2.4.12. Suppose $A$ and $B$ are two $m \times n$ matrices with the same reduced echelon form. Show that there exists an invertible matrix $E$ so that $EA = B$. Is the converse true?

Let $C$ denote the reduced echelon form of both $A$ and $B$. Since row operations can be realized by multiplying on the left by elementary matrices, there are elementary matrices $E_1, \ldots, E_k$ such that $C = E_k \cdots E_1 A$ and elementary matrices $E'_1, \ldots, E'_l$ such that $C = E'_l \cdots E'_1 B$. Equating these two, and using the fact that elementary matrices are invertible, we have:

$$E_k \cdots E_1 A = E'_l \cdots E'_1 B$$

Hence we define $E = (E_1)^{-1} \cdots (E_k)^{-1} E'_l \cdots E'_1$, which is invertible since it is the product of invertible matrices.

To see the converse, suppose $EA = B$, where $E$ is invertible. This means that $E$ can be row reduced to the identity. It follows that $E$ is equal to a product of elementary matrices. Thus, there is a sequence of row operations taking $A$ to $B$. By the uniqueness of reduced echelon forms, we see that $A$ and $B$ must have the same reduced echelon form.

2.5.4. Suppose $a, b, c, d \in \mathbb{R}^n$. Check that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -c^T & - \\ -d^T & - \end{bmatrix} = ac^T + bd^T.$$ 

We can just write this out:

$$\begin{bmatrix} a_1 & b_1 \\ \vdots & \vdots \\ a_n & b_n \end{bmatrix} \begin{bmatrix} c_1 & \cdots & c_n \\ d_1 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} a_1 c_1 + b_1 d_1 & \cdots & a_1 c_n + b_1 d_n \\ \vdots & \vdots \\ a_n c_1 + b_n d_1 & \cdots & a_n c_n + b_n d_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1 c_1 & \cdots & a_1 c_n \\ \vdots & \vdots \\ a_n c_1 & \cdots & a_n c_n \end{bmatrix} + \begin{bmatrix} b_1 d_1 & \cdots & b_1 d_n \\ \vdots & \vdots \\ b_n d_1 & \cdots & b_n d_n \end{bmatrix}$$

$$= ac^T + bd^T.$$

2.5.8. Suppose $A$ is invertible. Check that $(A^{-1})^T A^T = I$ and $A^T (A^{-1})^T = I$, and deduce that $A^T$ is likewise invertible with inverse $(A^{-1})^T$.

Since in general we have $(AB)^T = B^T A^T$, we have $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$ and $A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I$, as required.
2.5.12. Suppose $A$ is a symmetric $n \times n$ matrix. If $x, y \in \mathbb{R}^n$ are vectors satisfying $Ax = 2x$ and $Ay = 3y$, show that $x$ and $y$ are orthogonal.

Using the fact that $A$ is symmetric, we have $x \cdot Ay = x^T Ay = y^T Ax = y^T Ax = x \cdot y$. Also, $x \cdot Ay = x \cdot 3y = 3(x \cdot y)$, and $Ax \cdot y = 2x \cdot y = 2(x \cdot y)$. Putting these together, we see that $2(x \cdot y) = 3(x \cdot y)$, and hence $x \cdot y = 0$.

2.5.19. We say an $n \times n$ matrix is orthogonal if $A^T A = I_n$.

(a) Prove that the column vectors $a_1, \ldots, a_n$ of an orthogonal matrix $A$ are unit vectors that are orthogonal to one another.

By definition, the $(i, j)$ entry of $A^T A$ is the dot product between the $i^{th}$ column of $A$ (i.e. the $i^{th}$ row of $A^T$) and the $j^{th}$ column of $A$. Thus, we see right away that

$$a_i \cdot a_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

This means that $\|a_i\| = 1$ for each $i$, and $a_i$ and $a_j$ are orthogonal for $i \neq j$.

(b) Fill in the missing columns in the following matrices to make them orthogonal.

For $\begin{bmatrix} \sqrt{\frac{3}{2}} & a \\ -\frac{1}{2} & b \end{bmatrix}$: Since the dot product of the two columns must be $0$, we must have $\sqrt{\frac{3}{2}} + a - \frac{1}{2} b = 0$, i.e. $b = \sqrt{3} a$. Moreover, we must have $a^2 + b^2 = 1$, so $a^2 + (3a)^2 = 1$, so $a^2 = \frac{1}{4}$, so $a = \pm 12$. Thus, the two options are $\begin{bmatrix} \sqrt{\frac{3}{2}} & \frac{1}{2} \\ -\frac{1}{2} & \sqrt{\frac{3}{2}} \end{bmatrix}$ and $\begin{bmatrix} \sqrt{\frac{3}{2}} & -\frac{1}{2} \\ -\frac{1}{2} & -\sqrt{\frac{3}{2}} \end{bmatrix}$.

For $\begin{bmatrix} 1 & 0 & a \\ 0 & -1 & b \\ 0 & 0 & c \end{bmatrix}$: Since the dot product of the third column with either of the first two columns must be $0$, we must have $a = b = 0$. Since the third column must be a unit vector, we must have $c = \pm 1$.

For $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$: Taking the dot products of the second column with the first and third columns shows that $\frac{1}{2} + 2b + 2c = 0$ and $2a - 2b + c = 0$, which implies that $c = -a$ and $b = a/2$. Since the second column must be a unit vector, we deduce that $b = \pm 13$, so the matrix must be $\begin{bmatrix} \frac{1}{2} & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & -\frac{1}{2} & \frac{1}{3} \\ \frac{3}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ or $\begin{bmatrix} \frac{1}{2} & -\frac{2}{2} & \frac{2}{2} \\ -\frac{1}{2} & \frac{1}{3} & \frac{1}{3} \\ \frac{3}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$.

(c) Show that any $2 \times 2$ orthogonal matrix $A$ must be of the form $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ or $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{bmatrix}$ for some real number $\theta$.

Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^T A = I_2$. Writing this out, this implies that $a^2 + c^2 = 1$, $b^2 + d^2 = 1$, and $ab + cd = 0$. By using polar coordinates, there are some angles $\theta, \phi$ such that $(a, c) = (\cos \theta, \sin \theta)$ and $(b, d) = (\cos \phi, \sin \phi)$. Since the vectors $(a, c)$ and $(b, d)$ are
orthogonal, these angles must be $\pi/2$ apart: either $\phi = \theta + \pi/2$ or $\phi = \theta - \pi/2$. Using angle addition formulas, in the first case, we get $\cos \phi = -\sin \theta$ and $\sin \phi = \cos \theta$; in the latter case, we get $\cos \phi = \sin \theta$ and $\sin \phi = \cos \theta$, corresponding to the two cases above.

(d) Show that if $A$ is an orthogonal $2 \times 2$ matrix, then $\mu_A: \mathbb{R}^2 \to \mathbb{R}^2$ is either a rotation or the composition of a rotation and a reflection.

If $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, we have already seen that $\mu_A$ is counterclockwise rotation by $\theta$. If $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, notice that we can write $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which is the product of a rotation matrix and a reflection matrix (namely reflection across the $x$ axis). Thus, $\mu_A$ has the form described above.

(e) Prove that the row vectors of an orthogonal matrix $A$ are unit vectors that are orthogonal to one another.

In light of part (a), this is just the same as saying that if $A$ is an orthogonal matrix, then $A^T A = I_n$, so $A A^T = I_n$. Therefore, $A A^T = I_n$. We can write this as $(A^T)^T A^T = I_n$, which means that $A^T$ is an orthogonal matrix.

2.5.22. (a) Show that the only matrix that is both symmetric and skew-symmetric is $O$.

If $A$ is both symmetric and skew-symmetric, we have $A^T = A$ and $A^T = -A$, so $A = -A$, so $2A = O$, so $A = O$.

(b) Given any square matrix $A$, show that $S = \frac{1}{2}(A + A^T)$ is symmetric and $K = \frac{1}{2}(A - A^T)$ is skew-symmetric.

In general, $(A + B)^T = A^T + B^T$ and $(cA)^T = cA^T$. Using, this, we have $S^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A + A^T) = S$ and $K^T = \frac{1}{2}(A^T - (A^T)^T) = \frac{1}{2}(-A + A^T) = -K$.

(c) Deduce that any square matrix $A$ can be written in the form $A = S + K$, where $S$ is symmetric and $K$ is skew-symmetric.

We have $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

(d) Prove that the expression in part (c) is unique.

If $A = S + K = S' + K'$, where $S$ and $S'$ are symmetric and $K$ and $K'$ are skew-symmetric, then we can reorganize to get $S - S' = K - K'$. Therefore, the matrix on either side of this equation is both symmetric and skew-symmetric, so it is $O$. Hence $S = S'$ and $K = K'$, as required.

2.5.23. (a) Suppose $A$ is an $m \times n$ matrix and $A \bar{x} \cdot y = 0$ for every vector $x \in \mathbb{R}^n$ and every vector $y \in \mathbb{R}^m$. Prove that $A = O$.

Let us write the standard basis for $\mathbb{R}^n$ as $e_1, \ldots, e_n$, and the standard basis for $\mathbb{R}^m$ as $d_1, \ldots, d_m$ (to avoid notational confusion). Then $A e_i$ is the $i$th column of $A$, and $A e_i \cdot d_j$ is the $j$th entry of $A e_i$, i.e. $a_{ji}$. The hypothesis thus says that $A e_i \cdot d_j = 0$ for all $i$ and $j$; thus, all the entries of $A$ are zero, as required.

(b) Suppose $A$ is a symmetric $n \times n$ matrix and $A \bar{x} \cdot x = 0$ for every vector $x \in \mathbb{R}^n$. Prove that $A = O$. 


Following the hint, for any vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have:

$$
0 = A(\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y})
$$

$$
= A\mathbf{x} \cdot \mathbf{x} + A\mathbf{x} \cdot \mathbf{y} + A\mathbf{y} \cdot \mathbf{x} + A\mathbf{y} \cdot \mathbf{y}
$$

$$
= A\mathbf{x} \cdot \mathbf{y} + A\mathbf{y} \cdot \mathbf{x}
$$

where we used the given information again to get to the last line. On the other hand, let’s notice that $A\mathbf{y} \cdot \mathbf{x} = (A\mathbf{y})^T \mathbf{x} = \mathbf{y}^T A^T \mathbf{x} = \mathbf{y}^T A \mathbf{x} = \mathbf{y} \cdot A \mathbf{x} = A \mathbf{x} \cdot \mathbf{y}$. (This only works because $A$ is symmetric.) Plugging this in above, we deduce that $A\mathbf{x} \cdot \mathbf{y} = 0$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and thus $A = O$ by part (a).