Math 107: Linear Algebra and Differential Equations

Test #1

Name: _____________________________  Thursday, October 1, 2009

All answers must be justified.

Question 1. [10 points]

(a) Find the inverse of the following matrix

\[
A = \begin{bmatrix}
1 & -2 \\
2 & -3
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 & 2 \\
-2 & 1
\end{bmatrix}
\]

(b) Use the inverse to compute the solution to the system of equations

\[
x - 2y = 5
\]
\[
2x - 3y = 7
\]

\[
x = -1
\]
\[
y = -3
\]
Question 2. [16 points] Suppose that $A$ and $B$ are two $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = 7$. Compute the following:

(a) $\det(ABA) =$
\[
3 \times 7 \times 3 = 63
\]

(b) $\det(A^{-1}) =$
\[
\frac{1}{3}
\]

(c) $\det(BAB^{-1}) =$
\[
7 \times 3 \times \frac{1}{7} = 3
\]

(d) $\det(A^TB^{-1}) =$
\[
\frac{3}{7}
\]

(e) $\det(2A) =$
\[
2^n \cdot 3
\]
Question 3. [20 points] Consider the system of questions with unknowns $x_1, x_2, x_3$:

\[
ax_1 + x_3 = 1 \\
2(b-a)x_2 + 2x_3 = 4 \\
2ax_1 + (b-a)x_2 + x_3 = 2
\]

(a) Determine the conditions on $a$ and $b$ so that the above system of equations admits a unique solution.

\[
\begin{bmatrix} a & 0 & 1 \\ 0 & b-a & 1 \\ 2a & b-a & 1 \end{bmatrix}^2 \rightarrow \begin{bmatrix} a & 0 & 1 \\ 0 & b-a & 1 \\ 2a & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & b-a & 1 \\ a & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & b-a & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[b \neq a \text{ and } a \neq 0\]

(b) Determine the conditions on $a$ and $b$ so that the above system of equations has no solution.

\[b = a\]
Question 4. [10 points] Under what condition on the constants $a$ and $b$ is the vector
\[
\begin{bmatrix}
a \\ b
\end{bmatrix}
\]
in the linear span of the vectors:
\[
\begin{bmatrix}
2 \\ 1 \\ 7
\end{bmatrix}, \quad
\begin{bmatrix}
-1 \\ 2 \\ 4
\end{bmatrix}, \quad
\begin{bmatrix}
3 \\ 1 \\ 9
\end{bmatrix}
\]
\[
\begin{bmatrix}
2 & -1 & 3 & a \\ 1 & 2 & 1 & 1 \\ 7 & 4 & 9 & b
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 1 & 1 \\ 2 + 3 & 3 & a \\ 7 & 4 & 9 & b
\end{bmatrix} \rightarrow
\begin{bmatrix}
1 & 2 & 1 & 1 \\ 0 & -3 & 1 & a-2 \\ 0 & 0 & 0 & b-7-2(a-2)
\end{bmatrix}
\]
\[
b - 7 - 2(a - 2) = 0
\]
or \[
2a - b + 3 = 0
\]
Question 5. [14 points] Given

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \]

Find the rank and the dimension of the null space of \( A \).

\[ A \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -6 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \]

One free variable \( \Rightarrow \dim(\text{NS}(A)) = 1 \).

Two pivots \( \Rightarrow \text{rank}(A) = 2 \).
Question 6. [15 points]

(a) Show that the set

\[ U = \left\{ \begin{bmatrix} x \\ x-y \\ y \end{bmatrix} : x, y \in \mathbb{R} \right\} \]

satisfies the conditions for being a subspace of \( \mathbb{R}^3 \).

1. It contains \( \mathbf{0} \) (\( x=0, y=0 \))

2. Closed under addition

   if \( U = \begin{bmatrix} x_1 \\ x_1-y_1 \\ y_1 \end{bmatrix} \) and \( V = \begin{bmatrix} x_2 \\ x_2-y_2 \\ y_2 \end{bmatrix} \) then \( U+V = \begin{bmatrix} x_1+x_2 \\ (x_1+x_2)-(y_1+y_2) \\ y_1+y_2 \end{bmatrix} \)

3. Closed under scalar multiplication

   \( \alpha U = \begin{bmatrix} \alpha x_1 \\ \alpha(x_1-y_1) \\ \alpha y_1 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_1-\alpha y_1 \\ \alpha y_1 \end{bmatrix} \)

(b) Find a basis for \( U \). What is the dimension of \( U \)?

\[
\begin{bmatrix} x \\ x-y \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}
\]

basis = \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}

\text{dimension} = 2
Question 7. [20 points] (a) Determine whether

\[
V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ -1 \end{bmatrix} \right\}
\]

form a spanning set for \( \mathbb{R}^3 \). Explain your answer.

Find \( c_1, c_2, c_3 \) so that

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ 10 \\ -1 \end{bmatrix}
\]

\[
\begin{bmatrix} 1 \\ -1 \\ 7 \\ a \\ 2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 0 \\ 5 \\ -4 \\ b-2a \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} 0 \\ 5 \\ -4 \\ 0 \\ 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

The system has a unique solution.

\( V \) form a spanning set for \( \mathbb{R}^3 \).

(b) Does \( V \) above form a basis for \( \mathbb{R}^3 \)? Why or why not.

Yes because there are 3 vectors in \( V \).