2.17.6. Let $A$, $B$, and $A_\alpha$ denote subsets of a space $X$.

(a) Show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.

By definition, $\overline{A}$ is the intersection of all closed sets containing $A$. Since $\overline{B}$ is a closed set that contains $B$ and hence $A$, it must therefore contain $\overline{A}$.

(b) Show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

The set $A \cup B$ is a closed set that contains $A \cup B$, so it contains both $A$ and $B$, and therefore it contains both $\overline{A}$ and $\overline{B}$. Therefore, $\overline{A \cup B} \supset \overline{A} \cup \overline{B}$.

Conversely, $\overline{A} \cup \overline{B}$ is a closed set (since it is the union of two closed sets) that contains $A \cup B$, so it contains $A \cup B$. (Note that we had to use the fact that finite intersections of closed sets are closed!)

(c) Show that $\bigcup A_\alpha \supset \bigcup \overline{A_\alpha}$; give an example where equality fails.

Just as in (b), the set $\bigcup A_\alpha$ is a closed set that contains each $A_\alpha$, so it contains each $\overline{A_\alpha}$, and thus it contains $\bigcup \overline{A_\alpha}$. (But we can’t do the converse because an arbitrary union of closures isn’t necessarily closed!)

As a counterexample, let $A_i$ be the closed set $[\frac{1}{i}, 1] \subset \mathbb{R}$. Then $\bigcup_{i \in \mathbb{N}} A_i = \bigcup_{i \in \mathbb{N}} A_i = (0, 1]$, while $\overline{\bigcup_{i \in \mathbb{N}} A_i} = [0, 1]$.

2.17.11. Show that the product of two Hausdorff spaces is Hausdorff.

Let $X$ and $Y$ be Hausdorff spaces. For any two distinct points $(x, y)$ and $(x', y')$ in $X \times Y$, we may assume that they differ in at least one coordinate. If $x \neq x'$, choose disjoint open sets $U, U' \subset X$ with $x \in U$ and $x' \in U'$. Then $U \times Y$ and $U' \times Y$ are disjoint open sets in $X \times Y$ containing $(x, y)$ and $(x', y')$ respectively. Likewise, if $y \neq y'$, choose disjoint open sets $V, V' \subset Y$ with $y \in V$ and $y' \in V'$; then $X \times V$ and $X \times V'$ are the needed open sets.

2.17.12. Show that a subspace of a Hausdorff space is Hausdorff.

Let $X$ be Hausdorff, and let $A \subset X$ be a subspace. For any two distinct points $x, x' \in A$, we may find disjoint open sets $U, U'$ with $x \in U$ and $x' \in U'$. Then $U \cap A$ and $U' \cap A$ are disjoint open sets in $A$ containing $x$ and $x'$ respectively.

2.17.13. Show that $X$ is Hausdorff if and only if the diagonal $\Delta = \{x \times x \mid x \in X\}$ is closed in $X \times X$.

We will show an equivalent condition: that $X \times X - \Delta$ is open.

If $X$ is Hausdorff, then for any $x \times x' \in X \times X - \Delta$, we have $x \neq x'$. Choose disjoint open sets $U, U'$ with $x \in U$ and $x' \in U'$. Then $U \times U'$ is open in $X \times X$, and it does not contain any points of $\Delta$. Therefore, $X \times X - \Delta$ is open.
Conversely, if $X \times X \setminus \Delta$ is open, then for any point $x \times x' \in X \times X - \Delta$, there is a basic open set $U \times U'$ containing $x \times x'$ and disjoint from $\Delta$. Therefore, $U$ and $U'$ are disjoint open sets containing $x$ and $x'$ respectively.

2.17.17. Consider the lower limit topology and the topology given by the basis $\mathcal{C} = \{(a, b) \mid a, b \in \mathbb{Q}, a < b\}$. Determine the closures of the intervals $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$ in these two topologies.

In the lower limit topology, we claim that the closure of any interval $(a, b)$ is $[a, b)$. If $x < a$ or $x \geq b$, we can easily find an open set $[x, x+\epsilon)$ disjoint from $(a, b)$. On the other hand, a is a limit point: any open set containing a must contain an interval $[a, a+\epsilon)$, which intersects $(a, b)$. Thus, $\bar{A} = [0, \sqrt{2})$ and $\bar{B} = [\sqrt{2}, 3)$.

In the topology given by $\mathcal{C}$, the only difference is $\sqrt{2}$ is a limit point of $A$. Indeed, if $(a, b)$ is an element of $\mathcal{C}$ containing $\sqrt{2}$, we must have $a < \sqrt{2}$. Thus, $\bar{A} = [0, \sqrt{2}]$ and $\bar{B} = [\sqrt{2}, 3)$.

2.17.18. Determine the closures of the following subsets of the ordered square:

(a) $A = \left\{ \frac{1}{n} \times 0 \mid n \in \mathbb{Z}_+ \right\}$

We claim that $\bar{A} = A \cup \{0 \times 1\}$. The point $0 \times 1$ is a limit point because any open set containing $0 \times 1$ must contain $(0, \epsilon) \times [0, 1]$ for some $\epsilon > 0$, and therefore meets $A$. Any other point $x \times y \in I \times I - A$ can be seen to have a neighborhood (specifically, an interval in the dictionary ordering) that is disjoint from $A$. Namely, if $x = 0$ and $y < 1$, then we can use $[0 \times 0, 0 \times 1)$. If $0 < x < 1$, then choose $n \in \mathbb{Z}_+$ with $\frac{1}{n+1} < x < \frac{1}{n}$, and take the interval $(\frac{1}{n+1} \times 0, \frac{1}{n} \times 0)$. If $x = 1$ and $y > 0$, then take $(1 \times 0, 1 \times 1)$.

(b) $B = \{(1 - \frac{1}{n}) \times \frac{1}{2} \mid n \in \mathbb{Z}_+ \}$

We claim that $\bar{B} = \bar{B} \cup \{1 \times 0\}$. The argument is similar to the previous example.

(c) $C = \{x \times 0 \mid 0 < x < 1\}$

Any point $x \times y$ with $0 < y < 1$ has a neighborhood distinct from $C$, as do the points $0 \times 0$ and $1 \times 1$. On the other hand, any neighborhood of a point $x \times 0$ (with $0 < x \leq 1$) or $x \times 1$ (with $0 \leq x < 1$) must include an entire vertical strip (as in (a)), and thus intersect $C$. Thus,

$$\bar{C} = (0, 1] \times \{0\} \cup [0, 1) \times \{1\}$$

(d) $D = \{x \times \frac{1}{2} \mid 0 < x < 1\}$

Any point $x \times y$ with $0 < y < \frac{1}{2}$ or $\frac{1}{2} < y < 1$, or with $x = 0$ and $0 \leq y < 1$, or with $x = 1$ and $0 < y \leq 1$, has a neighborhood disjoint from $D$. On the other hand, any neighborhood of $x \times 0$ (with $0 < x \leq 1$) or of $x \times 1$ (with $0 \leq x < 1$) must include entire vertical segments, and thus intersects $D$. Thus,

$$\bar{D} = D \cup \{x \times 0 \mid 0 < x \leq 1\} \cup \{x \times 1 \mid 0 \leq x < 1\}.$$ 

(e) $E = \{\frac{1}{2} \times y \mid 0 < y < 1\}$

Any point $x \times y$ with $0 \leq x < \frac{1}{2}$ or $\frac{1}{2} < x \leq 1$ has a neighborhood disjoint from $E$. The points $\frac{1}{2} \times 0$ and $\frac{1}{2} \times 1$ are limit points. Therefore, $\bar{E} = \{\frac{1}{2} \times y \mid 0 \leq y \leq 1\}$. 

2.18.2. Suppose that \( f: X \to Y \) is continuous. If \( x \) is a limit point of \( A \subset X \), is it necessarily true that \( f(x) \) is a limit point of \( f(A) \)?

Suppose \( f \) is a constant function: \( f(x) = y_0 \) for all \( x \in X \). Then for any limit point \( x \) of \( A \), we have \( f(x) = y_0 \). Since \( f(A) \) has only one point, that point cannot be a limit point.

2.18.7(a). Suppose that \( f: \mathbb{R} \to \mathbb{R} \) is “continuous from the right,” i.e., \( \lim_{x \to a^+} f(x) = f(a) \) for each \( a \in \mathbb{R} \). Show that \( f \) is continuous when considered as a function from \( \mathbb{R}_f \) to \( \mathbb{R} \).

As a review from calculus, we say that \( \lim_{x \to a^+} f(x) = b \) if for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that for all \( x \) with \( a < x < a + \delta \), we have \( |f(x) - b| < \epsilon \).

Now, assume that \( \lim_{x \to a^+} f(x) = f(a) \) for each \( a \in \mathbb{R} \). Let \( U \subset \mathbb{R} \) be an open set. For any \( a \in f^{-1}(U) \), choose some \( \epsilon > 0 \) such that \( (f(a) - \epsilon, f(a) + \epsilon) \subset U \). By assumption, there is a \( \delta > 0 \) such that \( f([a, a + \delta)) \subset (f(a) - \epsilon, f(a) + \epsilon) \subset U \), and therefore \( [a, a + \delta) \subset f^{-1}(U) \). Thus, \( f^{-1}(U) \) is open in \( \mathbb{R}_f \).

Note: We will see in Chapter 3 that the only continuous maps from \( \mathbb{R} \) to \( \mathbb{R}_f \) are constant maps.

2.18.8. Let \( Y \) be an ordered set in the order topology. Let \( f, g: X \to Y \) be continuous.

(a) Show that the set \( \{ x \mid f(x) \leq g(x) \} \) is closed in \( X \).

In class we proved the following: Given an ordered set \( Y \), for any two distinct elements \( a, b \in Y \) with \( a < b \), there are open sets \( U, V \) such that \( a \in U \), \( b \in V \), and for all \( c \in U \) and \( d \in V \), \( c < d \).

Now, let us show that the set \( A = \{ x \mid f(x) > g(x) \} \) (which is the complement of the one above) is open in \( X \). Suppose \( x \in A \). We may find open sets \( U \) and \( V \) as above with \( g(x) \in U \) and \( f(x) \in V \). Then \( g^{-1}(U) \) and \( f^{-1}(V) \) are open in \( X \), and \( g^{-1}(U) \cap f^{-1}(V) \) contains \( x \) and is contained in \( A \). Thus, \( A \) is open as required.

(b) Let \( h: X \to Y \) be the function \( h(x) = \min\{f(x), g(x)\} \). Show that \( h \) is continuous.

By the preceding discussion, the sets \( C_1 = \{ x \mid f(x) \leq g(x) \} \) and \( C_2 = \{ x \mid f(x) \geq g(x) \} \) are closed, and their union equals \( X \). The function \( h \) equals \( f \) on \( C_1 \) and \( g \) on \( C_2 \), and these agree on the overlap. Hence, by the pasting lemma, \( h \) is continuous.