Mathematics 690 syllabus: Spring 2019
Topics in Algebraic number theory: Modular forms on $G_2$

1 About the instructor

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2 Basic course information

1. Class meeting: Tuesdays and Thursdays, 4:40pm-5:55pm, in Physics 227
2. My office hours: TBD, and by appointment
3. Final exam: There will not be a final exam
4. Class presentation: Each enrolled student will prepare one lecture sometime throughout the term (from a list of possibilities)
5. Homework: There will be occasional short homework assignments.

3 Course Synopsis

The study of modular forms on $G_2$ was initiated by Gan, Gross, and Savin, following work of Gross and Wallach and Wallach on quaternionic discrete series. This course will be an introduction to this circle of ideas. In particular, the course will cover: the octonions and $G_2$; the arithmetic invariant theory of cubic rings; the explicit Fourier expansion of modular forms on $G_2$; theta functions and Eisenstein series; and the Dirichlet series for the standard $L$-function of modular forms. We will begin by discussing analogous facts on $GSp_4$. For the student only familiar with classical modular forms, the course may serve as an introduction to automorphic forms on larger groups.

4 Material covered

This course is aimed at graduate students in number theory/automorphic forms. Another title for this course could have been “Modular forms: Examples, Fourier expansions, and $L$-functions”. On the one hand, the goal from this course is to discuss examples, Fourier expansions, and $L$-functions for modular forms on $G_2$. However, the other main goal for this course is to introduce you to various techniques and ideas that are prevalent in modern number theory, through some interesting examples and applications. As a consequence, the course will be a bit of a hodge-podge of different things, although hopefully many of these things will be useful for your research or understanding of modern number theory.
A rough plan for the course\footnote{This is subject to change} is as follows:

1. Introduction

2. Review of modular forms on $\text{GL}_2$:
   
   (a) $L$-functions on $\text{GL}_2$: Rankin-Selberg and Hecke
   (b) Eisenstein series; Casselman-Shalika

3. Siegel modular forms:
   
   (a) Definition, examples, basic properties
   (b) Eisenstein series: Siegel, Klingen, constant term; Hecke summation.
   (c) Theta functions and Poisson summation
   (d) Hecke operators; Satake transform; definition of $L$-functions
   (e) The Spin $L$-function on $\text{GSp}_4$
   (f) The standard $L$-function on $\text{GSp}_4$

4. Quaternions, octonions, and $G_2$

   (a) Review of quaternions
   (b) Octonions; definition of $G_2$; special subgroups of $G_2$
   (c) The Lie algebra of $G_2$
   (d) Parabolic subgroups of $G_2$; flag varieties;
   (e) Arithmetic invariant theory of binary cubics

5. Modular forms on $G_2$

   (a) Differential equations for modular forms on $\text{GL}_2$, $\text{GSp}_4$
   (b) Fourier expansion of modular forms on $G_2$
   (c) Klingen Eisenstein series

6. The standard $L$-function on $G_2$

   (a) $\text{Spin}_8$; the Rankin integral
   (b) Satake on $G_2$; local calculations

7. Time permitting: Bigger exceptional groups

   (a) $F_4$, $E_6$, $E_7$, $E_8$
   (b) The minimal representation on $E_8$
   (c) The dual pair $G_2 \times F_4 \subseteq E_8$