1.2.18. Prove the triangle inequality: For any vector \( x, y \in \mathbb{R}^n \), \( \|x + y\| \leq \|x\| + \|y\| \).

We have:

\[
\|x + y\|^2 = (x + y) \cdot (x + y) \\
= (x \cdot x) + (y \cdot y) + 2(x \cdot y) \\
= \|x\|^2 + \|y\|^2 + 2(x \cdot y) \\
\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \quad \text{(by Cauchy–Schwarz)} \\
= (\|x\| + \|y\|)^2
\]

Taking square roots, we have \( \|x + y\| \leq (\|x\| + \|y\|) \) as required. \( \square \)

1.3.10. (a) The equations \( x_1 = 0 \) and \( x_2 = 0 \) each describe planes in \( \mathbb{R}^3 \) that contain the \( x_3 \) axis. Write down the Cartesian equation of a general such plane.

The Cartesian equation of a plane in \( \mathbb{R}^3 \) is \( ax + by + cz = d \), where \( a, b, c \) are not all 0. If a plane contains the \( x_3 \) axis, it must contain both \((0, 0, 0)\) and \((0, 0, 1)\). Hence \( 0a + 0b + 0c = d \), which gives \( d = 0 \); and \( 0a + 0b + 1c = d = 0 \), so \( c = 0 \). To show that these are the only restrictions, any plane with the equation \( ax_1 + bx_2 = 0 \) contains every point of the form \((0, 0, x_3)\) for any \( x_3 \), since such a point necessarily satisfies the equation irrespective of \( x_3 \). Thus, the general form is \( ax_1 + bx_2 = 0 \) (for any \( a, b \in \mathbb{R} \) not both zero).

(b) The equations \( x_1 - x_2 = 0 \) and \( x_1 - x_3 = 0 \) describe planes that contain the line through the origin with direction vector \((1, 1, 1)\). Write down the Cartesian equation of a general such plane.

If a plane contains this line, it contains \((0, 0, 0)\), so again we have \( d = 0 \). It also contains \((1, 1, 1)\), so we must have \( 1a + 1b + 1c = 0 \), hence \( c = -a - b \). Thus, the general form is \( ax_1 + bx_2 - (a + b)x_3 = 0 \), as before.

1.3.12. Suppose \( a \neq 0 \) and \( P \subset \mathbb{R}^3 \) is the plane through the origin with normal vector \( a \). Suppose \( P \) is spanned by \( u \) and \( v \), and assume that \( u \cdot v = 0 \).

(a) Show that for every \( x \in P \), we have \( x = \text{proj}_u(x) + \text{proj}_v(x) \).

Since \( x \in P \), and \( P = \text{Span}(u, v) \), we can write \( x = su + tv \) for some scalars \( s, t \). We will try to figure out what \( s \) and \( t \) must be. Compute the dot product of \( x \) with \( u \):

\[
x \cdot u = (su + tv) \cdot u \\
= s(u \cdot u) + t(v \cdot u) \\
= s(u \cdot u)
\]
so \( s = \frac{x \cdot u}{u \cdot u} \). A similar computation shows that \( t = \frac{x \cdot v}{v \cdot v} \). Plugging in these values for \( s \) and \( t \), we have:

\[
x = \frac{x \cdot u}{u \cdot u} u + \frac{x \cdot v}{v \cdot v} v
\]

\[= \text{proj}_u(x) + \text{proj}_v(x)
\]

by definition. \(\Box\)

(b) Show that for any \( x \in \mathbb{R}^n \), we have \( x = \text{proj}_a(x) + \text{proj}_u(x) + \text{proj}_v(x) \).

Following the hint, let \( w = x - \text{proj}_a(x) \), which is just the “perpendicular part” of \( x \) with respect to \( a \). Thus, \( w \cdot a = 0 \), so \( w \in \mathcal{P} \). Hence, by applying the previous part, we can write

\[w = \text{proj}_u(w) + \text{proj}_v(w),\]

and therefore

\[x = \text{proj}_a(x) + \text{proj}_u(w) + \text{proj}_v(w).\]

We just have to check that \( \text{proj}_u(x) = \text{proj}_u(w) \) and \( \text{proj}_v(x) = \text{proj}_v(w) \), and then we’ll be done.

For the first of these equations, let’s write \( \text{proj}_a(x) = ca \), where \( c = \frac{x \cdot a}{a \cdot a} \). We have:

\[
\text{proj}_u(x) = \frac{x \cdot u}{u \cdot u} u
\]

\[= \frac{(w + ca) \cdot u}{u \cdot u}
\]

\[= \frac{w \cdot u}{u \cdot u} + \frac{ca \cdot u}{u \cdot u}
\]

\[= \frac{w \cdot u}{u \cdot u}
\]

\[= \text{proj}_u(w)
\]

as required. Here we used the fact that \( a \cdot u = 0 \), which is true since \( u \in \mathcal{P} \). A similar reasoning applies to show that \( \text{proj}_v(x) = \text{proj}_v(w) \). \(\Box\)

1.4.12. Find all the unit vectors in \( \mathbb{R}^4 \) that make an angle of \( \pi/3 \) with \( (1, 1, 1, 1) \) and an angle of \( \pi/4 \) with both \( (1, 1, 0, 0) \) and \( (1, 0, 0, 1) \).

Let \( a_1 = (1, 1, 1, 1) \), \( a_2 = (1, 1, 0, 0) \), and \( a_3 = (1, 0, 0, 1) \). If \( x \) makes an angle of \( \pi/3 \) with \( a_1 \), we must have

\[
\frac{x \cdot a_1}{\|x\|\|a_1\|} = \cos(\pi/3) = \frac{1}{2}.
\]

If we assume further that \( x \) is a unit vector then (since \( \|a_1\| = \sqrt{1+1+1+1} = 2 \) this simplifies to \( x \cdot a_1 = 1 \). Similar analysis likewise shows that \( x \cdot a_2 = x \cdot a_3 = 1 \). Thus, \( x \) must satisfy the system of linear equations

\[
x_1 + x_2 + x_3 + x_4 = 1
\]

\[x_1 + x_2 = 1
\]

\[x_1 + x_4 = 1.
\]
If we solve this by Gaussian elimination, we get:

\[
\begin{align*}
  x_1 &= 1 - x_4 \\
  x_2 &= x_4 \\
  x_3 &= -x_4 \\
  x_4 &= \text{free}
\end{align*}
\]

There is an additional constraint that enables us to solve for \(x_4\): the fact that \(\mathbf{x}\) must be a unit vector. This gives:

\[
\mathbf{x} \cdot \mathbf{x} = 1
\]
\[
(1 - x_4)^2 + x_4^2 + (-x_4)^2 + x_4^2 = 1
\]
\[
4x_4^2 - 2x_4 + 1 = 1
\]
\[
2x_4(2x_4 - 1) = 0
\]
\[
x_4 = 0 \text{ or } \frac{1}{2}
\]

These two values correspond to the vectors \((1, 0, 0, 0)\) and \((\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})\). As a check, it is easy to see that both of these are unit vectors and that their dot products with \(a_1, a_2,\) and \(a_3\) are all 1, as required.

1.4.15. (a) Prove or give a counterexample: If \(A\) is an \(m \times n\) matrix and \(\mathbf{x} \in \mathbb{R}^n\) satisfies \(A\mathbf{x} = \mathbf{0}\), then either every entry of \(A\) is zero or \(\mathbf{x} = \mathbf{0}\).

This is definitely false: it would be saying that no homogeneous linear equation can have any nonzero solutions! As a very basic counterexample, let \(A = \begin{bmatrix} 1 & 1 \end{bmatrix}\), and \(\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\), so that \(A\mathbf{x} = [1 - 1] = [0]\) (which we’re considering as a vector in \(\mathbb{R}^1\)).

(b) Prove or give a counterexample: If \(A\) is an \(m \times n\) matrix, and \(A\mathbf{x} = \mathbf{0}\) for every vector \(\mathbf{x} \in \mathbb{R}^n\), then every entry of \(A\) is 0.

Following the hint, notice that the entries of \(A\mathbf{x}\) are the dot products \(\mathbf{A}_i \cdot \mathbf{x}\), where \(\mathbf{A}_i\) are the rows of \(A\). If \(A\mathbf{x} = \mathbf{0}\) for all \(\mathbf{x}\), then \(\mathbf{A}_i \cdot \mathbf{x} = 0\) for all \(\mathbf{x}\), and therefore \(\mathbf{A}_i = \mathbf{0}\) by Problem 1.2.16 (from last week, solved above). Hence we deduce that \(A\) is the zero matrix. \(\square\)