Homework 7

Remember that it is OK to work with your peers on the homework problems, but you should write up the solutions yourself. Also, if you do work with someone else on the homework, you should acknowledge that you worked with them on your write-up.

Problem numbers below refer to the course textbook, “An introduction to the theory of numbers”, by Niven, Zuckerman, and Montgomery.

1. In class we discussed that one key piece of the proof that every quadratic irrational number has a periodic continued fraction was the fact that the denominators \(q_n\) of the quadratic irrationals \(\xi_n\) (in the notation of our proof) are eventually positive. The point of this homework problem is to prove that fact. Let us recall the setup: We have \(\xi_0 = \xi = \frac{m_0 + \sqrt{d}}{q_0}\), with \(q_0 | d - m_0^2\).

One defines the \(\xi_n\) recursively as \(a_n = \lfloor \xi_n \rfloor\) and \(\xi_{n+1} = \frac{1}{\xi_n - a_n}\). We proved in class that \(\xi_n = m_n + \sqrt{d}q_n\) for integers \(m_n, q_n\) with \(q_n | d - m_n^2\). Prove that the \(q_n\)'s are eventually positive, as follows.

(a) Let \(\xi'_n = \frac{m_n - \sqrt{d}}{q_n}\). To prove that the \(q_n\)'s are eventually positive, it suffices to prove that \(\xi_n - \xi'_n\) is eventually a positive real number.

(b) Suppose the continued fraction expansion of \(\xi = \xi_0 = \langle c_0, c_1, \ldots, c_{n-1}, \ldots \rangle\). Define the \(2 \times 2\)-matrix

\[
m_n = \begin{pmatrix} P_n & Q_n \\ R_n & S_n \end{pmatrix} = \begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_{n-1} & 1 \\ 1 & 0 \end{pmatrix}.
\]

Express \(\xi_0\) in terms of \(P_n, Q_n, R_n, S_n\) and \(\xi_n\).

(c) Use your expression from the previous part to express \(\xi'_0\) in terms of \(P_n, Q_n, R_n, S_n\) and \(\xi'_n\), and then solve for \(\xi'_n\) in terms of \(\xi'_0, P_n, Q_n, R_n, S_n\). You should get

\[
\xi'_n = -\frac{S_n \left( \xi'_0 - \frac{Q_n}{S_n} \right)}{R_n \left( \xi'_0 - \frac{P_n}{R_n} \right)}.
\]

(d) Explain that \(S_n, R_n\) are positive for all \(n \geq 1\). Conclude using Lemma 5 from the “Continued fractions II” lecture notes (that we proved in class) that \(\xi'_n\) is negative for \(n\) sufficiently large, because

\[
\lim_{n \to \infty} \frac{\xi'_0 - \frac{Q_n}{S_n}}{\xi'_0 - \frac{P_n}{R_n}} = 1.
\]

(Prove this using that lemma.)

(e) Explain that \(\xi_n > 0\) for \(n \geq 1\). Conclude that \(\xi'_n\) is negative for all \(n\) sufficiently large, and thus \(\xi_n - \xi'_n\) is positive for all \(n\) sufficiently large.