MATH 411, HW 7 SOLUTIONS

3.24.3. Let \( f : X \to X \) be continuous. Show that if \( X = [0,1] \), there is a point with \( f(x) = x \). What happens if \( X = [0,1] \) or \((0,1)\)?

Consider the continuous function \( g : [0,1] \to [-1,1] \) defined by \( g(x) = f(x) - x \). We have \( g(0) = f(0) \ge 0 \) and \( g(1) = f(1) - 1 \le 0 \). By the intermediate value theorem, there is some \( t \in [0,1] \) such that \( g(t) = 0 \), and hence \( f(t) = t \).

On the other hand, for \( X = [0,1] \) or \((0,1)\), the function \( f(x) = \frac{x+1}{2} \) is continuous, takes values in \( X \), and has no fixed point (if \( \frac{x+1}{2} = x \), then \( x = 1 \)).

3.24.5. Consider the following sets in the dictionary order. Which are linear continua?

   (a) \( \mathbb{Z}_+ \times [0,1] \)

   We claim that this is a linear continuum. Given any \( n \times x \) and \( n' \times x' \) with \( n \times x < n' \times x' \), if \( n = n' \), then \( x < x' \), so \( n \times x < n \times \frac{x+x'}{2} < n \times x' \); and if \( n < n' \), then \( n \times x < n \times \frac{x+1}{2} < n' \times x' \). To prove the least upper bound property, suppose \( A \) is a bounded-above subset of \( \mathbb{Z}_+ \times [0,1] \). The first coordinates of all elements of \( A \) must be bounded above, so there are only finitely many different values; let \( n_0 \) be the largest value that occurs. Let \( B = \{ y \in [0,1] \mid n_0 \times y \in A \} \). Since \( B \) is bounded above, \( \sup(B) \) exists. If \( \sup(B) < 1 \), then \( n_0 \times \sup(B) \) is the least upper bound for \( A \); otherwise, \( (n_0 + 1) \times 0 \) is the least upper bound.

   (b) \([0,1] \times \mathbb{Z}_+ \)

   This is not a linear continuum: there is no element between \( x \times n \) and \( x \times n + 1 \).

   (c) \([0,1] \times [0,1] \)

   This is a linear continuum; the proof of Example 1 of §24 follows through almost verbatim. Let \( A \) be a bounded subset of \([0,1] \times [0,1] \); let \( x_0 \times y_0 \) be an upper bound. We must have \( x_0 < 1 \), so \( A \subseteq [0,x_0] \times [0,1] \). Thus, there is no problem in finding \( b = \sup(\pi_1(A)) \).

   (d) \([0,1] \times [0,1] \)

   Let \( A = \{ \frac{1}{2} \} \times [0,1] \). This set is bounded above by any \( x \times 0 \) with \( \frac{1}{2} < x \le 1 \), but not by any \( \frac{1}{2} \times y \). Therefore, there is no least upper bound. Thus, the set is not a linear continuum.

3.24.7.

   (a) Let \( X \) and \( Y \) be ordered sets in the order topology. Show that if \( f : X \to Y \) is order preserving and surjective, then \( f \) is a homeomorphism.

   Note that order-preserving means that \( x < x' \) implies \( f(x) < f(x') \).

   First, we check that \( f \) is injective: For any distinct \( x \neq x' \in X \), either \( x < x' \) (so \( f(x) < f(x') \)) or \( x' < x \), (so \( f(x') < f(x) \)). In either case, we see that \( f(x) \neq f(x') \). Thus, \( f \) has an inverse function \( f^{-1} \). Moreover, the inverse of \( f \) is also order-preserving: if \( y < y' \), we may rewrite this as \( f(f^{-1}(y)) < f(f^{-1}(y')) \), and therefore \( f^{-1}(y) < f^{-1}(y') \) since \( f \) is order preserving.

   To see that \( f \) is continuous, we just observe that \( f^{-1}((a,b)) = (f^{-1}(a), f^{-1}(b)) \), which is open. Likewise, \( f^{-1} \) is also continuous since it is order-preserving.
(b) Let \( X = Y = \mathbb{R}_+ \) (i.e. the set of nonnegative real numbers). Given a positive integer \( n \), show that the function \( f(x) = x^n \) is order preserving and surjective. Conclude that its inverse, the \( n \)th root function, is continuous.

If \( 0 \leq x < y \), then
\[
f(y) - f(x) = y^n - x^n = (y - x)(y^{n-1} + y^{n-2}x + \cdots + xy^{n-2} + x^{n-1}) > 0
\]
since each of the two factors is \( > 0 \). Thus, \( f(x) < f(y) \). Thus, \( f \) is order preserving. (You can also prove this using calculus.)

To show that \( f \) is surjective, let \( y \in \mathbb{R}_+ \). There is some positive integer \( m \) such that \( 0 \leq y \leq m^n \). By the intermediate value theorem, there exists a real number \( x \in [0, n] \) such that \( x^n = y \), as required.

By part (a), it follows that the inverse function of \( f \) is continuous.

(c) Let \( X = (-\infty, 1) \cup [0, \infty) \subset \mathbb{R} \) with the subspace topology. Show that the function \( f: X \rightarrow \mathbb{R} \) defined by
\[
f(x) = \begin{cases} 
  x + 1 & x < -1 \\
  x & x > 0 
\end{cases}
\]
is order preserving and surjective. Is \( f \) a homeomorphism?

If \( x < x' \), there are three cases: If \( x < x' < -1 \), then \( f(x) = x + 1 \) and \( f(x') = x' + 1 \). If \( 0 \leq x < x' \), then \( f(x) = x \) and \( f(x') = x' \). If \( x < -1 \) and \( 0 \leq x' \), then \( f(x) = x + 1 < 0 \) and \( f(x') = x' \geq 0 \). In all three cases, we see that \( f(x) < f(x') \). Moreover, \( f \) is surjective: if \( y \geq 0 \), then \( y = f(y) \), while if \( y < 0 \), then \( y = f(y - 1) \).

However, \( f \) is not a homeomorphism: \( X \) is not connected, while \( Y \) is connected. The point is that the subspace topology on \( X \) is different from the order topology. (See p. 90 in Munkres.) In particular, sets of the form \([0, a)\) are open in the subspace topology but not in the order topology.

3.24.8.

(a) Is a product of path-connected spaces necessarily path-connected?

Yes, assuming that we use the product topology; it’s false otherwise. If \( x = (x_\alpha) \) and \( y = (y_\alpha) \) are points of \( \prod X_\alpha \), choose paths \( f_\alpha: [0, 1] \rightarrow X_\alpha \) with \( f_\alpha(0) = x_\alpha \) and \( f_\alpha(1) = y_\alpha \) for each \( \alpha \). Then the function \( f: [0, 1] \rightarrow \prod X_\alpha \) with coordinate functions \( f_\alpha \) is continuous (since we are using the product topology), so it is a path from \( x \) to \( y \).

(b) If \( A \subset X \) and \( A \) is path-connected, is \( A \) necessarily path-connected?

No: consider the example of the topologist’s sine curve. The set \( S \) is just an embedded copy of \((0, \infty)\), which is path-connected, but \( \bar{S} = S \cup \{0\} \times [-1, 1] \) is not path-connected.

(c) If \( f: X \rightarrow Y \) is continuous and \( X \) is path-connected, is \( f(X) \) necessarily path-connected?

Yes. For any \( y_0, y_1 \in f(X) \), choose points \( x_0, x_1 \in X \) with \( f(x_0) = y_0 \) and \( f(x_1) = y_1 \). There is a path \( \gamma: [0, 1] \rightarrow X \) with \( \gamma(0) = x_0 \) and \( \gamma(1) = x_1 \). Then \( f \circ \gamma \) is a path from \( y_0 \) to \( y_1 \).

(d) If \( \{X_\alpha\} \) is collection of path-connected subspaces of \( X \), and if \( \bigcap A_\alpha \neq \emptyset \), is \( \bigcup A_\alpha \) necessarily path-connected?

Yes. Choose \( x_0 \in \bigcap A_\alpha \). For any points \( x, y \in \bigcup X_\alpha \), assume that \( x \in X_\alpha \) and \( y \in X_\beta \) for some \( \alpha, \beta \). Choose a path from \( x \) to \( x_0 \) in \( X_\alpha \) and from \( x_0 \rightarrow y \) in \( X_\beta \). The concatenation of these paths is then a path from \( x \) to \( y \) in \( \bigcup X_\alpha \).
3.25.1. What are the components and path components of \( \mathbb{R}_t \)? What are the continuous functions \( f : \mathbb{R} \rightarrow \mathbb{R}_t \)?

For any \( x \in \mathbb{R} \), the sets \( (-\infty, x) \) and \( [x, \infty) \) are both open in \( \mathbb{R}_t \), giving a separation. Thus, for any \( a < b \), we may find a separation (take \( x = \frac{a+b}{2} \)) in which \( a \) and \( b \) are on opposite sides. It follows that the only connected subspaces of \( \mathbb{R}_t \) are the one-point sets. These are therefore the components and path components.

If \( f : \mathbb{R} \rightarrow \mathbb{R}_t \) is continuous, then the image of \( f \) must be connected, hence a one-point set, so \( f \) is a constant function.

3.25.2.

(a) What are the components and path-components of \( \mathbb{R}^\omega \) (in the product topology)?

By problem 3.24.8(a), \( \mathbb{R}^\omega \) is path-connected, so it has only one component and one path-component. To be explicit, for any sequences \( x \) and \( y \), the function

\[
f(t) = (1-t)x + ty = ((1-t)x_1 + ty_1, (1-t)x_2 + ty_2, \ldots)
\]

is a path from \( x \) to \( y \).

(b) Consider \( \mathbb{R}^\omega \) in the uniform topology. Show that \( x \) and \( y \) lie in the same component of \( \mathbb{R}^\omega \) iff the sequence \( x - y = (x_1 - y_1, x_2 - y_2, \ldots) \) is bounded.

First, we prove the statement in the case where \( y = 0 \).

If \( x \) is bounded, then the function \( f(t) = tx \) is continuous, as seen in problem 3.20.4(a) in HW 5, and thus gives a path from \( 0 \) to \( x \). (The problem only gave a few examples, but I wrote up a more general proof in the solutions.) Thus, \( x \) is in the same component as \( 0 \).

Next, we show that if \( x \) is not bounded, then \( x \) lies in a different component than \( 0 \). Let \( B \) be the set of bounded sequences and \( U \) the set of unbounded sequences. We claim that \( B \) and \( U \) are both open in the uniform topology. More generally, let \( u \) and \( v \) be any sequences with \( \bar{\rho}(u, v) = r < 1 \). If one of the two sequences is bounded (say \( |u_i| \leq M \) for some \( M \)), then since \( |u_i - v_i| \leq r \) for all \( i \), we have \( |v_i| \leq |u_i| + |v_i - u_i| \leq M + r \), so the other sequence is bounded as well. Thus, the sets \( B \) and \( U \) are both open:

for any \( x \in B \), \( B(x, 1) \subset B \), and for any \( x \in U \), \( B(x, 1) \subset U \). In particular, if \( x \in U \), then \( \mathbb{R}^\omega = B \cup U \) is a separation of \( \mathbb{R}^\omega \) separating \( 0 \) from \( x \), so \( 0 \) and \( x \) cannot lie in the same component.

Now, we return to the general case. For any fixed \( y \in \mathbb{R}^\omega \), the function \( g_y : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega \) given by \( g_y(x) = x - y \) is an isometry with respect to the uniform metric, hence a homeomorphism. (It is also a homeomorphism with respect to the box and product topologies, as was established in problem 2.19.8 on HW 4.) Thus, \( x \) and \( y \) lie in the same component iff \( g_y(x) = x - y \) and \( g_y(y) = 0 \) lie in the same component. By the preceding discussion, this holds iff \( x - y \) is bounded.

(c) Give \( \mathbb{R}^\omega \) the box topology. Show that \( x \) and \( y \) lie in the same component of \( \mathbb{R}^\omega \) iff \( x - y \) is eventually zero.

Just as in the previous problem, it suffices to consider the case where \( y = 0 \), since the functions \( g_y \) are homeomorphisms with respect to the box topology.

If \( x \) is eventually 0, then the function \( f(x) = tx \) is continuous, as seen in my writeup of problem 2.20.4(a), and therefore \( x \) lies in the same component as \( y \).

Suppose \( x \) is not eventually 0. Following the hint, let \( h : \mathbb{R}^\omega \rightarrow \mathbb{R}^\omega \) be the function defined in coordinates by

\[
h_i(u) = \begin{cases} u_i & \text{if } x_i = 0 \\ i(1 - \frac{u_i}{x_i}) & \text{if } x_i \neq 0. \end{cases}
\]
That is, the $i$\textsuperscript{th} coordinate of $h(\mathbf{u})$ only depends on $u_i$, as a linear function with nonzero slope. By problem 2.19.8, $h$ is a homeomorphism. Observe that $h(x) = 0$, while $h(0)$ has $i$\textsuperscript{th} coordinate equal to 0 if $x_i = 0$ and equal to $i$ if $x_i \neq 0$. Thus, if $x$ is not eventually 0, then $h(0)$ is not bounded. The sets $B$ and $U$ from part (b) are open in the uniform topology, hence also in the box topology. Therefore, $h^{-1}(B)$ and $h^{-1}(U)$ form a separation of $\mathbb{R}^\omega$, with $x \in h^{-1}(B)$ and $0 \in h^{-1}(U)$. Thus, $x$ and 0 are not in the same component of $\mathbb{R}^\omega$, as required.

3.25.3. Show that the ordered square $I_o^2$ is locally connected but not locally path connected. What are the path components of this space?

In Example 1 of §24, it is established that $I_o^2$ is a linear continuum. For any $x \times y \in I_o^2$, and any open set $U$ containing $x \times y$, there is an open interval or ray $V$ containing $x \times y$ and contained in $U$. By Theorem 24.1, $V$ is connected. Thus, $I \times I$ is locally connected.

On the other hand, we show that the ordered square is not locally connected at say $(0,0)$. The sets $B = \{ x \times y : 0 < x \leq 1 \}$ and $B' = \{ x \times y : 0 < y \leq 1 \}$ are open in $I_o^2$ and $B \cap B' = \emptyset$. Then $I_o^2 = B \cup B'$, and we see that $I_o^2$ is not locally connected at $(0,0)$.

3.26.3. Show that a finite union of compact subspaces of $X$ is compact.

Let $A = A_1 \cup \cdots \cup A_n$, where each $A_i$ is a compact subspace of $X$. Let $\mathcal{U}$ be any collection of open sets in $X$ covering $A$. For each $i = 1, \ldots, n$, there is a finite subcollection $U_1^{(i)}, \ldots, U_k_i^{(i)}$ of sets in $\mathcal{U}$ whose union contains $A_i$. Then

$$A \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} U_j^{(i)},$$

which is the union of a finite subcollection of sets in $\mathcal{U}$.

3.26.4. Show that every compact subspace of a metric space is bounded in that metric and is closed. Find a metric space in which not every closed bounded subspace is compact.

Let $(X,d)$ be a metric space, and $A$ a compact subspace of $X$. Choose some $x_0 \in X$, and consider the open sets $\{ B(x_0,r) \mid r \in \mathbb{R} \}$. The set $A$ is contained in the union of these sets, so there is some $r$ such that $A \subset B(x_0,r)$. For any $a_1, a_2 \in A$, we have $d(a_1, a_2) \leq d(a_1, x_0) + d(x_0, a_2) < 2r$, so $A$ is bounded.

On the other hand, let $\bar{d}$ be the standard bounded metric on $\mathbb{R}$: $\bar{d}(x,y) = \min\{|x-y|, 1\}$. We have already established that $\bar{d}$ induces the standard topology on $\mathbb{R}$; that $\mathbb{R}$ is bounded in this metric; and that $\mathbb{R}$ is not compact.

3.26.5. Let $A$ and $B$ be disjoint compact subspaces of the Hausdorff space $X$. Show that there exist disjoint open sets $U$ and $V$ containing $A$ and $B$, respectively.

By Lemma 26.4, for each $b \in B$, there are disjoint open sets $U_b, V_b$ such that $A \subset U_b$ and $b \in V_b$. The sets $V_b$ cover $B$, so there is a finite subcover: that is, $B \subset V_{b_1} \cup \cdots \cup V_{b_n}$ for some $b_1, \ldots, b_n \in B$. Let $U = U_{b_1} \cap \cdots \cap U_{b_n}$ and $V = V_{b_1} \cup \cdots \cup V_{b_n}$. Then $U$ and $V$ are each open, $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$, as required. (For the final statement, if $x \in U \cap V$, then $x \in V_{b_i}$ for some $i$, and also $x \in U_{b_i}$; contradiction.)