2.1.7. Find all $2 \times 2$ matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfying:

(a) $A^2 = I$

We need to solve four non-linear equations:

\[
\begin{align*}
    a^2 + bc &= 1 \\
    ab + bd &= 0 \\
    ac + cd &= 0 \\
    bc + d^2 &= 1.
\end{align*}
\]

Combining the first and fourth equations tells you that $a^2 = d^2 = 1 - bc$, which means that $a = \pm d$. We can consider two cases: either $a = d \neq 0$, or $a = -d$.

In the first case, the second and third equations tell us that $b = c = 0$, and therefore the first and fourth equations say that $a^2 = d^2 = 1$. Hence, the only two matrices we obtain are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

In the second case, the second and third equations don’t give any constraint on $b$ and $c$. The only constraint is that $1 - bc \geq 0$, since otherwise $a^2$ and $d^2$ would be negative. For any $b$ and $c$ with $bc \leq 1$, we obtain two possible matrices, namely $\begin{bmatrix} \pm \sqrt{1-bc} & b \\ c & \mp \sqrt{1-bc} \end{bmatrix}$.

(b) $A^2 = O$

We proceed similarly to the preceding problem, where now the right-hand sides of all four equations are 0. As before, we deduce that $a^2 = d^2$. If $a = d$, then the second and third equations give $b = c = 0$, and then the first and fourth give $a = d = 0$. If $a = -d$, then $b$ and $c$ can be any numbers with $bc \leq 0$ (i.e. they have opposite signs), and then we get $A = \begin{bmatrix} \pm \sqrt{-bc} & b \\ c & \mp \sqrt{-bc} \end{bmatrix}$.

(c) $A^2 = -I_2$

In this case, note that we can’t have $a = d \neq 0$, since that would force $a^2 = d^2 = -1$. Hence the only solutions are $\begin{bmatrix} \pm \sqrt{-1-bc} & b \\ c & -\mp \sqrt{-1-bc} \end{bmatrix}$.

2.1.8. For each of the following matrices, find a formula for $A^k$ for positive integers $k$.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

After experimenting with a few small values of $k$, we may observe the pattern that $A^k = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix}$. We now prove this claim by induction. For $k = 1$ this is automatic. For the induction step, suppose that we know that $A^k$ satisfies the formula; we must show that $A^{k+1}$
does. We have

\[
A^{k+1} = A^k A = \begin{bmatrix} 2^k & 0 \\ 0 & 3^k \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2^k \cdot 2 & 0 \\ 0 & 3^k \cdot 3 \end{bmatrix} = \begin{bmatrix} 2^{k+1} & 0 \\ 0 & 3^{k+1} \end{bmatrix}
\]
as required. Thus, by induction, the formula holds for all \(k\).

(b) \(A = \begin{bmatrix} d_1 & d_2 & \cdots & d_n \end{bmatrix} \).

By an argument as in part (a), we obtain

\[
A^k = \begin{bmatrix} d_1^k & & & \\ & d_2^k & & \\ & & \ddots & \\ & & & d_n^k \end{bmatrix}.
\]

(c) \(A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \).

Again, after a bit of experimenting, we observe the pattern that \(A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \). We prove
this claim by induction. The case \(k = 1\) is given. For the induction step, suppose we know
that \(A^k\) satisfies the formula. Then

\[
A^{k+1} = A^k A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}
\]
so \(A^{k+1}\) also satisfies the formula. Thus, the formula holds for all \(k\).

2.1.14. Find all \(2 \times 2\) matrices \(A\) that commute with all \(2 \times 2\) matrices \(B\).

Suppose \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), and that for every \(B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \), we have \(AB = BA\). This means
that for all possible \(e, f, g, h \in \mathbb{R}\), we have:

\[
\begin{align*}
ac + bg &= ac + cf \\
cc + dg &= ag + ch
\end{align*}
\]
\[
af + bh &= be + df \\
af + dh &= bg + dh.
\]

In particular, we can plug in some strategically chosen values of \(e, f, g, h\) to deduce some
simple facts about \(a, b, c, d\). Specifically, if we take \(e = 1, f = 0, g = 0, h = 0\), we see that
\(b = 0\) and \(c = 0\). And if we take \(e = 0, f = 1, g = 0, h = 0\), we see that \(a = d\). Thus, \(A\)
must be a multiple of the identity matrix. We already know that any multiple of the diagonal
matrix commutes with all matrices \(B\), so this description is complete. (That is, we won’t
learn anything new from considering other values of \(e, f, g, h\).)