3.1.1. Which of the following are subspaces? Justify your answer in each case.

(a) \{ \mathbf{x} \in \mathbb{R}^2 \mid x_1 + x_2 = 1 \}
This set does not contain \( \mathbf{0} \), so it is not a subspace.

(b) \{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} \text{ for some } a, b \in \mathbb{R} \}
This set is precisely \( \text{Span} \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \), and the span of any set of vectors is a subspace.

(c) \{ \mathbf{x} \in \mathbb{R}^2 \mid x_1 + 2x_2 < 0 \}
This set does not contain \( \mathbf{0} \), so it is not a subspace.

(d) \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1 \}
This set does not contain \( \mathbf{0} \), so it is not a subspace.

(e) \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 0 \}
The only vector in the set is \( \mathbf{0} \), since the condition implies that \( x_1 = x_2 = x_3 = 0 \). Thus, the set is a subspace (even though it’s written in a sneakly non-linear way).

(f) \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = -1 \}
This is the empty set, which is not a subspace since it doesn’t contain \( \mathbf{0} \).

(g) \{ \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R} \}
This set is precisely \( \text{Span} \left( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) \), and the span of any set of vectors is a subspace.

(h) \{ \mathbf{x} \in \mathbb{R}^3 \mid \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R} \}.
First, to determine whether \( \mathbf{0} \) is in this set, we must see whether there exist \( s \) and \( t \) such that
\[
\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]
i.e. \( 2s + t = -3 \), \( s + 2t = 0 \), and \( s + t = -1 \). If we try to solve these equations, we find that \( s = -2 \), \( t = 1 \) works.
Now, we can verify the addition and scalar multiplication properties explicitly, or make the observation that this set is exactly the same as the set in part (g). Namely, our argument above indicates that \( \frac{301}{2} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \). Thus, any vector of the form \( x = \frac{301}{2} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \) can be rewritten as \( (s + 2) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (t - 1) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \), so it is in the set from (g), and any vector of the form \( s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) can be rewritten as \( \frac{301}{2} (s - 2) \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + (t + 1) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \), so it is in the set from (h). Thus, the set in question is a subset.

(i) \( \{ x \in \mathbb{R}^3 \mid x = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ for some } s, t \in \mathbb{R} \}. \)

To determine whether 0 is in this set, we must see whether there exist \( s \) and \( t \) such that

\[
\begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

i.e. \( 2s + t = 2, s + 2t = 4, \) and \( s + t = -1. \) There is no solution, so we deduce that the set is not a subspace.

3.1.6. (a) Let \( U \) and \( V \) be subspaces of \( \mathbb{R}^n. \) Define the intersection of \( U \) and \( V \) to be

\[ U \cap V = \{ x \in \mathbb{R}^n \mid x \in U \text{ and } x \in V \}. \]

Show that \( U \cap V \) is a subspace of \( V. \) Give two examples.

We must check the three properties:

1. Since \( 0 \in U \) and \( 0 \in V, \) we have \( 0 \in U \cap V. \)
2. If \( x, y \in U \cap V, \) then \( x \) and \( y \) are in both \( U \) and \( V, \) so \( x + y \) is in \( U \) and in \( V, \) so \( x + y \in U \cap V. \)
3. If \( x \in U, \) then for any \( c \in \mathbb{R}, \) \( cx \) is in both \( U \) and \( V, \) so \( cx \in U \cap V. \)

As examples, if \( U \) and \( V \) are both lines through the origin in \( \mathbb{R}^n, \) then \( U \cap V \) is either \( \{0\} \) (if \( U \) and \( V \) are distinct lines), or equal to \( U \) (if \( U \) and \( V \) are the same line). Likewise, if \( U \) and \( V \) are both planes through the origin, then \( U \cap V \) is either \( \{0\}, \) a line, or all of \( U \) (if \( U \) and \( V \) are the same plane).

(b) Is \( U \cup V = \{ x \in \mathbb{R}^n \mid x \in U \text{ or } x \in V \} \) always a subspace? Give a proof or counterexample.

Let \( U = \text{Span}(e_1) \) and \( V = \text{Span}(e_2) \) in \( \mathbb{R}^2. \) (That is, \( U \) is the \( x_1 \) axis and \( V \) is the \( x_2 \) axis.) Then \( e_1 \) and \( e_2 \) are both in \( U \cup V, \) but \( e_1 + e_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) is not in \( U \cup V \) since it is neither \( U \) nor \( V. \) Thus, \( U \cup V \) is not a subspace.
3.1.9. Determine the intersection of the subspaces $P_1$ and $P_2$ in each case.

(b) $P_1 = \text{Span} \left( \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$, $P_2 = \text{Span} \left( \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$.

A vector is in $P_1 \cap P_2$ iff it can be written as a linear combination of the first two vectors and also as linear combination of the second two vectors. First, let us find all $a, b, c, d \in \mathbb{R}$ such

$$a \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = c \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$ 

That is, we must solve the system

$$a + 0b - 2c - d = 0$$
$$2a + b - c - 0d = 0$$

(the equation corresponding to the third entries is redundant). Reducing, we see that $c$ and $d$ are free and $a = 2c + d$ and $b = -4c - 2d$. This means that every vector in $P_2$ is in $P_1$. Moreover, we could also have solved for $c$ and $d$ in terms of $a$ and $b$, which says that every vector in $P_1$ is also in $P_2$. Thus, $P_1 \cap P_2 = P_1 = P_2$.

(c) $P_1 = \text{Span} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right)$, $P_2 = \{ \mathbf{x} \in \mathbb{R}^3 \mid x_1 - x_2 + x_3 = 0 \}$.

Vectors in $P_1 \cap P_2$ must be of the form $s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and must also satisfy the constraint of $P_2$. That is, we must have $(s + t) - (2t) + (-s + 3t) = 0$, i.e. $t = 0$ (with any $s$ allowed).

Thus, $P_1 \cap P_2 = \text{Span} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$.

3.1.10. Let $V \subset \mathbb{R}^n$ be a subspace. Show that $V \cap V^\perp = \{0\}$.

If $\mathbf{x} \in V \cap V^\perp$, then $\mathbf{x} \cdot \mathbf{x} = 0$ (since this is the dot product of something in $V$ with something in $V^\perp$), and hence $\mathbf{x} = \mathbf{0}$. Thus, $V = \{0\}$.

0.1. 3.1.12. Suppose $V$ and $W$ are orthogonal subspaces of $\mathbb{R}^n$, i.e. $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Prove that $V \subset W^\perp$.

For any vector $\mathbf{v} \in V$, we have $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{w} \in W$. Thus, $\mathbf{v}$ satisfies the defining property of being in $W^\perp$, so $\mathbf{v} \in W^\perp$. Thus, $V \subset W^\perp$.

3.1.13. Let $V \subset \mathbb{R}^n$ be a subspace. Show that $V \subset (V^\perp)^\perp$. Do you think more is true?

For any vector $\mathbf{v} \in V$, and any vector $\mathbf{w} \in V^\perp$, we have $\mathbf{v} \cdot \mathbf{w} = 0$ (since $\mathbf{w}$ is orthogonal to every vector in $V$). That is, $\mathbf{v}$ is orthogonal to every vector in $V^\perp$, so $\mathbf{v} \in (V^\perp)^\perp$. This shows that $V \subset (V^\perp)^\perp$. The reverse inclusion was shown in class.

3.1.14. Let $V$ and $W$ be subspaces of $\mathbb{R}^n$ with the property that $V \subset W$. Prove that $W^\perp \subset V^\perp$.

For any vector $\mathbf{u} \in W^\perp$, we have $\mathbf{u} \cdot \mathbf{w} = 0$ for every vector $\mathbf{w} \in W$. In particular, this is true for every $\mathbf{w} \in V$. Thus, $\mathbf{u} \in V^\perp$. 

3.2.10. Let $A$ be an $m \times n$ matrix, and let $B$ be an $n \times p$ matrix. (a) Prove that $N(B) \subset N(AB)$.

For any vector $v \in N(B)$, we have $Bv = 0$, and hence $AB(v) = A(Bv) = 0$. (Note: here the 0 in the first equation is in $\mathbb{R}^n$, and the 0 in the second is in $\mathbb{R}^m$.) Hence, $v \in N(AB)$.

(b) Prove that $C(AB) \subset C(A)$.

If $v \in C(AB)$, then there is a vector $w \in \mathbb{R}^p$ such that $(AB)w = v$. (I.e., the system $(AB)x = v$ is consistent.) Rewriting this, we see that $A(Bw) = v$, which means that the system $Ax = v$ is consistent as well. Hence, $v \in C(A)$.

(c) If $A$ is $n \times n$ and nonsingular, prove that $N(B) = N(AB)$.

We already know from part (a) that $N(B) \subset N(AB)$; we need to prove that $N(AB) \subset N(A)$. For any $v \in N(AB)$, we have $ABv = 0$. Multiplying both sides on the left by $A^{-1}$ (which exists because $A$ is nonsingular), we see that $Bv = 0$, so $v \in N(B)$.

(d) If $B$ is $n \times n$ and nonsingular, prove that $C(AB) = C(A)$.

We already know from part (b) that $C(AB) \subset C(A)$; we just need to show that $C(A) \subset C(AB)$. For any vector $v \in C(A)$, there is a vector $w \in \mathbb{R}^n$ such that $Aw = v$. Then $AB(B^{-1}w) = A\cdot w = v$, so $v \in C(AB)$ as well.

3.2.11. Let $A$ be an $m \times n$ matrix. Prove that $N(A^T A) = N(A)$.

The inclusion $N(A) \subset N(A^T A)$ follows from the previous problem; we need to see that $N(A^T A) \subset N(A)$. If $v \in N(A^T A)$, then $A^T Av = 0$, so $Av \in N(A^T)$. Also, $Av \in C(A)$, by definition. And since $C(A) = N(A^T)^\perp$, we deduce that $Av = 0$, so $v \in N(A)$, as required.