1. Introduction

My research is in differential geometry and the geometry of partial differential equations. I am primarily interested in special geometric structures and, in particular, problems related to special holonomy.

The holonomy group of a Riemannian manifold is a basic invariant of the manifold defined as the group generated by parallel transport around closed loops. In the 1950s, Berger classified the possible holonomy groups of (simply connected, irreducible, non-symmetric) Riemannian manifolds [5] (see table 1). A Riemannian manifold with holonomy group one of $SU(m), Sp(q) \cdot Sp(1), Sp(q), G_2,$ or $Spin(7)$ is said to have special holonomy. Such manifolds are tantalising objects for differential geometers from several points of view:

1. Manifolds with special holonomy are all Einstein manifolds and, except in the $Sp(q) \cdot Sp(1)$ case, are Ricci-flat. In fact, all currently known examples of compact Ricci-flat manifolds have special holonomy.
2. They are host to special gauge theoretical equations, analogous to the instanton equations central to Donaldson’s work on the topology of smooth 4-manifolds (see §2.1).
3. They carry a distinguished class of submanifolds, analogous to complex submanifolds of a Kähler manifold, the calibrated submanifolds, which are volume minimising in their homology class (see §3.1).
4. They play an important role in physics as the model for the extra dimensions of space in string theory and M theory.

The construction and study of metrics with special holonomy has been one of the major themes in differential geometry in the second half of the 20th century and beyond, and many compelling problems remain. Recently, the field has been given fresh impetus by the establishment of the ‘Simons Collaboration on Special Holonomy in Geometry, Analysis and Physics,’ a collaboration with which I have been involved since 2016.

In my research, I am interested in constructing examples of spaces related to special holonomy, as well as constructing examples of the geometric objects (instantons and calibrated submanifolds) associated with them. Most of my work is conducted using the language and techniques of the theory of exterior differential systems [8]. This theory, initiated by Élie Cartan at the start of the 20th century, is particularly suited for the analysis of the overdetermined PDE systems that tend to appear in these geometric situations.

Table 1. Berger’s list of holonomy groups

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Holonomy group</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$SO(n)$</td>
<td>Generic holonomy</td>
</tr>
<tr>
<td>$2m$</td>
<td>$U(m)$</td>
<td>Kähler</td>
</tr>
<tr>
<td>$2m$</td>
<td>$SU(m)$</td>
<td>Calabi-Yau</td>
</tr>
<tr>
<td>$4q$</td>
<td>$Sp(q) \cdot Sp(1)$</td>
<td>Quaternion-Kähler</td>
</tr>
<tr>
<td>$4q$</td>
<td>$Sp(q)$</td>
<td>Hyper-Kähler</td>
</tr>
<tr>
<td>7</td>
<td>$G_2$</td>
<td>$G_2$-holonomy</td>
</tr>
<tr>
<td>8</td>
<td>$Spin(7)$</td>
<td>$Spin(7)$-holonomy</td>
</tr>
</tbody>
</table>
2. Previous work

2.1. Gauge theory on Aloff-Wallach spaces (joint with Gonçalo Oliveira). A 7-dimensional Riemannian manifold \((M,g)\) is called a \(G_2\)-manifold if it has holonomy group \(G_2\). A \(G_2\)-manifold admits a special harmonic 3-form \(\varphi\), which in fact determines both the metric \(g\) and an orientation. In the language of \(G\)-structures, the 3-form \(\varphi\) defines a torsion-free \(G_2\)-structure on \(M\).

There is an intriguing gauge-theoretical equation that can be written on a \(G_2\)-manifold, the \(G_2\)-instanton equation. Let \(A\) be a connection on principal \(G\)-bundle \(P\), and let \(F_A\) denote the curvature of \(A\). Then \(A\) is said to be a \(G_2\)-instanton if

\[
F_A \wedge \star \varphi = 0,
\]

where \(\star \varphi\) denotes the Hodge star operator induced by \(\varphi\). This equation exhibits formal similarities to the anti-self-dual equations on a Riemannian 4-manifold, justifying the name instanton for solutions.

An important problem in \(G_2\)-geometry is to develop invariants to distinguish different \(G_2\)-manifolds. One suggestion, due to Donaldson and Thomas [13] and elaborated upon in [14], is to define an invariant by ‘counting’ \(G_2\)-instantons. This is considered a hard problem and there are several difficulties to be overcome. In particular, it is expected that one may need to consider \(G\)-structures \(\varphi\) that are not torsion-free, meaning that \(\varphi\) is not harmonic and the metric \(g\) does not have holonomy \(G_2\). There is in fact a larger class of \(G\)-structures for which equation (2.1) still lies in an elliptic complex. These are the so-called coclosed \(G_2\)-structures, where \(\varphi\) is only required to satisfy \(d \star \varphi = 0\). A particularly interesting subcase is when \(\varphi\) satisfies \(d \varphi = \lambda \varphi\), the nearly parallel case. These structures determine an Einstein metric with positive scalar curvature and the holonomy of the cone metric on \(\mathbb{R} \times M\) is contained in \(\text{Spin}(7)\).

In my paper [3], joint with Gonçalo Oliveira, we investigate \(G_2\)-instantons on the Aloff-Wallach spaces \(N_{k,l}\). These are homogeneous spaces given by the quotient \(SU(3)/U(1)\), where \((k,l) \in \mathbb{Z}^2 - (0,0)\) are integers parametrising the embedding of the circle inside the maximal torus. They carry many homogeneous coclosed \(G_2\)-structures, making them a good testing ground for ideas. In [3] we classify homogeneous instantons (with gauge group \(U(1)\) or \(SO(3)\)) with respect to all homogeneous coclosed \(G_2\)-structures for most Aloff-Wallach spaces \(N_{k,l}\). We find various examples where \(G_2\)-instantons can be used to distinguish between two different nearly parallel \(G_2\)-structures on the same Aloff-Wallach space. This is the first time that \(G_2\)-instantons have been systematically used to distinguish between \(G_2\)-structures. For example, the space \(N_{1,1}\) carries two homogeneous nearly parallel \(G_2\)-structures. One of these, \(\varphi^\text{ts}\), is 3-Sasakian, meaning the cone is hyper-Kähler, while the other, \(\varphi^\text{np}\), is strictly nearly parallel, meaning the cone has holonomy \(\text{Spin}(7)\).

**Theorem 2.1.** There are no irreducible invariant \(G_2\)-instantons for \(\varphi^\text{ts}\) with gauge group \(SO(3)\), while such instantons do exist for \(\varphi^\text{np}\).

We use our classification to give examples of other interesting phenomena, such as examples of \(G_2\)-instantons that are not locally area minimising (this cannot happen in the torsion-free case). We also find the following behavior of families of \(G_2\)-instantons.

**Theorem 2.2.** Let \(N_{k,l}\) be an Aloff-Wallach space. There is an \(SO(3)\)-bundle \(P\) and a continuous family \(\{\varphi(s)\}_{s \in \mathbb{R}}\) of coclosed \(G_2\)-structures such that as \(s\) decreases to 0, two irreducible \(G_2\)-instantons on \(P\) merge into the same reducible and obstructed \(G_2\)-instanton.

2.2. Quadratic closed \(G_2\)-structures. In my work [2], I study closed \(G_2\)-structures satisfying a certain natural PDE system, the quadratic condition. A \(G_2\)-structure \(\varphi \in \Omega^3(M)\) on a 7-manifold \(M\) is called closed if it satisfies \(d\varphi = 0\). This is another weakening of the torsion-free condition, complementary to the coclosed condition that features in §2.1. When \(\varphi\) is closed we have

\[
d \star \varphi = \tau \wedge \varphi,
\]

where \(\tau\) is a 2-form satisfying \(\tau \wedge \star \varphi = 0\), called the torsion of the closed \(G_2\)-structure.

Motivation for the study of closed \(G_2\)-structures comes from the construction of compact manifolds with holonomy \(G_2\). The only known method for constructing such manifolds is to start with a compact manifold \(M\) with a closed \(G_2\)-structure \(\varphi\), such that the torsion \(\tau\) is ‘small’ in some suitable sense. Then one may use a result of Joyce [22] to perturb \(\varphi\) to some torsion-free \(G_2\)-structure \(\hat{\varphi}\).
Closed $G_2$-structures are also important because they serve as the initial conditions for the Laplacian flow. This is the nonlinear flow equation for a $G_2$-structure $\varphi$ given by

$$\frac{d}{dt} \varphi = \Delta_\varphi \varphi,$$

where $\Delta_\varphi$ is the Laplacian induced by $\varphi$. In the case where $\varphi$ is closed we have $\Delta_\varphi \varphi = d\tau$ and the closed condition is preserved along the flow. The critical points of the flow are exactly the torsion-free $G_2$-structures, and one might expect to be able to use the flow to deform an initial closed $G_2$-structure into a torsion-free one. Of course, the situation may be more complicated than this, and while progress is being made (see [27], [25], [26] for example), much work remains to be done.

A closed $G_2$-structure induces a metric $g_\varphi$. In [7], Bryant gives formulas for the scalar and Ricci curvatures of $g_\varphi$,

$$\text{Scal}(g_\varphi) = -\frac{1}{2} |\tau|^2, \quad \text{Ric}(g_\varphi) = \frac{1}{4} |\tau|^2 - \frac{1}{4} j_\varphi (d\tau - \frac{1}{2} *_\varphi (\tau \wedge \tau)),$$

where $j_\varphi$ is a certain algebraic map $j_\varphi : \Omega^3(M) \rightarrow \Gamma(S^2T^*M)$. When $\varphi$ is torsion-free, the metric $g_\varphi$ has holonomy contained in $G_2$ and these formulas verify the fact, mentioned in §1, that such metrics are Ricci-flat. Given these formulas, a natural question is

**Question 2.3.** Can a closed $G_2$-structure induce an Einstein metric with non-zero scalar curvature?

One can use (2.5) and a little algebra to write the Einstein condition using differential forms as

$$d\tau = \frac{3}{4} |\tau|^2 \varphi + \frac{1}{2} *_\varphi (\tau \wedge \tau).$$

In the case where $M$ is assumed compact, question 2.3 was answered in the negative by Cleyton and Ivanov [10]. In [7], Bryant gives a simpler proof and extends the result to the following

**Theorem 2.4** (Bryant). Let $M$ be a compact 7-manifold with a closed $G_2$-structure $\varphi$. Suppose

$$|\text{Ric}(g_\varphi)|^2 \leq \frac{4}{7} C \text{Scal}(g_\varphi)^2,$$

for some constant $C \leq 1$. Then

- If $C < 1$, $\tau = 0$
- If $C = 1$, equality holds in (2.7) everywhere on $M$, and $\tau$ satisfies

$$d\tau = \frac{1}{6} |\tau|^2 \varphi + \frac{1}{6} *_\varphi (\tau \wedge \tau).$$

Equation (2.8) is equivalent to (2.7) with $C = 1$, and a closed $G_2$-structure satisfying this condition is called extremally Ricci-pinched, or ERP for short. Prior to my work there were essentially only two known examples, due to Bryant [7] and Lauret [24], both homogeneous.

Equations (2.6) and (2.8) have a similar form. Motivated by this, Bryant [7] introduced a generalisation of both conditions

**Definition 2.5.** Let $\lambda$ be a constant. A closed $G_2$-structure $\varphi$ is called $\lambda$-quadratic if it satisfies

$$d\tau = \frac{1}{2} |\tau|^2 \varphi + \lambda \left( \frac{1}{2} \varphi + *_\varphi (\tau \wedge \tau) \right).$$

The following cases are of particular interest:

- $\frac{1}{2}$-quadratic $\iff$ Einstein.
- $\frac{1}{6}$-quadratic $\iff$ ERP.
- $0$-quadratic $\iff$ $\varphi$ is an eigenform, i.e. an eigenfunction for $\Delta_\varphi$. These are a special type of soliton for the Laplace flow (2.3), analogous to Einstein metrics under Ricci flow.
- $-\frac{1}{4}$-quadratic $\iff$ Conformally flat metric. However, conformal flatness places more restrictions than just $-\frac{1}{8}$-quadratic.

In [2], I study equation (2.9) from the point of view of exterior differential systems. It turns out that the Cartan-Kähler theorem cannot be used to prove existence of solutions (at least without considerable further efforts). In the non-ERP case, I get the following results

**Results 2.6.**

- New examples of $\lambda$-quadratic closed $G_2$-structures for $\lambda = -1, -\frac{1}{8}, \frac{1}{3}, \frac{2}{5}, \frac{3}{4}$. These are the first known examples with $\lambda \neq \frac{1}{6}$. 
Conversely, if we have a $K$-structure on $M$, then a calibrated distribution on $M$ and a calibrated distribution. I study the geometry of calibrated distributions by studying the geometry of such $K$-structures.

The ERP case also yields noteworthy results. Using work of Bryant in [7], I define a canonical $U(2)$-structure refining an ERP closed $G_2$-structure. The torsion of a generic $U(2)$-structure takes values in a tensor bundle of rank 119, but in this restrictive situation I show that the torsion must take values in a subbundle of rank 26. This bundle splits in a natural way as

$$ ERP\text{ torsion space} = A \oplus P \oplus Z, $$

where $A$, $P$, and $Z$ are subbundles of rank 12, 10, and 4 respectively, each modeled on an irreducible representation of $U(2)$. I then investigate the situation when the torsion takes values in only one of these subbundles. An ERP closed $G_2$-structure is said to be of type $A$ (resp. $P$, $Z$) if its torsion takes values only in $A$ (resp. $P$, $Z$).

**Theorem 2.7.** There exists a Weierstrass-type formula for ERP closed $G_2$-structures of type $A$. Given a meromorphic function on the unit disc, one can construct an ERP closed $G_2$-structure of type $A$ — and conversely, every such structure can be locally described in this way.

**Theorem 2.8.** Every ERP closed $G_2$-structure of type $P$ is locally equivalent to a certain $\mathbb{R}^4$-bundle over a spacelike maximal 3-dimensional submanifold of the pseudo-Riemannian symmetric space $SO(3,3)/SO(3,2)$. Conversely, given a spacelike maximal 3-dimensional submanifold $N$ of $SO(3,3)/SO(3,2)$, there is a canonical $\mathbb{R}^4$-bundle over $N$ carrying an ERP closed $G_2$-structure of type $P$. These examples are all solutions for the Laplace flow (2.3).

The situation with structures of type $Z$ is not completely understood. I have identified a subclass of type $Z$ structures that admits a Weierstrass-type formula (à la theorem 2.7), but it remains unclear if every structure of type $Z$ must lie in this subclass.

3. Current work

3.1. Calibrated distributions and $SO(4)$-structures. Let $(M, g)$ be a Riemannian manifold of dimension $n$. A $k$-form $\theta$ is called a calibration if

- $d\theta = 0$
- for any $p \in M$ and any oriented $k$-dimensional subspace $E$ of $T_pM$, $\theta|_E = \lambda\text{vol}|_E$, where $\lambda \leq 1$.

A $k$-dimensional subspace $E$ satisfying $\theta|_E = \text{vol}|_E$ is called calibrated, and a $k$-submanifold is called a calibrated submanifold if all of its tangent planes are calibrated. Calibrated submanifolds have the remarkable property of being volume minimising in their homology class, and they occupy a special place in the study of minimal submanifolds.

The theory of calibrated submanifolds was initiated by Harvey and Lawson in the 80s [20]. Although they concentrated on calibrations in Euclidean space, the link with special holonomy was quickly established. Indeed, the covariant constant differential forms on manifolds with special holonomy provide important examples of calibrations, and the submanifolds calibrated by these forms are extremely important in the study of these manifolds.

In an ongoing project I am investigating the geometry of calibrated distributions on manifolds of special holonomy.

**Definition 3.1.** Let $(M, g)$ be a Riemannian manifold endowed with a calibration $\theta$. A calibrated distribution on $M$ is a subbundle $Q$ of $TM$ such that for all $p \in M$, $Q_p$ is calibrated by $\theta$.

Suppose $M$ has holonomy group $H$, and $\theta$ is a covariant constant differential form giving a calibration on $M$. Then a calibrated distribution on $M$ can be thought of as an extra geometric structure on $M$. Let $K < H$ be the subgroup of $H$ stabilising $\theta$. A calibrated distribution on $M$ induces a $K$-structure on $M$, the torsion of which must lie in a bundle modeled on $\mathfrak{h}/\mathfrak{t} \otimes \mathbb{R}^n$. Conversely, if we have a $K$-structure on $M$ with torsion lying in $\mathfrak{h}/\mathfrak{t} \otimes \mathbb{R}^n$, then we automatically get a metric with holonomy $H$ and a calibrated distribution. I study the geometry of calibrated distributions by studying the geometry of such $K$-structures.
In the $G_2$ case we may take either the 3-form $\varphi$ or the 4-form $\ast \varphi$ as the calibration. Then $H = G_2$ and $K = SO(4)$. The torsion space $\mathfrak{g}_2/\mathfrak{so}(4) \otimes \mathbb{R}^7$ splits as
\begin{equation}
V_{1,5} \oplus V_{1,3} \oplus V_{1,1} \oplus V_{2,4} \oplus V_{2,2} \oplus V_{0,4} \oplus V_{0,2},
\end{equation}
where $V_{a,b}$ denotes the bundle modeled on the $SO(4)$-module of highest weight $(a, b)$. Using this splitting I am able to obtain characterisations of the Bryant-Salamon manifolds, the first known complete manifolds with holonomy $G_2$.

**Theorem 3.2.** Let $M$ be a 7-manifold with an $SO(4)$-structure.

- If the torsion of the structure takes values in $V_{1,1}$, then $M$ is locally equivalent to a Bryant-Salamon example on the spinor bundle over a 3-dimensional space form.
- If the torsion of the structure takes values in $V_{0,2}$, then $M$ is locally equivalent to a Bryant-Salamon example $\Lambda_5^1 X$, where $X$ is a self-dual Einstein 4-manifold.

In the cases where the torsion takes values in either $V_{1,5}$ or $V_{2,4}$ there is a link with the theory of semi-flat $G_2$-manifolds, i.e. $G_2$-manifolds endowed with a calibrated fibration where the fibres are flat tori. The semi-flat coassociative case was considered by Baraglia in [4], and a semi-flat coassociative fibration was shown to be equivalent to a maximal spacelike 3-submanifold in $\mathbb{R}^{3,3}$ (cf. §2.2). In my work I intend to classify algebraically special semi-flat $G_2$-manifolds, i.e. semi-flat $G_2$-manifolds where the torsion tensor (which must in these cases take values in either $V_{1,5}$ or $V_{2,4}$) has a non-trivial stabiliser. This should lead to novel explicit examples of $G_2$-manifolds, and I expect well-known integrable systems will appear in some of the cases.

### 3.2. Irreducible $SO(3)$-Geometry in Dimension 5

The action of the group $SO(3)$ by conjugation on the space of traceless symmetric $3 \times 3$ matrices gives an embedding $SO(3) \subset SO(5)$. A 5-manifold whose structure group reduces to this copy of $SO(3)$ is said to carry an irreducible $SO(3)$-structure. The integrable examples of these structures are the symmetric spaces $\mathbb{R}^5, SU(3)/SO(3)$ and $SL(3)/SO(3)$, and general irreducible $SO(3)$-structures may be thought of as non-integrable analogues of these examples.

The investigation of irreducible $SO(3)$-structures structures was suggested by Friedrich in [18], and the first results were obtained in [6]. The differential geometry and topology of such structures was further studied in [1], and homogeneous examples on 5-dimensional Lie groups were described in [9]. In an ongoing project I continue the investigation into the geometry of these structures.

The reduction of structure group of a Riemannian 5-manifold $(M, g)$ from $SO(5)$ to $SO(3)$ can be encoded in a symmetric 3-tensor $\Upsilon$ satisfying certain algebraic properties. Nurowski and Bobiński single out the irreducible $SO(3)$-structures satisfying
\begin{equation}
\nabla_X \Upsilon (X, X, X) = 0 \text{ for all } X \in TM
\end{equation}
for particular attention, and call them nearly integrable. The terminology nearly integrable is meant to bring to mind an analogy with nearly Kähler structures. Indeed, a nearly Kähler manifold is an almost-Hermitian manifold $(N^{2n}, g, J)$ where the almost-complex structure $J$ satisfies
\begin{equation}
\nabla_X J (X, X) = 0 \text{ for all } X \in TN.
\end{equation}
However, while nearly Kähler structures exist in abundance locally, I have found that the local existence theory of nearly integrable $SO(3)$-structures is much more complicated and restricted.

**Theorem 3.3.** The space of germs of nearly integrable $SO(3)$-structures is finite dimensional.

Let $\mathcal{H}^n$ denote the irreducible $SO(3)$-module of dimension $2n + 1$. The nearly integrable condition (3.2) is equivalent to requiring that the torsion of the $SO(3)$-structure takes values in the space $\mathcal{H}^3 \oplus \mathcal{H}^3$, whereas the torsion of a generic $SO(3)$-structure takes values in
\begin{equation}
\frac{\mathfrak{so}(5)}{\mathfrak{so}(3)} \otimes \mathbb{R}^5 \cong \mathcal{H}^5 \oplus \mathcal{H}^4 \oplus \mathcal{H}^3 \oplus \mathcal{H}^2 \oplus \mathcal{H}^1,
\end{equation}
and I study the cases where the torsion takes values in only one of the spaces in this splitting.
4. Future work

4.1. Cohomogeneity-one nearly parallel $G_2$-structures. Following work of Podestà and Spiro [29], Conti and Salamon [12], and Fernandez, Ivanov, Muñoz and Ugarte [15], Haskins and Foscolo [17] have recently constructed two examples of non-homogeneous nearly Kähler metrics on compact manifolds, thereby answering a long-standing open question about the existence of such examples. Their two examples are cohomogeneity-one, meaning they carry a faithful action of a Lie group $G$ with orbits of codimension one, and Haskins and Foscolo conjecture that these are the only compact nearly Kähler metrics of cohomogeneity-one.

Nearly Kähler structures are analogous to nearly parallel $G_2$-structures in the sense that the cone over a nearly Kähler manifold has holonomy contained in $G_2$, while the cone over a nearly parallel $G_2$-manifold has holonomy contained in Spin(7). As a consequence, both structures induce metrics that are Einstein with positive scalar curvature. However, there are differences between the two cases. For example, the homogeneous examples have been classified for both cases—there are infinitely many diffeomorphism types of homogeneous nearly parallel $G_2$-manifolds, while there are just four types in the nearly-Kähler case. In fact, the infinite family of Aloff-Wallach spaces $N_{k,l}$ introduced in §2.1 all carry homogeneous nearly parallel $G_2$-structures.

In a future project, I propose to study nearly parallel $G_2$-structures of cohomogeneity one. Mirroring the approach of Haskins and Foscolo, there will be four main steps involved in such a project:

1. Classify the possible groups $G$ and $G$-actions on a complete nearly parallel $G_2$-manifold.
2. A hypersurface in a manifold with $G_2$-structure is equipped with a canonical SU(3)-structure. Thus, the principal orbits of the $G$-action will come equipped with homogeneous SU(3)-structures of a certain type, called nearly half-flat. Thus, for each $G$, one will need to classify the possible homogeneous nearly half-flat SU(3)-structures on the principal $G$-orbits.
3. Away from the singular orbits, a cohomogeneity-one nearly parallel $G_2$-structure may be thought of as a curve in the space of homogeneous nearly half-flat SU(3)-structures on the principal $G$-orbits.
4. Finally, the ODE system must be solved (or at least solutions shown to exist), and the issue of extension over the singular $G$-orbits needs to be understood. This is likely to prove the most difficult step.

Cleyton and Swann [11] have classified the possible principal orbits for any cohomogeneity-one $G_2$-structure, and completely worked out the cohomogeneity-one case when $G$ is simple. Thus, the remaining cases to consider all have non-simple $G$, the possibilities being

\[ SU(2)^3, SU(2)^2 \times U(1), SU(2)^2, SU(2) \times T^3, SU(3) \times U(1), T^6, \]

and preliminary calculations seem to rule out the latter two groups. The $SU(2)^2$ case stands out as particularly alluring because of the relatively large number of cohomogeneity-one actions of this group on compact 7-manifolds [21]. Here the works [28] and [16] will function as inspiration for my approach to steps (2) – (4).

If successful, this project will provide new examples of Spin(7) holonomy cones, as well as possibly new examples of Einstein 7-manifolds. Depending on the exact nature of the result, there is also the possibility to address the interesting open problem

**Question 4.1** ([19]). Do there exist compact nearly parallel $G_2$-manifolds with non-zero $b_3$?

4.2. Calibrated submanifolds in spaces with torsion (joint with Jesse Madnick). As discussed in §3.1, calibrated submanifolds play an important role in the study of metrics with special holonomy. Similarly to the situation with gauge theory described in §2.1, it is the case that for certain problems one may want to consider (pseudo-)calibrated submanifolds in the non-torsion-free case. A simple example occurs when considering calibrated cones. The construction of calibrated cones reduces to the construction of their cross-sections, and these cross-sections are given by (pseudo-)calibrated submanifolds in some space with torsion. These links are minimal in their own right, despite the defining differential form not being closed. With this in mind, a question I wish to answer, in collaboration with Jesse Madnick, is
Question 4.2. Let $H$ be a special holonomy group appearing on Berger’s list. What conditions on a $H$-structure imply that all calibrated submanifolds (of a given type) are minimal?

As well as answering this question, we intend to construct various examples of calibrated submanifolds in geometries with torsion, especially in the nearly parallel $G_2$ case. It is possible to adapt the approach of Landsberg in [23] to reduce the construction of a certain class of pseudo-associative submanifolds in the Berger space $SO(5)/SO(3)$ (endowed with its nearly parallel $G_2$-structure) to the solution of a single elliptic PDE in two variables; we intend to use this reduction to give new examples in this case. It should also be possible to use this approach to derive a Weierstrass-type formula for a certain class of associative submanifolds in $\mathbb{R}^7$ with its flat $G_2$-structure.

References

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