PART 3: MODULAR FORMS ON $G_2$

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We will now move on to discussing modular forms on the exceptional group $G_2$. Recall from the beginning of the semester modular forms on $G_2$ of weight $n$ are very special automorphic functions $F : G_2(\mathbb{Q}) \backslash G_2(\mathbb{A}) \rightarrow \text{Sym}^{2n}(\mathbb{C})$ that satisfy

1. a $K_{\infty} = (\text{SU}(2) \times \text{SU}(2))/\mu_2$-equivariance condition $F(gk) = k^{-1} \cdot F(g)$ for all $g \in G_2(\mathbb{A})$ and $k \in K_{\infty}$ and
2. are annihilated by a special first order linear differential operator $D_n$.

For the expert, for $n \gg 0$ the right regular action of $G_2(\mathbb{R})$ on such an $F$ generates a quaternionic discrete series representation $\pi_n$, and $F$ corresponds to the minimal $K_{\infty}$-type in this representation. These particular discrete series representations were singled out and studied by Benedict Gross and Nolan Wallach. However, it is not simply that $\pi_n$ is a discrete series representation (and thus cohomological) that makes these $F$ very special. Rather, it turns out that the $\pi_n$ behave very similarly to the holomorphic discrete series representations on Hermitian symmetric spaces. And consequently, the associated automorphic functions $F$ seem to behave like holomorphic modular forms on Hermitian symmetric spaces. In particular, these $F$ have a robust Fourier expansion (which we will prove) and appear to be closely connected to arithmetic (we will give evidence for this). Just how arithmetic these modular forms are remains to be seen, although I am optimistic.

1. THE ALGEBRAIC GROUP $G_2$

Let us now begin describing the algebraic group $G_2$. To do this, we will first construct the group $\text{PGL}_2$ in a slightly unusual fashion, and then mimic this construction to obtain $G_2$. Throughout, $k$ denotes an arbitrary field of characteristic 0.

We begin with some notation. For $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k)$ a two-by-two matrix, set $m' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ the cofactor matrix of $m$. Thus $mm' = \det(m)1_2$.

Lemma 1. One has

- $(mm')' = m$
- $(m_1m_2)' = m'_2m'_1$
- $m + m' = \text{tr}(m)1_2$
- $mm' = \det(m)1_2$
- If $\det(m) \neq 0$, then $m' = \det(m)m^{-1}$
- If $g \in \text{GL}_2$, then $(gmg^{-1})' = gm'g^{-1}$.

Proof. These are all immediate. \hfill \square

Now, consider the algebraic group $H_1$ of all linear automorphisms of $M_2(k)$ that preserve the identity $1_2$, multiplication, and involution $'$:

$H_1 := \{ h \in \text{GL}(M_2) : h(1_2) = 1_2, h(xy) = h(x)h(y), h(x)' = h(x') \text{ for all } x, y \in M_2 \}$.

Denote by $H$ the connected component of the identity of $H_1$.

Lemma 2. The group $H$ defined above is $\text{PGL}_2$. 

Proof. If $g \in \text{GL}_2$, then the linear automorphism $m \mapsto gmg^{-1}$ on $M_2(k)$ is clearly in $H$. As the center of $\text{GL}_2$ acts trivially via this action, one obtains a map $\text{PGL}_2 \rightarrow H$.

For the converse, denote by $M^{tr=0}$ the trace 0 elements of $M_2$. Note that elements of $H$ preserve the quadratic form $\det(\cdot)$ on $M_2$, and because they preserve 1, elements of $H$ preserve the orthogonal complement $(k1_2)^\perp$ of 1 for this quadratic form. This orthogonal complement is exactly the three-dimensional space $M_2^{tr=0}$. Thus we obtain an inclusion $H \rightarrow O(M_2^{tr=0}, \det)$ of $H$ into this three-dimensional orthogonal space. As is well-known, $\text{PGL}_2 = \text{SO}(M_2^{tr=0}, \det)$. This gives the lemma.

To construct the group $G_2$, we proceed as in our above construction of $\text{PGL}_2$, except with $M_2$ replaced by what are called the octonions. The octonions are an 8-dimensional vector space, that comes equipped with certain extra algebraic structures, that we now describe.

We denote by $\Theta$ the octonions. As mentioned, this is an 8-dimensional $k$ vector space that has the following extra structures:

1. An element $1 \in \Theta$
2. A multiplication $\Theta \otimes \Theta \to \Theta$ that is neither commutative nor even associative. One has $1 \cdot x = x \cdot 1 = x$ for all $x \in \Theta$
3. A non-degenerate quadratic form $n_\Theta : \Theta \to k$ that satisfies the important property $n_\Theta(xy) = n_\Theta(x)n_\Theta(y)$ for all $x, y \in \Theta$.
4. An involution $\ast$ that satisfies $(xy)^\ast = y^\ast x^\ast$, $xx^\ast = n_\Theta(x)1 = x^x x$, and $x + x^\ast \in k1$ for $x, y \in \Theta$.

We will shortly construct the octonions $\Theta$. However, with the above definition, one defines

$$G_2 := \{ g \in \text{GL}(\Theta) : g1 = g, (gx)^\ast = g(x^\ast), g(xy) = g(x)g(y) \text{ for all } x, y \in \Theta \}$$

the automorphisms of $\Theta$ that preserve the various structures. Note that because $xx^\ast = n(x)1$, if follows from the definition of $G_2$ that $n_\Theta(gx) = n_\Theta(x)$ for all $x \in \Theta$.

We will give two constructions of $\Theta$. The first one is called the Cayley-Dickson construction. Fix $\gamma \in k^\times$. The construction proceeds as follows. One sets $\Theta = M_2 \oplus M_2$; we write elements of $\Theta$ as pairs $(x, y)$ with $x, y \in M_2$. One sets $1 = (1_2, 0)$, $n_\Theta((x, y)) = \det(x) - \gamma \det(y)$, and $(x, y)^\ast = (x^\ast, -y)$. Note that $(x, y) + (x, y)^\ast = \text{tr}(x)1$. Finally, the multiplication is given by

$$(x, y_1)(x, y_2) = (x_1x_2 + \gamma y_2y_1, y_2x_1 + y_1x_2^\gamma).$$

Observe that

$$(x, y)(x, y)^\ast = (x, y)(x^\ast, -y) = (xx^\ast - \gamma y^\ast y, -yx + yx) = (\det(x) - \gamma \det(y))(1, 0) = n_\Theta((x, y))1$$

as desired. Moreover, it is straightforward to check that $(z_1z_2)^\ast = z_2^\ast z_1^\ast$ for all $z_1, z_2 \in \Theta$. We must check the important property of the multiplicativity of the norm, $n_\Theta(z_1z_2) = n_\Theta(z_1)n_\Theta(z_2)$ for all $z_1, z_2 \in \Theta$.

Lemma 3. One has $n_\Theta((x_1, y_1)(x_2, y_2)) = n_\Theta((x_1, y_1))n_\Theta((x_2, y_2))$.

Proof. We checked this in class, but it would be a good exercise to do it yourself. To do it, use that fact that for all $x, y \in M_2$, $\det(x + y) = \det(x) + \text{tr}(xy^\ast) + \det(y)$.

We have thus constructed the octonions. What makes them interesting, however, is that these algebraic structures possess automorphisms, i.e., that $G_2$ is not the trivial group. In fact, we can use the Cayley-Dickson construction to see that easily that $\text{SL}_2 \times \text{SL}_2$ maps to $G_2$, as stated precisely in the following proposition.

For $g, h \in \text{SL}_2$, let $(g, h)$ act on $\Theta$ as

$$(g, h) \cdot (x, y) = (gxg^{-1}, hyg^{-1}).$$
Proposition 4. The above action defines an map $\text{SL}_2 \times \text{SL}_2 \to G_2$ whose kernel is a diagonally embedded $\mu_2$.

Proof. It is easy to see that the map $\text{SL}_2 \times \text{SL}_2 \to \text{GL}(\Theta)$ has kernel exactly this diagonally embedded $\mu_2$. To check that the image is in $G_2$, one must verify that this action preserves the identity $1$ (immediate), the conjugation $\ast$ (easy), and the multiplication. We wrote this out in class. Again, you should try to write it out yourself that this action preserves the multiplication on $\Theta$. \hfill \Box