \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ −1 \end{bmatrix} = \begin{bmatrix} (−4) \cdot 3 \\ (−4) \cdot 2 \\ (−4) \cdot 1 \\ (−4) \cdot (−1) \end{bmatrix} = \begin{bmatrix} −12 \\ −8 \\ −4 \\ 4 \end{bmatrix}$
Definition (Algebraic)
Let $\vec{v} \in \mathbb{R}^n$ be a vector and let $c \in \mathbb{R}$ be a scalar. The scalar product of $c$ and $\vec{v}$ is

$$c \cdot \vec{v} = c \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} c \cdot v_1 \\ c \cdot v_2 \\ \vdots \\ c \cdot v_n \end{bmatrix}$$

Example
$$(-4) \cdot \begin{bmatrix} 3 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-4) \cdot 3 \\ (-4) \cdot 2 \\ (-4) \cdot 1 \\ (-4) \cdot (-1) \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \\ -4 \\ 4 \end{bmatrix}$$
Weighted Digraphs

Weighted Net Flow

We can determine the *weighted net flow* through each node by summing the weighted incidence vectors.
Weighted Digraphs

Weighted Net Flow

We can determine the *weighted net flow* through each node by summing the weighted incidence vectors.

\[
\langle v_1, v_2, v_3 \rangle = 5 \cdot \vec{a}_1 + 2 \cdot \vec{a}_2 + 7 \cdot \vec{a}_3
\]
We can determine the weighted net flow through each node by summing the weighted incidence vectors.

\[
\langle v_1, v_2, v_3 \rangle = 5 \cdot \vec{a}_1 + 2 \cdot \vec{a}_2 + 7 \cdot \vec{a}_3 \\
= 5 \cdot \langle -1, 1, 0 \rangle + 2 \cdot \langle 0, -1, 1 \rangle + 7 \cdot \langle -1, 0, 1 \rangle
\]
Weighted Digraphs

Weighted Net Flow

We can determine the *weighted net flow* through each node by summing the weighted incidence vectors.

\[
\langle v_1, v_2, v_3 \rangle = 5 \cdot \langle -1, 1, 0 \rangle + 2 \cdot \langle 0, -1, 1 \rangle + 7 \cdot \langle -1, 0, 1 \rangle \\
= \langle -5, 5, 0 \rangle + \langle 0, -2, 2 \rangle + \langle -7, 0, 7 \rangle
\]
We can determine the weighted net flow through each node by summing the weighted incidence vectors.

\[
\begin{align*}
\langle v_1, v_2, v_3 \rangle &= 5 \cdot \vec{a}_1 + 2 \cdot \vec{a}_2 + 7 \cdot \vec{a}_3 \\
&= 5 \cdot \langle -1, 1, 0 \rangle + 2 \cdot \langle 0, -1, 1 \rangle + 7 \cdot \langle -1, 0, 1 \rangle \\
&= \langle -5, 5, 0 \rangle + \langle 0, -2, 2 \rangle + \langle -7, 0, 7 \rangle \\
&= \langle -12, 3, 9 \rangle
\end{align*}
\]
Definition

A *linear combination* of vectors \( \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \} \) is a sum of the form

\[
c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + \cdots + c_n \cdot \vec{v}_n
\]

where \( c_1, c_2, \ldots, c_n \) are scalars.
Example
Consider the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ given by

$$
\vec{v}_1 = \langle -1, -1, 1 \rangle \quad \vec{v}_2 = \langle 2, 0, 1 \rangle \quad \vec{v}_3 = \langle -1, -4, 5 \rangle
$$
Example
Consider the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ given by

$$\mathbf{v}_1 = \langle -1, -1, 1 \rangle \quad \mathbf{v}_2 = \langle 2, 0, 1 \rangle \quad \mathbf{v}_3 = \langle -1, -4, 5 \rangle$$

By choosing $c_1 = 3$, $c_2 = -2$, and $c_3 = 4$, we may form the linear combination

$$c_1 \cdot \mathbf{v}_1 + c_2 \cdot \mathbf{v}_2 + c_3 \cdot \mathbf{v}_3 = 3 \cdot \langle -1, -1, 1 \rangle + (-2) \cdot \langle 2, 0, 1 \rangle + 4 \cdot \langle -1, -4, 5 \rangle$$

$$= \langle -3, -3, 3 \rangle + \langle -4, 0, -2 \rangle + \langle -4, -16, 20 \rangle$$

$$= \langle -11, -19, 21 \rangle$$

We say "$\langle -11, -19, 21 \rangle$ is a linear combination of \{ $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ \}."
Example

Consider the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ given by

$$\vec{v}_1 = \langle -1, -1, 1 \rangle \quad \vec{v}_2 = \langle 2, 0, 1 \rangle \quad \vec{v}_3 = \langle -1, -4, 5 \rangle$$

By choosing $c_1 = 3$, $c_2 = -2$, and $c_3 = 4$, we may form the linear combination

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3$$
Example
Consider the vectors \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \) given by

\[
\vec{v}_1 = \langle -1, -1, 1 \rangle \quad \vec{v}_2 = \langle 2, 0, 1 \rangle \quad \vec{v}_3 = \langle -1, -4, 5 \rangle
\]

By choosing \( c_1 = 3, c_2 = -2, \) and \( c_3 = 4, \) we may form the linear combination

\[
c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3 \\
= 3 \cdot \langle -1, -1, 1 \rangle + (-2) \cdot \langle 2, 0, 1 \rangle + 4 \cdot \langle -1, -4, 5 \rangle
\]
Example
Consider the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ given by

\[ \vec{v}_1 = \langle -1, -1, 1 \rangle \quad \vec{v}_2 = \langle 2, 0, 1 \rangle \quad \vec{v}_3 = \langle -1, -4, 5 \rangle \]

By choosing $c_1 = 3$, $c_2 = -2$, and $c_3 = 4$, we may form the linear combination

\[
c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3 = 3 \cdot \langle -1, -1, 1 \rangle + (-2) \cdot \langle 2, 0, 1 \rangle + 4 \cdot \langle -1, -4, 5 \rangle
\]
\[
= \langle -3, -3, 3 \rangle + \langle -4, 0, -2 \rangle + \langle -4, -16, 20 \rangle
\]
Weighted Digraphs

Linear Combinations

Example
Consider the vectors \( \vec{v}_1, \vec{v}_2, \) and \( \vec{v}_3 \) given by

\[
\vec{v}_1 = \langle -1, -1, 1 \rangle \quad \vec{v}_2 = \langle 2, 0, 1 \rangle \quad \vec{v}_3 = \langle -1, -4, 5 \rangle
\]

By choosing \( c_1 = 3, \ c_2 = -2, \) and \( c_3 = 4, \) we may form the linear combination

\[
c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3
\]

\[
= 3 \cdot \langle -1, -1, 1 \rangle + (-2) \cdot \langle 2, 0, 1 \rangle + 4 \cdot \langle -1, -4, 5 \rangle
\]

\[
= \langle -3, -3, 3 \rangle + \langle -4, 0, -2 \rangle + \langle -4, -16, 20 \rangle
\]

\[
= \langle -11, -19, 21 \rangle
\]
Example
Consider the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ given by

$$\vec{v}_1 = \langle -1, -1, 1 \rangle \quad \vec{v}_2 = \langle 2, 0, 1 \rangle \quad \vec{v}_3 = \langle -1, -4, 5 \rangle$$

By choosing $c_1 = 3$, $c_2 = -2$, and $c_3 = 4$, we may form the linear combination

$$c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2 + c_3 \cdot \vec{v}_3$$

$$= 3 \cdot \langle -1, -1, 1 \rangle + (-2) \cdot \langle 2, 0, 1 \rangle + 4 \cdot \langle -1, -4, 5 \rangle$$

$$= \langle -3, -3, 3 \rangle + \langle -4, 0, -2 \rangle + \langle -4, -16, 20 \rangle$$

$$= \langle -11, -19, 21 \rangle$$

We say “$\langle -11, -19, 21 \rangle$ is a linear combination of $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$.”
Question
Is it possible to set weights so that the net flow through the nodes is $\vec{b} = \langle 3, -1, 0 \rangle$?
**Question**

Is it possible to set weights so that the net flow through the nodes is \( \vec{b} = \langle 3, -1, 0 \rangle \)?

**Answer**

We want to determine if \( \vec{b} = \langle 3, -1, 0 \rangle \) is a *linear combination* of the incidence vectors.
Weighted Digraphs
Linear Combinations

Question
Is it possible to set weights so that the net flow through the nodes is \( \vec{b} = \langle 3, -1, 0 \rangle \)?

Answer
We want to determine if \( \vec{b} = \langle 3, -1, 0 \rangle \) is a linear combination of the incidence vectors. This means that we need to solve

\[
\langle 3, -1, 0 \rangle = c_1 \cdot \langle -1, 1, 0 \rangle + c_2 \cdot \langle 0, -1, 1 \rangle + c_3 \cdot \langle -1, 0, 1 \rangle
\]
Question
Is it possible to set weights so that the net flow through the nodes is \( \vec{b} = \langle 3, -1, 0 \rangle \)?

Answer
We want to determine if \( \vec{b} = \langle 3, -1, 0 \rangle \) is a linear combination of the incidence vectors. This means that we need to solve

\[
\langle 3, -1, 0 \rangle = c_1 \cdot \vec{a}_1 + c_2 \cdot \vec{a}_2 + c_3 \cdot \vec{a}_3 \\
= c_1 \cdot \langle -1, 1, 0 \rangle + c_2 \cdot \langle 0, -1, 1 \rangle + c_3 \cdot \langle -1, 0, 1 \rangle \\
= \langle -c_1 - c_3, c_1 - c_2, c_2 + c_3 \rangle
\]
Weighted Digraphs
Linear Combinations

Question
Is it possible to set weights so that the net flow through the nodes is $\overrightarrow{b} = \langle 3, -1, 0 \rangle$?

Answer
We want to determine if $\overrightarrow{b} = \langle 3, -1, 0 \rangle$ is a *linear combination* of the incidence vectors. This means that we need to solve

\[
\langle 3, -1, 0 \rangle = c_1 \cdot \overrightarrow{a}_1 + c_2 \cdot \overrightarrow{a}_2 + c_3 \cdot \overrightarrow{a}_3
\]

\[
= c_1 \cdot \langle -1, 1, 0 \rangle + c_2 \cdot \langle 0, -1, 1 \rangle + c_3 \cdot \langle -1, 0, 1 \rangle
\]

\[
= \langle -c_1 - c_3, c_1 - c_2, c_2 + c_3 \rangle
\]

for the “unknown” weights $c_1$, $c_2$, and $c_3$. 

\[ \]