2.2.7. (a) Calculate $A_\theta A_\phi$ and $A_\phi A_\theta$.

$$A_\theta A_\phi = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi - \sin \theta \sin \phi & -\sin \theta \cos \phi - \cos \theta \sin \phi \\ \sin \theta \cos \phi + \cos \theta \sin \phi & \cos \theta \cos \phi + \sin \theta \sin \phi \end{bmatrix}$$

And $A_\phi A_\theta$ equals the same matrix, as can be seen by swapping the roles of $\theta$ and $\phi$ everywhere.

(b) Use your answer to part (a) to derive the addition formulas for sine and cosine.

Geometrically, rotating the plane by $\theta$ and then rotating it by $\phi$ is the same as rotating it by $\theta + \phi$ all at once. Hence, the above matrix is equal to $A_{\theta + \phi}$. This means, in particular, that

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$
$$\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi.$$  

(Note: If, like me, you always have difficulty remembering the angle addition formulas, you can easily remember them using this method!)

2.2.8. For $0 \leq \theta \leq \pi$, prove that $\|A_\theta \mathbf{x}\| = \|\mathbf{x}\|$ and that the angle between $\mathbf{x}$ and $A_\theta \mathbf{x}$ equals $\theta$.

To avoid writing lots of square roots, let’s just compute $\|A_\theta \mathbf{x}\|^2$. We have:

$$\|A_\theta \mathbf{x}\|^2 = \left\| \begin{bmatrix} x_1 \cos \theta - x_2 \sin \theta \\ x_1 \sin \theta + x_2 \cos \theta \end{bmatrix} \right\|^2$$

$$= (x_1 \cos \theta - x_2 \sin \theta)^2 + (x_1 \sin \theta + x_2 \cos \theta)^2$$

$$= x_1^2 \cos^2 \theta - 2x_1 x_2 \cos \theta \sin \theta + x_2^2 \sin^2 \theta$$
$$- x_1^2 \sin^2 \theta + 2x_1 x_2 \cos \theta \sin \theta + x_2^2 \cos^2 \theta$$

$$= (x_1^2 + x_2^2)(\cos^2 \theta + \sin^2 \theta)$$

$$= x_1^2 + x_2^2$$

$$= \|\mathbf{x}\|^2.$$
If $\phi$ denote the angle between $x$ and $A_\theta x$, then

$$
\cos \phi = \frac{x \cdot A_\theta x}{\|x\| \|A_\theta x\|}
$$

$$
= \frac{x_1(x_1 \cos \theta - x_2 \sin \theta) + x_2(x_1 \sin \theta + x_2 \cos \theta)}{\|x\|^2}
$$

$$
= \frac{(x_1^2 + x_2^2) \cos \theta}{\|x\|^2}
$$

$$
= \cos \theta
$$

Since $\theta$ and $\phi$ are both between 0 and $\pi$ and both have the same cosine, we must have $\theta = \phi$.

2.3.8. Suppose $A$ is a square matrix satisfying the equation $A^3 - 3A + I = O$. Show that $A$ is invertible.

We may rewrite the equation as $-A^3 + 3A = I$, or in other words $A(-A^2 + 3I) = I$. Therefore, $-A^2 + 3I$ is a right inverse for $A$, which implies that $A$ is invertible (since it is square).

2.3.11. Suppose $A$ and $B$ are $n \times n$ matrices. Prove that if $AB$ is nonsingular, then both $A$ and $B$ are nonsingular.

Following the hint, we first show that $B$ is nonsingular. For every nonzero vector $x \in \mathbb{R}^n$, note that $(AB)\vec{x} \neq \vec{0}$. By associativity, this is the same as saying that $A(B\vec{x}) \neq 0$, and therefore $B\vec{x} \neq 0$ (since we can’t multiply a matrix by $\vec{0}$ and get a nonzero vector). Thus, the only solution to $B\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, which indicates that $B$ is nonsingular.

Now, observe that $A = (AB)(B^{-1})$, so $A$ is the product of two invertible matrices, hence it is invertible.

2.3.16. Suppose $A$ is an $n \times n$ matrix satisfying $A^{10} = O$. Prove that the matrix $I_n - A$ is invertible.

There’s actually nothing special about 10 here; let’s assume $A^m = O$ for some integer $m > 1$. The trick is first to remember some facts about factoring polynomials:

$$
t^2 - 1 = (t - 1)(t + 1)
$$

$$
t^3 - 1 = (t - 1)(t^2 + t + 1)
$$

$$
t^4 - 1 = (t - 1)(t^3 + t^2 + t + 1)
$$

and in general

$$
t^m - 1 = (t - 1)(t^{m-1} + t^{m-2} + \cdots + t + 1).
$$

Analogous formulas hold for matrices:

$$
A^m - I_n = (A - I_n)(A^{m-1} + A^{m-2} + \cdots + A + I_n).
$$

We can prove this by just expanding out the right side and canceling a lot of terms.

Now if $A^m = O$, we see that

$$
-I_n = (A - I_n)(A^{m-1} + A^{m-2} + \cdots + A + I_n)
$$

$$
I_n = (I_n - A)(A^{m-1} + A^{m-2} + \cdots + A + I_n)
$$
and hence

\[(I_n - A)^{-1} = A^{m-1} + A^{m-2} + \cdots + A + I_n.\]