1. Classical modular forms

The purpose of this course is to (eventually) define and discuss modular forms on the exceptional group \( G_2 \). I will not assume that you know what \( G_2 \) is, and certainly I will not assume that you know what a modular form is on \( G_2 \). Instead, the prerequisite for this course is basic graduate representation theory, complex analysis, algebra etc, and a familiarity with classical modular forms on \( \text{GL}_2 \) or the upper half plane. Today I’ll give an overview of what we will discuss over the course of the term, and the key results we will prove, time-permitting.

Let me recall the definition of classical modular forms: \( \text{SL}_2(\mathbb{R}) \) acts on the upper half-plane

\[
\mathfrak{h} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}
\]

by fractional linear transformations \( \gamma z = (az + b)(cz + d)^{-1} \) for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{R}) \). Suppose \( n \geq 1 \), and \( f : \mathfrak{h} \rightarrow \mathbb{C} \) is a holomorphic function. One says that \( f \) is a modular form of weight \( n \) if

- \( f(\gamma z) = (cz + d)^n f(z) \) for all \( \gamma \) in a congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \)
- \( f(x + iy) \) is \( O(y^N) \) for some \( N \) as \( y \rightarrow \infty \)

Recall that such an \( f \) has a Fourier expansion: if \( f \) is level 1 then \( f(z) = f(z + n) \) for all \( n \in \mathbb{Z} \) and this plus growth condition plus holomorphy implies \( f(z) = \sum_{n \geq 0} a_f(n)q^n \) for some complex numbers \( a_f(n) \). Here \( q = \exp(2\pi i) \), as usual.

**Examples**

1. Ramanujan’s function: \( \Delta(z) = q \prod_{n \geq 1} (1 - q^n)^24 = q - 24q^2 + 252q^3 - 1472q^4 + \cdots \)
2. Eisenstein series: \( E_{2k}(z) = \sum_{(m,n) \neq 0} \frac{1}{(mz + n)^{2k}} \), \( k \geq 2 \).

Modular forms as above are associated to \( \text{SL}_2 \) (actually, better to use \( \text{GL}_2 \), but for simplicity we will use \( \text{SL}_2 \) for a few lectures) as follows:

- \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathfrak{h} \) transitively
- The stabilizer of \( i \) is \( \text{SO}(2) \Rightarrow \text{SL}_2(\mathbb{R})/\text{SO}(2) \cong \mathfrak{h} \)
- Key point: \( \text{SO}(2) \subseteq \text{SL}_2(\mathbb{R}) \) is a compact subgroup, and is maximal among compact subgroups. (General theory: all maximal compact subgroups are conjugate)
- Thus: modular forms give \( f : \text{SL}_2(\mathbb{R})/\text{SO}(2) \rightarrow \mathbb{C} \) satisfying \( f(\gamma g) \approx f(g) \) for all \( \gamma \in \Gamma \), some congruence subgroup.

**Better** Given \( g \in \text{SL}_2(\mathbb{R}) \), \( f \) a modular form on of weight \( n \), define \( \text{varphi}_f : \text{SL}_2(\mathbb{R}) \rightarrow \mathbb{C} \)

\[
\varphi_f(g) = j(g, i)^{-n} f(g \cdot i)
\]

where \( j(g, z) = cz + d, g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) is the usual factor of automorphy. **Recall:** \( j(g_1g_2, z) = j(g_1, g_2z)j(g_2, z) \).

The function \( \phi_f \) has nice properties, just like \( f \) did:

**Lemma 1.** Let the notation be as above. Suppose \( \gamma \in \Gamma \), and \( k_\theta = \left( \begin{array}{cc} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{array} \right) \in \text{SO}(2) \). Then

1. \( \phi_f(\gamma g) = \phi_f(g) \). So, \( \phi_f \) is actually left-invariant under \( \Gamma \).
\(\phi_f(gk) = e^{in\theta} \phi_f(g)\). So, \(\phi_f\) now translates on the right by \(SO(2)\) under this nice 1-dimensional representation.

**Proof.** This are immediate checks, from the definition:

\[
\phi_f(\gamma g) = j(\gamma g, i)^{-n} f(\gamma g \cdot i) \\
= j(\gamma g, i)^{-n} j(g, i)^{-n} j(\gamma, gi)^n f(g) \\
= \phi_f(g).
\]

Similarly:

\[
\phi_f(gk) = j(gk, i)^{-n} f(gk \cdot i) \\
= j(gk, i)^{-n} j(k, i)^{-n} f(g \cdot i) \\
= (\cos(\theta) - i \sin(\theta))^{-n} \phi_f(g) \\
= e^{in\theta} \phi_f(g).
\]

Finally, the fact that \(f\) was holomorphic means that there is a linear differential operator (the Cauchy-Riemann equations) that kills \(f\). This translates into the fact that \(D_n \phi_f \equiv 0\), for some specific linear differential operator \(D_n\).

**2. Automorphic functions**

Via the above lemma, it is clear that we can replace \(SL_2\) with other semisimple Lie groups \(G\), and again ask for functions that are left-invariant under a discrete subgroup \(\Gamma\) of \(G\). For example, we can take

1. \(G = SL_n(\mathbb{R})\), and we can consider functions \(f : SL_n(\mathbb{Z}) \backslash SL_n(\mathbb{R}) \to \mathbb{C}\)
2. \(G = SO(p, q) = \{g \in SL_{p+1} : g \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} g^t = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \}\)
3. \(G = Sp_{2n} = \{g \in GL_{2n} : g \begin{pmatrix} 1 \end{pmatrix} g^t = \begin{pmatrix} 1 \end{pmatrix} \}\)

All three of the above \(G\) can be considered generalizations of \(G = SL_2\):

1. \(SL_n\) obviously generalizes \(SL_2\)
2. \(PGL_2 = SO(2, 1)\), so this generalizes \(GL_2\) or \(SL_2\)
3. \(Sp_2 = SL_2\), so this is a generalization as well

One nice thing about \(SL_2\) was that \(SL_2(\mathbb{R})/K\) was a complex manifold, where \(K\) was a maximal compact subgroup of \(SL_2(\mathbb{R})\). Which of the above groups \(G\) have the property that \(G/K\) has \((G\text{-equivariant})\) complex structure?

1. \(K\) for \(SL_n(\mathbb{R})\) is \(SO(n)\). \(SL_n(\mathbb{R})/SO(n)\) is complex manifold only for \(n = 2\)
2. \(K\) for \(SO(p, q)\) is (essentially) \(SO(p) \times SO(q)\). \(G/K\) only complex manifold only when \(p = 2\) or \(q = 2\)
3. For \(G = Sp_{2n}(\mathbb{R})\), \(K = U(n)\). Here \(G/K\) is a complex manifold for all \(n\).

**3. Siegel modular forms**

When we replace \(SL_2\) by \(Sp_{2n}\), the classical holomorphic modular forms get replaced by what are called Siegel modular forms. We now define these.

Set

\[h_n = \{Z \in M_n(\mathbb{C}) : Z^t = Z, Im(Z) > 0\}\]
the so-called Siegel upper half-space of degree \( n \). Thus \( Z \in \mathfrak{h}_n \) means \( Z = X + iY \) with \( X, Y \) real symmetric matrices, and \( Y \) is positive definite. The group \( \text{Sp}_{2n}(\mathbb{R}) \) acts on \( \mathfrak{h}_n \) as
\[
g = (\begin{pmatrix} a & b \\ c & d \end{pmatrix}) \text{ in } n \times n \text{ block form}, \ Z \in \mathfrak{h}_n, \ g \cdot Z = (aZ + b)(cZ + d)^{-1}.
\]
More carefully, if \( g \in \text{Sp}_{2n}(\mathbb{R}) \) and \( Z \in \mathfrak{h}_n \), then it is a fact (we will discuss later)
\begin{itemize}
  \item \( cZ + d \) is invertible, as an \( n \times n \) complex matrix
  \item \( (aZ + b)(cZ + d)^{-1} \) is symmetric with positive-definite imaginary part
  \item \( \text{Sp}_{2n}(\mathbb{R}) \times \mathfrak{h}_n \to \mathfrak{h}_n \) as \( (g, Z) \mapsto g \cdot Z \) defines an action
\end{itemize}
Thus, when \( n = 1 \), this reduces to the usual action of \( \text{SL}_2(\mathbb{R}) \) on the upper half plane.

Suppose \( k \geq 1 \) is an integer. A **Siegel modular form** of weight \( k \) is a holomorphic function
\[
f : \mathfrak{h}_n \to \mathbb{C}
\]
satisfying
\begin{enumerate}
  \item \( f(\gamma Z) = \det(cZ + d)^kf(Z) \) for all \( \gamma \in \Gamma \subseteq \text{Sp}_{2n}(\mathbb{Z}) \) some congruence subgroup
  \item when \( n = 1 \), some slow growth condition (which turns out to be automatic for \( n \geq 1 \)).
\end{enumerate}

Siegel modular forms have a nice Fourier expansion: One has \( \left( \begin{array}{cc} 1 & Y \\ 0 & 1 \end{array} \right) \in \text{Sp}_{2n} \) if and only if the \( n \times n \) matrix \( V \) is symmetric. Assume (for simplicity) that \( \Gamma = \text{Sp}_{2n}(\mathbb{Z}) \), so that we are discussing Siegel modular forms of level 1. Then
\[
f(Z) = f(Z + V) \quad \text{for all } V \in M_n(\mathbb{Z}), V^t = V
\]
which implies the Fourier expansion
\[
f(Z) = \sum_{T \in S_n(\mathbb{Z})^\vee} a_f(T) e^{2\pi i \text{tr}(TZ)}
\]
for complex numbers (the Fourier coefficients) \( a_f(T) \). Here \( S_n(\mathbb{Z}) = \{ V \in M_n(\mathbb{Z}), V^t = V \} \) and \( S_n(\mathbb{Z})^\vee \) is the dual lattice in \( S_n(\mathbb{Q}) \) (\( n \times n \) symmetric matrices with rational coefficients) for the trace pairing \( (X, V) = \text{tr}(XV) \). In other words, the \( T \) appearing in the Fourier expansion are the half-integral symmetric matrices.

**Fact:** \( a_f(T) \neq 0 \) implies \( T \geq 0 \), i.e., \( T \) is positive semi-definite. This is the analogue to Siegel modular forms of the Fourier expansion for regular modular forms only having non-negative terms.

**Summary:** Suppose \( G \) a semisimple group over \( \mathbb{Q} \). Can consider
\begin{enumerate}
  \item The automorphic forms, which are functions \( \varphi : \Gamma \backslash G(\mathbb{R}) \to \mathbb{C} \) satisfying some conditions having to do with slow growth and being annihilated by some differential operator.
  \item When \( G(\mathbb{R})/K \) has the structure of a complex manifold, there are **special** automorphic forms:
    \begin{enumerate}
      \item Namely, one can consider those \( \varphi \) that are related to holomorphic functions on \( G(\mathbb{R})/K \)
      \item (like we did for holomorphic modular forms \( \phi_f \leftrightarrow f \) on \( \text{SL}_2 \))
    \end{enumerate}
\end{enumerate}
In part 1 of course: we will take \( G = \text{SL}_2 \) and \( G = \text{Sp}_4 \), and consider the holomorphic modular forms, respectively the Siegel modular forms. In particular, we will discuss
\begin{enumerate}
  \item Eisenstein series and \( L \)-functions for \( \text{GL}_2 \)
  \item Basic properties and examples of Siegel modular forms on \( \text{Sp}_4 \), such as Eisenstein series and \( \Theta \) functions
  \item \( L \)-functions for Siegel modular forms on \( \text{Sp}_4 \) (technically, \( \text{GSp}_4 \) or \( \text{PGSp}_4 \))
\end{enumerate}

4. **MODULAR FORMS ON** \( G_2 \)

In the second half of this course, we will consider \( G = G_2 \). In particular
\begin{itemize}
  \item There is an algebraic group \( G_2 \) over \( \mathbb{Q} \); we will define this group.
  \item \( G_2(\mathbb{R}) \) is a 14-dimensional non-compact Lie group
• \( K \subseteq G_2(\mathbb{R}) \) a maximal compact subgroup is \((\text{SU}(2) \times \text{SU}(2))/\mu_2 \); so, we will discuss how this \( K \) sits inside \( G_2(\mathbb{R}) \).
• There is a lattice \( G_2(\mathbb{Z}) \subseteq G_2(\mathbb{R}) \) and \( G_2(\mathbb{Z}) \backslash G_2(\mathbb{R}) \) is non-compact, just like we saw with \( \text{SL}_2 \).
• But now, \( G_2(\mathbb{R})/K \) has no invariant \( \mathbb{C} \)-structure.
• This implies that there is no \textit{a priori} or obvious notion of modular forms or Fourier expansion.

However, it turns out that there is a notion of modular forms on \( G_2 \). Their study was initiated by Gan, Gross, Savin, and Wallach.

Let us now define modular forms on \( G_2 \), at least in the way I like to define them. We have \( \text{SU}(2) \) acts on \( \mathbb{C}^2 \), and then the symmetric powers \( \text{Sym}^\ell(\mathbb{C}^2) \).

**Definition 2.** Let \( n \geq 1 \) be an integer. A \textbf{modular form} on \( G_2 \) of weight \( n \) (and level 1) is a function

- \( \varphi : G_2(\mathbb{R}) \to \text{Sym}^{2n}(\mathbb{C}^2) \) satisfying
- \( \varphi(\gamma g) = \varphi(g) \) for all \( \gamma \in G_2(\mathbb{Z}) \) and \( g \in G_2(\mathbb{R}) \)
- \( \varphi(gk) = k^{-1} \cdot \varphi(g) \) for all \( k \in K, \ g \in G_2(\mathbb{R}) \). Here the first \( \text{SU}(2) \)-factor\(^1\) is acting by the \( n^{th} \) symmetric power representation, and the second \( \text{SU}(2) \)-factor is acting by the trivial representation
- \( D_n \varphi \equiv 0 \) for a certain linear first-order differential operator \( D_n \). This is the analogue of the fact that the holomorphic modular forms satisfied the Cauchy-Riemann equations.

We will explain how the differential operator \( D_n \) comes from the representation theory of \( G_2(\mathbb{R}) \).

After defining modular forms on \( G_2 \), our next task will be to understand their Fourier expansion. Indeed, the name “modular forms” for \( G_2 \) is justified by the fact that we will prove that the above-defined objects have very nice Fourier expansions.

Recall that the Fourier coefficients for Siegel modular forms on \( \text{Sp}_4 \) were parametrized by \( 2 \times 2 \), half-integral symmetric matrices, i.e. matrices of the form \( T = \left( \begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right) \) with \( a, b, c \in \mathbb{Z} \). In other words, the Fourier coefficients of a Siegel modular form on \( \text{Sp}_4 \) are parametrized by binary quadratic forms \( au^2 + buv + cv^2 \). Moreover, in order for the Fourier coefficient \( a_f(T) \) of a Siegel modular form on \( \text{Sp}_4 \) to be nonzero, the binary quadratic \( au^2 + buv + cv^2 \) must be positive semi-definite.

We can write the Fourier expansion of Siegel modular forms on \( \text{Sp}_4 \) this way: For a positive semidefinite binary quadratic form \( p(u,v) \), define the function \( W_p(Z) = \exp(2\pi i(T_p, Z)) \) where \( T_p = \left( \begin{array}{cc} a & b/2 \\ b/2 & c \end{array} \right) \) if \( p(u,v) = au^2 + buv + cv^2 \). Then if if \( f \) is a Siegel modular form, one has

\[
f(Z) = \sum_{p \geq 0, \text{integral}} a_f(p) W_p(Z)
\]

for some complex numbers \( a_f(p) \).

We will see that the Fourier expansion for modular forms on \( G_2 \) is similar, but one replaces binary quadratic forms with binary cubic forms. The first main theorem for modular forms on \( G_2 \) that we will prove is the following one. Say that a binary cubic \( p \geq 0 \) if \( p(u,v) = au^3 + bu^2v + uv^2 + dv^3 \) with \( a, b, c, d \in \mathbb{R} \) factors into linear factors over \( \mathbb{R} \).\(^2\)

**Theorem 3.** (Rough statement) \([\text{Pol18a}], \text{or } [\text{Pol18b}]\) Fix \( n \geq 1 \), and fix a basis \( x, y \) of \( \mathbb{C}^2 \), so that

\[
\{ x^{2n}, x^{2n-1}y, \cdots, xy^{2n-1}, y^{2n} \}
\]

\(^1\)For the more advanced reader: I am taking the long root \( \text{SU}(2) \) to be the first factor, and the short root \( \text{SU}(2) \) to be the second factor.

\(^2\)This is the analogue of the positive-definiteness condition.
is a basis of $\text{Sym}^{2n}(C^2)$. Suppose $p \geq 0$ is a binary cubic form. Then, there are explicit functions $W_p : G_2(R) \to \text{Sym}^{2n}(C^2)$ such that if $\varphi$ is a modular form on $G_2$ of weight $n$, then the Fourier expansion of $\varphi$ is

$$\varphi^\omega = fx^{2n} + \beta x^ny^n + \overline{fy}^{2n} + \sum_{p \geq 0, \text{integral}} a_\varphi(p)W_p(g)$$

for $a_\varphi(p), \beta \in C$ and $f$ a holomorphic modular form on $\text{GL}_2$ of weight $3n$.

There are (of course) various things we will have things we will have to discuss/answer about this theorem, such as

- What does the " = " mean in the theorem, more precisely?
- What are the explicit functions $W_p(g)$?
- Why do only modular forms on $\text{GL}_2$ of weight $3n$ show up? So, $\Delta$ and $E_{12}$ can show up, but not, say, $E_{14}$ or the level one cusp form of weight $16$
- Where does the condition $p \geq 0$ come from?

We will also want to give examples of modular forms on $G_2$, and their Fourier expansion. So, we will prove the following theorem:

**Theorem 4.** (Rough statement)[Pol18c] Up to nonzero rational numbers (because I can’t remember the exact rational constants off the top of my head), one has the following two facts:

1. There is a modular form $\varphi_\Delta$ on $G_2$ of weight $4$ with (in the notation of Theorem 3) $f = \Delta$, $\beta = 0$, and Fourier coefficients $a_{\varphi_\Delta}(p) \in \mathbb{Z}$ for all integral binary cubic $p \geq 0$.
2. There is a modular form $\varphi_{E_{12}}$ on $G_2$ of weight $4$ with (in the notation of Theorem 3) $f = E_{12}$, $\beta = \frac{\zeta(5)}{(2\pi)^7}$, and Fourier coefficients $a_{\varphi_{E_{12}}}(p) \in \mathbb{Z}$ for all integral binary cubics $p \geq 0$.

This theorem makes crucial use of [Gan00].

Note that the fact that so many of the Fourier coefficients are integers means that there is some arithmetic theory lurking behind modular forms on $G_2$. However, the presence of the (presumably transcendental) constant $\frac{\zeta(5)}{(2\pi)^7}$ hints that this arithmeticity is just slightly less good than what happens for classical holomorphic and Siegel modular forms.

We will also discuss $L$-functions and Dirichlet series on $G_2$, following [GS15], [Seg17], as in [Pol18a].

**References**


[Pol18b] ________, *The Fourier expansion of modular forms on quaternionic exceptional groups*.

[Pol18c] ________, *The minimal modular form on quaternionic $E_8$*.


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