Review Handout: Math 212 (Pollack) Exam 2

The purpose of this handout is to help you review for the midterm. Inclusion of topics on this handout does not guarantee or imply that they will be on the midterm, nor does exclusion of topics imply that they will not be on the midterm. You should know how to solve all the homework problems—and know all the ideas that went in to solving them—and review your notes from class.

The exam will cover material from the following sections:

- 13.5: Optimization
- 13.6: Increments and linear approximation
- 13.7: The chain rule
- 13.8: Directional derivatives and the gradient vector
- 14.1, 14.2, 14.3: Double integrals
- 10.2, 14.4: Polar coordinates, and double integrals in polar coordinates
- 14.5: Applications of double integrals
- 14.6: Triple integrals
- 14.9: Change of variables

Problem 1. Suppose \((u, u)\) and \((v, e^v)\) are the points on the curves \(y = x\) and \(y = e^x\) that are closest to one another. Prove that

\[
2(v + e^v) = (1 + e^v)(v + e^v).
\]

Problem 2. Suppose that \(T = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2\) is a smooth function, and \(T(3, 5) = (f(3, 5), g(3, 5)) = (1, 2)\). Assume that \(\frac{\partial f}{\partial x}(3, 5) = -2, \frac{\partial f}{\partial y}(3, 5) = 1, \frac{\partial g}{\partial x}(3, 5) = 3\), and \(\frac{\partial g}{\partial y}(3, 5) = 0\). Let \(h(x, y) = x + 3y^2\).

1. What is \(\frac{\partial (h \circ T)}{\partial x}(3, 5)\) and \(\frac{\partial (h \circ T)}{\partial y}(3, 5)\)?
2. Use linear approximation to estimate \(h(T(3.1, 4.8))\).

Problem 3. Denote by \(S_1\) the surface in \(\mathbb{R}^3\) with equation \(x^2 + 2y^2 + 3z^2 = 6\) and by \(S_2\) the surface in \(\mathbb{R}^3\) with equation \(y = 2x^2 - z\). Let \(C\) be the curve that is the intersection of these two surfaces. What is a nonzero tangent vector to \(C\) at the point \((1, 1, 1)\)?

Problem 4. You are walking along the graph of the surface \(z = e^x - e^{2y} + x^2 + 5y\). Assume you are at the point \(P = (2, 1, 9)\). In what direction \(\langle u, v \rangle\) should you walk to decrease your height most rapidly?

Problem 5. Denote by \(R\) the region in the \(xy\)-plane that is bounded by the curves \(x + y = 6, y = 2x - 3, x + y = 9\) and \(2x - y = e^{x+y}\). Compute the area of \(R\).

Problem 6. Denote by \(T\) the region in \(\mathbb{R}^3\) that is bounded by the surfaces \(z = 2\sqrt{x^2 + y^2}\) and \(z = 3 - x^2 - y^2\). What is the volume of \(T\)?

Problem 7. Sketch the curves in the \(xy\)-plane whose equations are given in polar coordinates by \(r = 1 + \sin(\theta)\) and \(r = 2\cos(\theta)\). The two curves intersect at a point \((x, y)\) with \(x > 0\) and \(y > 0\). What is this point of intersection?

Problem 8. Consider the following integral evaluations:

\[
\int_0^1 \int_0^1 \int_0^{1-x-y} dz \, dy \, dx = \frac{1}{6}
\]
\[ \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \, dz \, dy \, dx = \frac{1}{24} \]
\[ \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y \, dz \, dy \, dx = \frac{1}{24} \]
\[ \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z \, dz \, dy \, dx = \frac{1}{24} \]

Denote by \( T \) the domain of integration in the above integrals. Sketch the region \( T \). What do the facts above say about \( T \)?

**Problem 9.** Order the following quantities from least to greatest. Be sure to justify your answer.

1. \( \int_0^\infty \int_0^\infty \sqrt{1 + y^2} e^{-(x^2+y^2)} \, dx \, dy \)
2. \( \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy \)
3. \( \int_0^\infty \int_0^\infty \sin^2(x) e^{-(x^2+y^2)} \, dx \, dy \)
4. \( \int_0^\infty \int_0^\infty \sqrt{1 + x^2} e^{-(x^2+y^2)} \, dx \, dy \)

**Problem 10.** Suppose \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is the linear transformation \( T(u,v) = (au+bv, cu+dv) \), and \( \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0 \). Let \( S \) be a region in \( \mathbb{R}^2 \) and set \( R = T(S) \). Assume that both \( R \) and \( S \) have mass density 1. Use the definition of centroid and the change of variables formula to determine how the centroids of \( R \) and \( S \) are related.

**Problem 11.** Go to section 14.4 in your textbook. Do a random problem that you have not already done before. Do the same with sections 14.5, 14.6, and 14.9. REPEAT.