1.5.10. Let $A$ be an $m \times n$ matrix. Prove or give a counterexample: If $Ax = 0$ has only the trivial solution $x = 0$, then $Ax = b$ always has a unique solution.

This is false! The trap is that $Ax = b$ may not have any solutions (and the problem cleverly omitted the assumption $Ax = b$ is consistent). For instance, if 

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix}, \quad b = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]

then the equation $Ax = 0$ immediately gives $x_1 = x_2 = 0$ and hence $x = 0$, but $Ax = b$ has no solutions.

1.5.12. In each case, give positive integers $m$ and $n$ and an example of an $m \times n$ matrix $A$ with the stated property, or explain why none can exist.

(a) $Ax = b$ is inconsistent for every $b \in \mathbb{R}^m$.

If $b = 0$, then the solution $Ax = b$ always has at least one solution, namely $x = 0$. Therefore, this can’t happen.

(b) $Ax = b$ has one solution for every $b \in \mathbb{R}^m$.

This will be true for any nonsingular $n \times n$ matrix. The most basic example is $m = n = 1$ and $A = [1]$.

(c) $Ax = b$ has no solutions for some $b \in \mathbb{R}^m$ and one solution for every other $b \in \mathbb{R}^m$.

This definitely can’t happen. For instance, if $Ax = b$ has no solutions, then $Ax = 2b$ also has no solutions, since if $x$ were a solution to $Ax = 2b$, then $\frac{1}{2}x$ would be a solution to $Ax = b$.

(d) $Ax = b$ has infinitely many solutions for every $b \in \mathbb{R}^m$.

This will be true for any $m \times n$ matrix $A$ with $m < n$ and rank($A$) = $m$. An example is $m = 1, n = 2, A = [1 \ 1]$.

(e) $Ax = b$ is inconsistent for some $b \in \mathbb{R}^m$ and has infinitely many solutions whenever it is consistent.

This will be true for any $m \times n$ matrix $A$ where rank($A$) < $m$ (which guarantees that it’s sometimes inconsistent) and rank($A$) < $n$ (which guarantees that there are infinitely many solutions). For instance, we could take $m = n = 2$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(f) There are vectors $b_1, b_2, b_3 \in \mathbb{R}^m$ such that $Ax = b_1$ has no solutions, $Ax = b_2$ has one solution, and $Ax = b_3$ has infinitely many solutions.

We saw in class that this cannot happen.

1.5.13. Suppose $A$ is an $m \times n$ matrix with rank $m$, and $v_1, \ldots, v_k \in \mathbb{R}^n$ with span($v_1, \ldots, v_k$)$= \mathbb{R}^n$. Prove that span($Av_1, \ldots, Av_k$)$= \mathbb{R}^m$.

We need to show that every vector in $\mathbb{R}^n$ can be written as a linear combination of $Av_1, \ldots, Av_k$. Since rank($A$) = $m$, for any $b \in \mathbb{R}^n$, we know that the system $Ax = b$ has a
solution, which means that \( b = Aw \) for some vector \( w \in \mathbb{R}^n \). Since the vectors \( v_1, \ldots, v_k \) span all of \( \mathbb{R}^n \), we can write \( w = c_1v_1 + \cdots + c_kv_k \) for some scalars \( c_1, \ldots, c_k \). We then observe:

\[
\begin{align*}
\mathbf{b} &= \mathbf{A}w \\
&= \mathbf{A}(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \\
&= c_1\mathbf{A}v_1 + \cdots + c_k\mathbf{A}v_k.
\end{align*}
\]

So \( \mathbf{b} \in \text{Span}(\mathbf{A}v_1, \ldots, \mathbf{A}v_k) \), as required. \( \square \)

1.5.14. Let \( A \) be an \( m \times n \) matrix with row vectors \( A_1, \ldots, A_m \). (a) Suppose \( A_1 + \cdots + A_m = 0 \). Deduce that \( \text{rank}(A) < m \).

First proof: For any vector \( \mathbf{x} = (x_1, \ldots, x_n) \), we have \( 0 = (A_1 + \cdots + A_m) \cdot \mathbf{x} = A_1 \cdot \mathbf{x} + \cdots + A_m \cdot \mathbf{x} \), which is the sum of the entries of \( \mathbf{A} \mathbf{x} \). This means that if \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has solutions, then the sum of the entries in \( \mathbf{b} \) must be zero. Equivalently, if \( \mathbf{b} \) is any vector whose sum of entries is nonzero, then \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has no solutions. This implies that \( \text{rank}(A) < m \).

Second proof: Let’s perform some row operations to \( A \). First, add a multiple of each of the first \( m-1 \) rows to the last row, and call the resulting matrix \( B \). By assumption, the new \( m^\text{th} \) row will be all zeros. If we then perform Gaussian elimination to obtain any echelon form of \( B \) (which is an echelon form of \( A \)), there will be at least one row of zeros at the bottom, and therefore \( \text{rank}(A) < m \). \( \square \)

(b) More generally suppose there is some linear combination \( c_1A_1 + \cdots + c_mA_m = 0 \), where some \( c_i \neq 0 \). Show that \( \text{rank}(A) < m \).

We can easily adapt the first proof from above to show that if \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has solutions, then \( c_1b_1 + \cdots + c_mb_m = 0 \). If we choose \( \mathbf{b} = (0, \ldots, 1, \ldots, 0) \), where the 1 is in the \( i^\text{th} \) entry, then we see there are no solutions. \( \square \)

1.6.9. A circle \( C \) passes through the points \((2, 6), (-1, 7), \) and \((-4, -2)\). Find the center and radius of \( C \).

To begin, the equation of a circle is \((x - x_0)^2 + (y - y_0)^2 = r^2\). This can also be written as

\[
x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 = r^2,
\]
or equivalently as

\[
x^2 + y^2 + ax + by + c = 0,
\]

where \( a = -2x_0 \), \( b = -2y_0 \), and \( c = x_0^2 + y_0^2 - r^2 \).

Now, if the circle contains the given points, we must have:

\[
\begin{align*}
2a + 6b + c &= -4 - 36 = -40 \\
-a + 7b + c &= -1 - 49 = -50 \\
-4a - 2b + c &= -16 - 4 = -20.
\end{align*}
\]

If we solve this by Gaussian elimination, which I leave up to you, we see that \( a = 2 \), \( b = -4 \), and \( c = -20 \). Hence, we must have \( x_0 = -1 \), \( y_0 = -2 \), and \( r^2 = x_0^2 + y_0^2 - c = 25 \), so \( r = 5 \). Thus, the points all lie on a circle of radius 5 about \((-1, 2)\) (which you can easily check).
1.6.11. Let \( P_i = (x_i, y_i) \in \mathbb{R}^2 \) for \( i = 1, 2, 3 \). Let
\[
A = \begin{bmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1
\end{bmatrix}.
\]

(a) Show that the points \( P_1, P_2, P_3 \) are collinear if and only if the equation \( Ax = 0 \) has a nontrivial solution.

As the hint suggests, a general line in \( \mathbb{R}^2 \) has the Cartesian equation \( ax + by + c = 0 \), where \( a \) and \( b \) are not both zero. The condition that \( P_1, P_2, \) and \( P_3 \) all lie on a particular line is precisely that \( ax_i + by_i + c = 0 \) for each \( i \), or in other words
\[
\begin{bmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1
\end{bmatrix} \begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}.
\]

If such a line exists, then those values of \((a, b, c)\) provide a nonzero solution to the equation \( Ax = 0 \). Conversely, if there is a nonzero vector \( x = (a, b, c) \) with \( Ax = 0 \), this vector must have either first or second entry nonzero because
\[
\begin{bmatrix}
  x_1 & y_1 & 1 \\
  x_2 & y_2 & 1 \\
  x_3 & y_3 & 1
\end{bmatrix} \begin{bmatrix}
  0 \\
  0 \\
  c
\end{bmatrix} = \begin{bmatrix}
  c \\
  c \\
  c
\end{bmatrix}.
\]

Thus, the line \( ax + by + c = 0 \) contains all three points.

(b) Deduce that if the three given points are not collinear, then there is a unique circle passing through them.

According to problem 9, finding a circle through the three points is equivalent to finding \( a, b, \) and \( c \) for which we have
\[
x_1^2 + y_1^2 + ax_1 + by_1 + c = 0 \\
x_2^2 + y_2^2 + ax_2 + by_2 + c = 0 \\
x_3^2 + y_3^2 + ax_3 + by_3 + c = 0.
\]

Bringing the square terms over the the right, we can write this condition as \( Ax = b \), where
\[
x = \begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix}
  -x_1^2 - y_1^2 \\
  -x_2^2 - y_2^2 \\
  -x_3^2 - y_3^2
\end{bmatrix}.
\]

By part (a), if the points are non-collinear, then the matrix \( A \) is nonsingular. By Proposition 5.5, this implies that \( Ax = b \) has a unique solution, as required.